AN INTRODUCTION TO MATHEMATICS

AN INTRODUCTION TO MATHEMATICS

With Applications to Science and Agriculture

BY

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F. S. CROFTS & CO.

NEW YORK -:- -:- MCMXXX

MANUFACTURED IN THE UNITED STATES OF AMERICA
BY BRAUNWORTH & CO., INC., BROOKLYN, NEW YORK

PREFACE

After some fourteen years of teaching in American colleges and universities the author finds that the average high school graduate has not developed in himself a mathematical type of reasoning. He therefore hopes that this treatment may in some measure accomplish this purpose.

The first few chapters are devoted to a thorough review of high school algebra, for the author is convinced that most college freshmen need considerable drill on the fundamental processes of algebra before attempting a very extensive study of mathematics.

In preparing this book the author has kept in mind two types of students: *first,* those who will never take additional work in mathematics, and *second,* those who will continue the work in science or agriculture for advanced degrees and will doubtless desire to pursue additional courses in mathematics. He has therefore attempted to write a book basic in the fundamental principles of mathematics and at the same time has endeavored to make practical applications to the fields of science and agriculture, wherever possible. He feels that a thorough knowledge of the material covered in this work will enable the second type of student to successfully pursue a course in analytical geometry followed by a course in the calculus.

The author gratefully acknowledges his indebtedness to his colleagues, Professor Wm. Asker for preparing the chapter on statistics, and Mr. H. B. MacDougal for checking much of the material, to Professor I. W. Smith of the North Dakota Agricultural College for using the material in mimeographed form and offering many valuable suggestions, to Dean D. A. Rothrock of Indiana University for reading most of the manuscript and to Professor Wm. Marshall of Purdue University for encouraging him in the work.

The author also desires to thank Professor E. S. Crawley of the University of Pennsylvania for his generous permission to use the greater part of his *Tables of Logarithms* as a portion of this book.

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AN INTRODUCTION TO MATHEMATICS

CHAPTER I

ALGEBRAIC OPERATIONS

1. Four fundamental operations. The operations with numbers are made up of additions, subtractions, multiplications and divisions. These operations are known as the four fundamental operations of algebra.

2. Addition and subtraction.

a. Addition is commutative. This means that $a + b = b + a$.

b. The sum of two or more numbers is the same, irrespective of the way in which they are grouped. Thus:

$$a + b + c = (a + b) + c = a + (b + c).$$

3. Use of parentheses. Signs of aggregation. The signs of aggregation are:

Parentheses	( )
Brackets	[ ]
Braces	{ }
Vinculum	——
Bar	|

Signs of aggregation may be removed with or without change of sign of each term included within the signs of aggregation, according as the sign − or + precedes the sign of aggregation. Thus:

$$\begin{aligned} & x - \{3y - 2[z - (y - 3) - (2y + 4)]\} \\ &= x - \{3y - 2[z - y + 3 - 2y - 4]\} \\ &= x - \{3y - 2z + 6y + 2\} \\ &= x - 9y + 2z - 2 \end{aligned}$$

Exercises

Add:

1. $3x - 2y,\ 7x + 6y,\ -5x + 4y.$

2. $2x + 3y - 7z,\ 4x - 9y + 6z,\ 8x + 7y + 3z.$

3. $4a^2b^2 + 5ac^2 - 2c^3,\ 7a^2b^2 - 2ac^2,\ 7ac^2 + 4c^3.$

4. $2x^2 - 3ax + 3c,\ 3x^2 + ax - c,\ x^2 - 2ax - 5c.$

5. $7x^3 - 4x + 2x^2 - 5,\ -2x + 5x^3 + 1 - 2x^2,\ 4 + 3x + 2x^2 + x^3.$

Subtract the first expression from the second in the following:

6. $3m + 2n,\ 4m - 5n.$

7. $2a^2 + 3a - 5,\ 4a^2 - 2a + 4.$

8. $3x^2 + 5xy - 4y^2 - 3x,\ 4x^2 - 2xy + y^2 + 2x.$

9. $5a^3 + 6ay^2 + 3ay - 2a + 3,\ 2a^3 + ay - 5ay^2 + 3a - 7.$

10. From the sum of $3a - 4b + 5c$ and $8b - 2a - 3c$, subtract the sum of $3a - 2b + 4c$ and $4b - 5a - 2c$.

Combine coefficients of similar terms:

11. $ay + by + cy.$

Solution. $ay + by + cy = (a + b + c)y.$

12. $2ax + 6bx - 3cx.$

13. $4x - 2abx + 7cx.$

14. $am + bm + an + bn.$

Simplify by removing signs of aggregation and combining like terms:

15. $2a - 3 + (x - 5a) - 2(3a - 2x).$

16. $2(a - 3) - 7(a + 2) + 8(a - 3).$

17. $7y - 3[4 - 2(y + 1) + 3(y - 4) - (y - 7)].$

18. $3a - [2a - b - \{3a - 2b - (2a + \overline{b - a}) + 3a\} - 3b].$

19. $x - (-2x - \{-5x + [x - \overline{3x - 2}] - 3\} - [3x - \overline{2x + 3}])$.
20. $7y - 5[4 - 3(y - 4 - b) - 4\{b - (5y + b) - 5\}]$.
21. $4z - \{-5z + (2z - 4w - \overline{3z + 2}) - 3(2z - 5w) - 8\}$.

4. Multiplication.

a. The factors of a product may be taken in any order. Thus, $cd = dc$.

b. The factors of a product may be grouped in any manner. Thus, $abc = (ab)c = a(bc) = b(ac)$.

c. When m and n are positive integers, $a^m \cdot a^n = a^{m+n}$.

That is, the exponent of the product of two or more powers of a number is the sum of the exponents of the powers taken singly. This is known as the first law of exponents for positive integers.*

Exercises

Multiply:

1. $2abc,\ -3a^2bc^2,\ 5ab^3c$.

Solution. $(2abc)(-3a^2bc^2)(5ab^3c)$

$= (2 \cdot -3 \cdot 5)(a \cdot a^2 \cdot a)(b \cdot b \cdot b^3)(c \cdot c^2 \cdot c)$ [(*a*) and (*b*), Art. 4]

$= -30a^4b^5c^4$. [(*c*), Art. 4]

2. $3x^2,\ 7xy^2 + 3x^2y - 2x^3y^2$.

3. $5a^2 + 2ay^2,\ 3a - 4a^2y$.

4. $(x + y)(x + y)(x + y)$.

5. $(a - b - c)(a + b + c)$.

6. $(x + y + 2z)^2$.

7. $\left(x - \dfrac{y}{3}\right)\left(x + \dfrac{y}{3}\right)$.

8. $(x^2 - y)^3$.

9. $(m^2 - mn + n^2)(m^2 + mn + n^2)$.

10. $(a - b)^n(a - b)^3$.

* For a complete discussion of exponents see Chapter VIII.

5. Division. If a and b are any given numbers and b is not *zero*, there is only one number x such that $a = bx$. The process of finding x is the process of dividing a by b. a is called the dividend, b the divisor and x the quotient.

Example. $\dfrac{a^m}{a^n} = a^{m-n}$, where $m > n$.

That is, the exponent in the quotient of two powers of a number is the exponent of the dividend minus the exponent of the divisor.

Note the condition that b is not to be zero. This means that the divisor can not be zero.

6. Division of a polynomial by a polynomial. Before performing the indicated division the dividend and divisor should be arranged according to ascending or descending powers of some letter.

Example. Divide $19a - 9a^2 + a^4 + 3a^3 - 6$ by $3 + a^2 - 2a$.

Solution.

$$\begin{array}{llll}
a^4 + 3a^3 - 9a^2 + 19a - 6 & \quad | \; a^2 - 2a + 3 \\
a^4 - 2a^3 + 3a^2 & \quad | \; a^2 + 5a - 2 \\
\hline
\quad 5a^3 - 12a^2 & \\
\quad 5a^3 - 10a^2 + 15a & \\
\hline
\qquad - 2a^2 + 4a - 6 & \\
\qquad - 2a^2 + 4a - 6 &
\end{array}$$

7. Zero in division. Division by zero is excluded from the operations in algebra. That is to say, the divisor can not be zero. If the dividend is zero, the quotient is zero. That is, $\dfrac{0}{a} = 0$.

Where is the fallacy in the following?

Let $\qquad x = m. \qquad (1)$

Multiply both sides by x, $\qquad x^2 = mx. \qquad (2)$

Subtract m^2 from both sides, $x^2 - m^2 = mx - m^2$. (3)

Divide both sides by $x - m$, $x + m = m$. (4)

But by (1) $x = m$. (5)

By (4) and (5) $2m = m$. (6)

Hence $2 = 1$. (7)

Exercises

Divide:

1. $33a^3b^3c^3 - 9a^2bc^2 + 15ab^2c$ by $3abc$.
2. $20c4d^3 + 15c^2d^2 - 10cd$ by $5cd$.
3. $\dfrac{x^4 - 13x^3y + 5x^2}{x^2}$.
4. $\dfrac{-5y^3 + 15y^4 - 10y_2}{5y^2}$.
5. $\dfrac{16x^3y^2 - 8xy^3 + 12x^2y^2}{4xy^2}$.
6. $x^2y^2 - y^4 + x^4$ by $x^2 - xy + y^2$.

The solution of this example gives $x^2 + xy + y^2$ as a quotient and $-2y^4$ as a remainder. And as in arithmetic, the complete quotient may be written.

$$x^2 + xy + y^2 + \frac{-2y^4}{x^2 - xy + y^2}.$$

7. $y^3 + 27$ by $y + 3$.
8. $a^3 - 8$ by $a - 2$.
9. $x^3 - 5x^2 - 17x + 66$ by $x - 6$.
10. $2x^4 - 8x^3 + 7 + 3x^2 + 10x$ by $2x^2 - 4x - 7$.
11. $a^4 + a^2b^2 + b^4$ by $a^2 + ab + b^2$.
12. $2x^3 - 9x^2y - 12y^3 + 17xy^2$ by $2x - 3y$.
13. $2x^4 - x^3y - 3x^2y^2 + xy^3$ by $x^2 + xy$.
14. $4a^3 - 3a - 15a^2 + 4$ by $a^2 - 3a - 3$.
15. $8a^3 + 27b^3$ by $2a + 3b$.

CHAPTER II

FACTORING

8. Important type products. The following type forms have already been treated in high school algebra. They should be reviewed here and memorized.

a. Common monomial factor.

$$ab + ac = a(b + c).$$

Example. $$2ax - 6a^3 = 2a(x - 3a^2.$$

b. Trinomial square.

$$a^2 + 2ab + b^2 = (a + b)^2.$$

Example. $$9 + 6a + a^2 = (3 + a)^2.$$

c. Difference of two squares.

$$m^2 - n^2 = (m - n)(m + n).$$

Example. $$(a + 2b)^2 - c^2 = (a + 2b - c)(a + 2b + c).$$

d. Trinomial of the form.

$$x^2 + (m + n)x + mn = (x + m)(x + n).$$

Example. $$x^2 + 5x + 6 = (x + 2)(x + 3).$$

e. Difference of two cubes.

$$m^3 - n^3 = (m - n)(m^2 + mn + n^2).$$

Example. $$8m^6 - n^3s^3 = (2m^2)^3 - (ns)^3$$
$$= (2m^2 - ns)(4m^4 + 2m^2ns + n^2s^2).$$

f. Sum of two cubes.

$$m^3 + n^3 = (m + n)(m^2 - mn + n^2).$$

Example. $8x^3 + 27y^6 = (2x)^3 + (3y^2)^3$

$$= (2x + 3y^2)(4x^2 - 6xy^2 + 9y^4).$$

g. Trinomial of the form.

$$ax^2 + bx + c.$$

Certain expressions of this form may be factored by inspection. The factors are two binomials whose first terms are factors of ax^2 and whose last terms are factors of c. Now we must choose the terms of binomials so that the algebraic sum of the cross products is bx.

Example. Factor $6x^2 - x - 15$.

The first terms of the factors are $3x$ and $2x$ or $6x$ and x, and the last terms of the factors are ± 3 and ∓ 5, or ± 1 and ∓ 15. Choosing the terms so that the algebraic sum of the cross products is $-x$, we find the factors to be $2x + 3$ and $3x - 5$.

Hence $$6x^2 - x - 15 = (2x + 3)(3x - 5).$$

h. Grouping of terms.

$$mx + ny + nx + my = (m + n)(x + y).$$

Example.

$$14ax + 21bx - 4ay - 6by = 7x(2a + 3b) - 3y(2a + 3b)$$
$$= (7x - 2y)(2a + 3b).$$

Exercises

Factor the following:

1. $2m + 3mn$.

2. $a^2 - 9b^2$.

3. $x^2 - 9x + 8$.

4. $t^2 + 9t - 36$.

5. $x^3 - 8y^6$.
6. $a^2 - c^2 - b^2 + 2bc$.
7. $7mx - 9ny + 7nx - 9my$.
8. $x^2 - mx + 2nx - 2mn$.
9. $8m^3n^6 + 27p^3$.
10. $(m + 2)^2 - 5(m + 2) - 176$.
11. $(a - b)^3 - 3(a - b)^2 - 4(a - b)$.
12. $y^2 - 8yz - 9x^2 + 16z^2$.
13. $21x^2 - 26x - 15$.
14. $a^4 - 16b^4 = (a^2)^2 - (4b)^2$.
15. $16a^2 + 56ab + 49b^2$.
16. $x^{3n} - y^3 = (x^n)^3 - y^3$. (Find two factors only.)
17. $(x + y)^3 - (v - w)^3$.
18. $x^4 - 3x^3y - 10x^2y^2$.

9. Other important products.

i. Square of a polynomial. The square of a polynomial equals the sum of the squares of the terms of the polynomial plus twice the product of each term by every term that follows. Thus,

$$(x + y + z + w)^2 = x^2 + y^2 + z^2 + w^2 + 2xy + 2xz + 2xw + 2yz + 2yw + 2zw.$$

j. Expressions that can be written as the difference of two squares.

Example. Factor $4a^4 + b^4$.

By the addition of $4a^2b^2$ and the subtraction of the same term we have

$$\begin{aligned} 4a^4 + b^4 &= 4a^4 + 4a^2b^2 + b^4 - 4a^2b^2 \\ &= (2a^2 + b^2)^2 - (2ab)^2 \\ &= (2a^2 + b^2 - 2ab)(2a^2 + b^2 + 2ab). \end{aligned}$$

k. Cube of a binomial. By actual multiplication, we find that

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

and

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

NOTE.—If a polynomial can be put into the form of the product under (*i*), it can be factored.

Example. $a^2 + b^2 + 16c^2 + 2ab + 8ac + 8bc$

$$= (a)^2 + (y)^2 + (4c)^2 + 2\cdot a\cdot b + 2\cdot a\cdot 4c + 2b\cdot 4c$$

$$= (a + b + 4c)^2.$$

Exercises

1. $x^2 + 9y^2 + 4z^2 + 6xy - 4xz - 12yz$. (See 9 (*i*).)
2. $l^2 + 4m^2 - 4lm + 4l - 8m + 4$.
3. $x^3 - 9x^2y + 27xy^2 - 27y^3$. (See 9 (*k*).)
4. $a^3 + 6a^2b + 12ab^2 + 8b^3$.
5. $x^4 - 6x^2 + 1$.
6. $a^4 - a^2 + 1$.
7. $x^4 + 4y^4$.
8. $x^2 + y^2 + 16z^2 - zxy + 8xz - 8yz$.

10. Highest common factor. A number or expression which will divide two or more expressions without a remainder, is called a common factor of those expressions.

The product of all the common prime factors of two or more expressions is called their highest common factor (H.C.F.).

To find the H.C.F. of two or more expressions, resolve each into its prime factors, and then find the product of the common prime factors.

Example. Find the H.C.F. of $a^2 - b^2$ and $a^2 - 5ab + 4b^2$.

Solution.

$$a^2 - b^2 = (a - b)(a + b),$$

$$a^2 - 5ab + 4b^2 = (a - b)(a - 4b).$$

$$\therefore \quad (a - b) = \text{H.C.F.}$$

Exercises

Find the H.C.F. of the following sets of expressions:

1. ax^2, $2abx$, $3a^2b^2$.

2. 52, 117, 78.

3. $x^2 + 2xy + y^2$, $x^2 + xy$, and $x^2 - 7xy - 8y^2$.

4. $x^3 - 1$, $x^2 + 13x - 14$, $x^2 - 1$.

5. $x^3 + 3x^2y + 3xy^2 + y^3$, $x^3 + 2x^2y + xy^2$, and $x^2y + 2xy^2 + y^3$.

6. $r^2 - 6r + 9$, $r^2 + 5r - 24$, and $r^2 - 9r + 18$.

7. $(x^2y - xy^2)^2$, $xy(x^2 - y^2)$.

8. $x^2 - 3x - 40$, $x^2 - x - 30$, $x^2 + 3x - 10$.

9. $x^2 - (y + c)^2$, $(y - x)^2 - z^2$, $y^2 - (x - z)^2$.

10. $(x^2 - 1)(x^2 + 5x + 6)$, $(x^2 + 3x)(x^2 - x - 6)$.

11. Lowest common multiple. *The lowest common multiple* (L.C.M.) *of two or more expressions is defined as the product of all their prime factors, each taken the greatest number of times that it occurs in any of the expressions.* It is evident that the L.C.M. of two or more expressions is the expression of lowest degree which contains each of the given expressions as a factor.

Example. Find the L.C.M. of $x^2 - x - 2$; $x^2 - 8x + 12$; $x^2 - 5x - 6$.

Solution.

$$x^2 - x - 2 = (x - 2)(x + 1),$$

$$x^2 - 8x + 12 = (x - 2)(x - 6),$$

$$x^2 - 5x - 6 = (x - 6)(x + 1).$$

$$\therefore \quad (x - 2)(x + 1)(x - 6) = \text{L.C.M.}$$

Exercises

Find the L.C.M. of each of the following sets of expressions:

1. $3ax^2$; a^2bx; $2ab^2x$.

2. $x^2 + xy$; $x^3 + y^3$; $x^2 - 3xy - 4y^2$.

3. $x + 1$; $x + 2$; $x^2 + 3x + 2$.

4. $a^2 + 7a + 10$; $a^2 + 4a - 5$.

5. $a^3 + 8$; $a^3 - 8$; $a^2 - 4$.

6. $x^2 - x - 6$; $x^2 - 6x + 9$; $6x - 18$.

7. $a^2 - b^2$; $a^2 - 2ab - 3b^2$; $(a + b)^2$.

8. $6x^2 + 18x - 60$; $3x^2 + 24x + 45$; $8x^2 - 24x + 16$.

CHAPTER III

LINEAR EQUATIONS IN ONE UNKNOWN

12. Equalities. A statement that two expressions are equal is called an *equality*.

There are two kinds of equalities, identical equalities or identities, and conditional equalities or equations.

In an identity the two members are equal for all values of the symbols for which the expressions are defined. Thus,

$$x^2 - 4 = (x - 2)(x + 2) \text{ is an identity.}$$

A conditional equality or an equation is true for only certain values of the letters involved. Thus, $x - 3 = 7$ is an equation and is true for the value $x = 10$ only.

Exercises

Which are the following, equations or identities?

1. $(l + m)^2 = l^2 + 2lm + m^2$.

2. $\dfrac{y^2 - b^2}{y - b} = y + b$.

3. $x^3 - 3x + 2 = 0$.

4. $y^2 - 2y + 3 = 0$.

13. Solution or root of an equation. *By the solution or root of an equation in one unknown we mean the value of the unknown that reduces the equation to an identity.* Thus, 6 is a solution of $x - 2 = 4$, for when $x = 6$ the equation becomes the identity $4 = 4$.

14. Equivalent equations. Two equations having the same roots are said to be equivalent. Thus the equations $x - 5 = 0$ and $2x - 10 = 0$ are equivalent.

15. Operations on equations. The following operations may be performed on the members of an equation:

(1) Adding the same number to both members.

(2) Subtracting the same number from both members.

(3) Multiplying both members by the same number, zero excluded.

(4) Dividing both members by the same number, zero excluded.

16. Type form of the linear equation in one unknown. The linear equation in a single unknown is of the form:

$$Ax + B = 0, \quad A \neq 0. \tag{1}$$

In fact every linear equation in one unknown can be reduced to the form of (1). Its solution is $x = -\frac{B}{A}$, as may be verified by substitution.

17. Verification by substitution. The operations of Art. 15 are useful in finding solutions but the solution is not complete until the values of the unknown are substituted in the equation to be solved. If such substitution produces an identity the solution is correct.

Exercises and Problems

Solve the following for x and verify the results:

1. $4x + 5 = 2x - 3.$

Solution. $$4x + 5 = 2x - 3. \tag{1}$$

Transpose and collect $$2x = -8. \tag{2}$$

Divide bv 2 $$x = -4. \tag{3}$$

Check. Substitute (3) in (1) and

$$4(-4) + 5 = 2(-4) - 3,$$

$$-11 = -11.$$

2. $3x + 5 = 7x - 9.$

3. $2x(3x + 2) = 6x^2 - 8.$

4. $4(x + 2) + x^2 = x^2 - 8.$

5. $(x + 1)(x + 2) = x(x + 4).$

6. $2[x + x(x - 1) + 1] = (x + 2)(2x - 1).$

7. $(a + b)x + (a - b)x = ab.$

8. $\frac{x}{2} + \frac{2x}{5} = \frac{3x}{10} + 6.$

9. $1.5x + 3.2x = 2.3x + 12.72.$

10. $\frac{x}{m} + \frac{y}{n} = 1.$

11. Given $s = vt$, solve for v and t.

12. Given $s = \frac{1}{2}gt^2$, solve for g.

13. Given $F = \frac{9}{5}C + 32$, solve for C.

14. A miller has wheat worth $2.20 per bushel and another lot worth $1.80 per bushel. He wishes to mix these to make 40 bushels of wheat which shall be worth $2.10 per bushel. How much of each shall he take?

15. A farmer has a cow whose milk contains 4% of butter fat (called a 4% milk) and another one which gives 5% milk. How shall he mix the milks to obtain 40 pounds of a $4\frac{1}{4}$% milk?

16. How much cream that contains 30% butter fat should be added to 500 pounds of milk that contains $3\frac{1}{2}$% butter fat to produce a standard milk with 4% of butter fat?

17. The milk from a certain cow contains $3\frac{1}{2}$% butter fat while that of another cow contains $4\frac{3}{4}$% butter fat. What will be the percentage of fat in an equal mixture?

18. A man made two investments amounting together to $5,000. On the first he gained 8%, and on the second he lost 6%. His net gain on the two was $120. What was the amount of each investment?

19. How heavy a stone can a man, by exerting a force of 175 pounds, lift with a crow-bar 6 feet in length if the fulcrum be six inches from the stone (neglect the weight of the crow-bar)?

Remember that $W \cdot w = F \cdot f$, where W is the weight to be lifted, F the force applied, w the distance between weight and fulcrum and f the distance between force and fulcrum.

20. A man can do a piece of work in 5 days, another in 6 days, and a third in 12 days. How many days will it require all to do it when working together?

21. The milk from a cow that gives 4 gallons of milk containing 3% butter fat is mixed with 9 gallons of milk containing 5% butter fat. What is the percentage of butter fat in the mixture?

CHAPTER IV

FRACTIONS

18. Algebraic fraction. An algebraic fraction is the indicated quotient of two expressions. Thus, $\frac{m}{n}$ means m divided by n.

19. Operations. The following operations and principles are used in the treatment of fractions:

I. The value of a fraction is not changed by multiplying or dividing both numerator and denominator by the same number. That is,

$$\frac{m}{n} = \frac{am}{an} \quad \text{and} \quad \frac{m}{n} = \frac{\frac{m}{a}}{\frac{n}{a}}.$$

II. Changing the sign of either the numerator or denominator of a fraction is equivalent to changing the sign of the fraction. That is,

$$\frac{-a}{b} = -\frac{a}{b} = \frac{a}{-b}.$$

III. Adding two fractions having a common denominator gives a fraction whose numerator is the sum of the numerators and whose denominator is the common denominator. That is,

$$\frac{l}{n} + \frac{m}{n} = \frac{l+m}{n}.$$

Also, $$\frac{l}{n} - \frac{m}{n} = \frac{l - m}{n}.$$

IV. The sum and the difference of any two fractions, $\frac{a}{b}$ and $\frac{c}{d}$, are equal to $\frac{ad + bc}{bd}$ and $\frac{ad - bc}{bd}$ respectively.

For, by I, $$\frac{a}{b} = \frac{ad}{bd}, \quad \text{and} \quad \frac{c}{d} = \frac{bc}{bd}.$$

Hence by III, $$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \quad \text{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

V. The product of two or more fractions is a fraction whose numerator is the product of their numerators and whose denominator is the product of their denominators. That is,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

VI. To divide one fraction by another, invert the divisor and then multiply. That is,

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

The reciprocal of a number is 1 divided by the number. Thus the reciprocal of a is $\frac{1}{a}$; of $\frac{m}{n}$ is $\frac{n}{m}$.

20. Reduction of a fraction to its lowest terms. Separate the numerator and the denominator into their prime factors and then cancel common factors by division.

Reduce to its lowest terms, $\frac{x^2 + 6x + 9}{x^2 - 9}$.

Solution. $$\frac{x^2 + 6x + 9}{x^2 - 9} = \frac{(x + 3)(x + 3)}{(x - 3)(x + 3)} = \frac{x + 3}{x - 3}.$$

Exercises

Reduce to lowest terms:

1. $\dfrac{51x}{85y}$.

2. $\dfrac{a^2 + ab}{a^2 - ab}$.

3. $\dfrac{a^3 - b^3}{a^2 - b^2}$.

4. $\dfrac{(m + n)^3}{m^3 + n^3}$.

5. $\dfrac{(a - b)(c - d)(b - c)}{(a - c)(c - b)(a - b)}$.

6. $\dfrac{x + 1}{(x + 1) + (x + 1)^2}$.

7. $\dfrac{(x^4 - y^4)(x^3 - y^3)}{(x^3 + y^3)(x^2 + y^2)}$.

8. $\dfrac{a^2 - 3a + 2}{a^2 + 4a - 5}$.

9. $\dfrac{x^2 + x - 20}{x^2 + 4x - 5}$.

10. $\dfrac{m - m^2 - n + mn}{m - mn + n^2 - n}$.

21. Addition and subtraction. Reduce the fractions to be added or subtracted to a common denominator and then add or subtract numerators.

Example. Add $\dfrac{3}{a^2 + 2a + 1} + \dfrac{4a}{a^2 - 1}$.

Solution.

$$\frac{3}{a^2 + 2a + 1} + \frac{4a}{a^2 - 1} = \frac{3}{(a + 1)(a + 1)}$$

$$+ \frac{4a}{(a + 1)(a - 1)} = \frac{3(a - 1)}{(a + 1)(a + 1)(a - 1)}$$

$$+ \frac{4a(a + 1)}{(a + 1)(a + 1)(a - 1)} = \frac{4a^2 + 7a - 3}{(a + 1)(a + 1)(a - 1)}.$$

Exercises

Perform the following additions and subtractions:

1. $\dfrac{x}{5} + \dfrac{y}{6}$.

2. $\dfrac{a}{x} - \dfrac{a + b}{2x}$.

3. $\dfrac{1}{x} + \dfrac{1}{x^2} + \dfrac{1}{x^3}$.

4. $\dfrac{1}{x^2 - 4} - \dfrac{1}{(x - 2)^2}$.

5. $\frac{7}{a+b} - \frac{2}{a} - \frac{5}{b}$.

6. $\frac{a+b}{a-b} - \frac{a-b}{a+b}$.

7. $\frac{x}{a+1} - \frac{x}{a-1}$.

8. $x + y + \frac{x}{y} - 1$.

9. $\frac{4}{x+1} - \frac{x-2}{x^2-x} - \frac{3x}{x^2-1}$.

10. $\frac{5}{3x-3} - \frac{8}{5x-15}$.

11. $\frac{1}{x} + \frac{1}{y} - \frac{1}{x+y} + \frac{1}{x-y}$.

12. $\frac{x^2+8x+13}{x^2+7x+10} - \frac{x-1}{x+2}$.

13. $\frac{3}{2(x+2)} - \frac{2}{(x+2)^2} + \frac{1}{2(x-2)}$.

14. $\frac{a+b}{a-b} - \frac{a-b}{a+b} - \frac{6a^2-2b^2}{a^2-b^2}$.

15. $\frac{a+4}{a^2+a+1} - \frac{a^2+4a-2}{1-a^3} - \frac{-1}{1-a}$.

16. $\frac{3a}{a^2+a-20} + \frac{2}{a^2-6a-55} - \frac{a-1}{a^2-15a+44}$.

17. $\frac{x+3}{x^2+5x+6} + \frac{x+2}{x^2+8x+12}$.

18. $\frac{1}{(a-b)(b-c)} + \frac{1}{(a-c)(c-b)} - \frac{1}{(b-a)(c-a)}$.

22. Multiplication and division. See principles V and VI. (Art. 19.)

Example (a). Find $\frac{x^2-2x+1}{x^2-1} \cdot \frac{x+1}{x^2+1}$.

Solution.

$$\frac{\overset{x-1}{\cancel{x^2-2x}+1}}{\underset{\cancel{x-1}}{\cancel{x^2-1}}} \cdot \frac{\cancel{x+1}}{x^2+1} = \frac{x-1}{x^2+1}.$$

Example (b). $\dfrac{(a-b)^2}{a+b} \div \dfrac{a^2-ab}{b}$.

Solution. $\dfrac{(a-b)^2}{a+b} \div \dfrac{a^2-ab}{b} = \dfrac{(a-b)^2}{a+b} \cdot \dfrac{b}{a^2-ab}$

$$= \frac{\cancel{(a-b)}(a-b)}{a+b} \cdot \frac{b}{a\cancel{(a-b)}} = \frac{b(a-b)}{a(a+b)}.$$

Exercises

Perform the following multiplications and divisions:

1. $\dfrac{2}{(1+x)^2} \cdot \dfrac{x+1}{x-1}$.

2. $\dfrac{1}{a-b}\left(\dfrac{1}{y-a} - \dfrac{1}{y-b}\right)$.

3. $\dfrac{m^2-mn}{a^2-ab} \cdot \dfrac{a^2+ab}{m^2+mn}$.

4. $\dfrac{6ab}{a-b} \div \dfrac{8ax}{a+b}$.

5. $\dfrac{m^2+2mn}{m^2+4n^2} \div \dfrac{m^2-4n^2}{mn-2n^2}$.

6. $\dfrac{a+b}{c} \div \dfrac{a^2-b^2}{2c^2}$.

7. $\left(\dfrac{x^2}{a^2} - \dfrac{x}{a} + 1\right) \div \left(\dfrac{x^2}{a^2} + \dfrac{x}{a} + 1\right)$.

8. $\dfrac{n^2-n-20}{n^2-25} \cdot \dfrac{n^2-25}{n+1} \div \dfrac{n^2+2n-8}{n^2-n-2}$.

9. $\left(a + \dfrac{ab}{a-b}\right)\left(b - \dfrac{ab}{a+b}\right)$.

10. $\dfrac{x^2+y^2}{x^3-y^3} \cdot \dfrac{x^4+y^4}{x^4-y^4} \div \dfrac{x+y}{(x-y)^2}$.

23. Complex fractions. A fraction with a fraction in its numerator or denominator or in both is called a complex fraction.

To simplify a complex fraction multiply both the numerator and the denominator of the complex fraction by the L.C.M. of the denominators of the simple fractions that make up the terms.

Example. $$\frac{\dfrac{1}{x}+\dfrac{1}{y+x}}{\dfrac{1}{y}-\dfrac{1}{y-x}}=\frac{xy(y^2-x^2)\left[\dfrac{1}{x}+\dfrac{1}{y+x}\right]}{xy(y^2-x^2)\left[\dfrac{1}{y}-\dfrac{1}{y-x}\right]}$$

$$=\frac{y(y^2-x^2)+xy(y-x)}{x(y^2-x^2)-xy(y+x)}=-\frac{y(y^2-2x^2+xy)}{x^2(x+y)}.$$

Exercises

Simplify the following fractions:

1. $\dfrac{\frac{1}{3}+\frac{1}{4}}{\frac{7}{6}}.$

2. $\dfrac{x-\dfrac{1}{y}}{y-\dfrac{1}{x}}.$

3. $\dfrac{p}{1-\dfrac{1}{p+1}}.$

4. $\dfrac{\dfrac{1}{1-2x}}{\dfrac{x^2}{1-2x}+1}.$

5. $\dfrac{\dfrac{x}{x-1}-1}{1+\dfrac{x}{1-x}}.$

6. $\dfrac{2x+3-\dfrac{1}{x-1}}{x-1}.$

7. $\dfrac{1-\dfrac{x-1}{2}-x}{1-\dfrac{1-x}{2}-x}.$

8. $\dfrac{x-y+\dfrac{1}{x}}{\dfrac{1}{x^2}+\dfrac{1}{x}+1}.$

9. $\dfrac{m-n}{m-n+\dfrac{1}{m+n+\dfrac{1}{m-n}}}.$

10. $\dfrac{\left(\dfrac{a+b}{a^2+ab+b^2}\right)\left(a^2+\dfrac{b^4}{a^2+b^2}\right)}{1\div\left(\dfrac{a}{a+b}+\dfrac{b}{a-b}\right)}.$

24. Fractional equations. To solve an equation that involves fractions, clear it of fractions by multiplying each member by the lowest common denominator (L.C.D.) of the fractions. (See Art. 23.)

When the unknown occurs in the denominator, multiplying by the L.C.D. may or may not introduce new roots that do not satisfy the equation to be solved. Such roots that do not satisfy the original equation are called *extraneous* roots.

Example 1. Solve $\dfrac{5}{x-1} + \dfrac{3}{x-5} = 1.$

Solution.
$$\frac{5}{x-1} + \frac{3}{x-5} = 1. \qquad (1)$$

Multiplying (1) by $(x-1)(x-5)$ gives

$$5(x-5) + 3(x-1) = (x-1)(x-5). \qquad (2)$$

Simplifying (2), $\quad x^2 - 14x + 33 = 0, \qquad (3)$

$$(x-11)(x-3) = 0. \qquad (4)$$

Hence,
$$x = 11$$
or
$$x = 3.$$

The roots of (2) are 11 and 3 and both satisfy (1).

Example 2. Solve $\dfrac{x-3}{x^2-9} = \dfrac{1}{7}.$

Solution.
$$\frac{x-3}{x^2-9} = \frac{1}{7}. \qquad (1)$$

Multiplying (1) by $(x^2-9)7$ gives

$$x^2 - 7x + 12 = 0. \qquad (2)$$

or
$$(x-3)(x-4) = 0.$$

Hence,
$$x = 3, \quad x = 4. \qquad (3)$$

The roots of (2) are 3 and 4. Now $x = 4$ satisfies (1), but

$x = 3$ does not satisfy (1) since the left hand member has no meaning when $x = 3$. Hence the extraneous root $x = 3$ is introduced in clearing of fractions.

The above example shows the importance of checking each solution by substituting the original equation.

Exercises

Solve the following equations and check the results:

1. $\dfrac{3}{x-2} = \dfrac{2}{x-3}.$

2. $\dfrac{5x}{x+1} - \dfrac{2}{x-3} = 2.$

3. $\dfrac{x-9}{x-5} + \dfrac{x-5}{x-8} = 2.$

4. $\dfrac{3x}{x-2} = \dfrac{14}{x+2} + 3.$

5. $\dfrac{4x+17}{x+3} - \dfrac{10-3x}{x-4} - 7 = 0.$

6. $\dfrac{3}{x-7} - \dfrac{4}{x-8} + \dfrac{1}{x-9} = 0.$

7. $\dfrac{m+x}{m-x} = \dfrac{m+n}{m-n}.$

8. $\dfrac{x^2-4}{x-2} = \dfrac{x^2+1}{x-1}.$

CHAPTER V

FUNCTIONS

25. Constants and variables. A constant is a symbol which represents the same number throughout a discussion.

A variable is a symbol which may represent different numbers in the discussion or problem into which it enters.

Thus, in the formula for the volume V of a sphere of radius r, $V = \frac{4}{3}\pi r^3$, the symbol π is a constant, whatever values V and r may have, while V and r are variables.

In most cases the letters $a, b, c, \ldots$ from the beginning of the alphabet are used to denote constants while the letters x, y, z at the end of the alphabet are used to denote variables.

26. Definition of a Function. *When two variables, x and y, are so related that to definitely assigned values of x there correspond definite values of y, then y is said to be a function of x.*

Thus in the equation, $V = \frac{4}{3}\pi r^3$, volume V is a function of r, the radius, for to every value of r there corresponds a definite volume. The expression, $x^2 + 3x - 5$ is a function of x, for to every value of x there corresponds a definite value for the expression. If we make $x = 2$, the expression takes the value 5, and when $x = 3$, the expression equals 13.

27. Functional Notation. When the same function of x occurs several times in a single algebraic discussion, we may simplify the work by representing the given function by some symbol. It is the custom to represent a function of x by the symbol of $f(x)$ which is read "f" function of "x" If another function of x occurs in the same discussion it can be represented by $F(x)$, which is read "F major function of x," while $f(x)$ is read "f minor function of x."

Thus, we may let,

$$f(x) = x^2 + 3x - 5.$$

Then, $$f(2) = 2^2 + 3.2 - 5 = 5$$

and $$f(a) = a^2 + 3a - 5.$$

that is, $f(2)$ and $f(a)$ mean the values of the function when $x = 2$ and a, respectively.

Exercises

1. Given $f(x) = 2x - 5$, find $f(1), f(3), f(-2)$.
2. Given $F(x) = x^3 - x^2 + 3$, find $F(1)$, $F(-a)$.
3. Given $f(n) = \dfrac{n^2 + n + 2}{n^2 - n - 1}$, find $f(1), f(2), f(\frac{1}{2})$.
4. If $f(x) = x^3 + x$ and $F(x) = 2x^2 - 4x - 5$, find the quotients $\dfrac{f(1)}{F(1)}$ and $\dfrac{f(3)}{F(2)}$.

28. Functional relations. *Whenever two variables are so related that one depends, for its value, on the value of the other there is said to exist a functional relation between these two variables.* There are many examples of functional relations in most every line of endeavor. However, it is possible to express only a few of these relations in the form of an algebraic equation.

Illustration (*a*). There exists a functional relation between the area and radius of a circle. The algebraic equation expressing this relation is, $A = \pi r^2$.

Illustration (*b*). The temperature of a place depends upon the time, altitude and latitude of the place. Hence, we have a functional relation existing, but this relation can not be expressed by an algebraic equation.

Exercises

1. Does there exist a functional relation among the volume, altitude and radius of base of a cylinder? Can this relation be expressed by an algebraic equation? If so, what is the equation?

2. What functional relation exists between the Fahrenheit and centigrade temperatures?

29. Formulas taken from geometry. Most of the formulas of mensuration are algebraic equations expressing functional relations.

The following is a list of useful common formulas:

1. Area A of a rectangle of sides a and b.

$$A = ab.$$

2. Area A of a parallelogram of base b and altitude h.

$$A = bh. \quad \text{(Fig. 1.)}$$

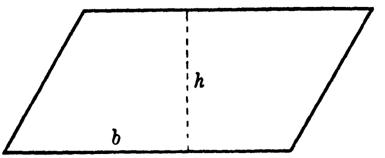

FIG. 1.

3. Area A of triangle of base b and altitude h.

$$A = \tfrac{1}{2}bh. \quad \text{(Fig. 2.)}$$

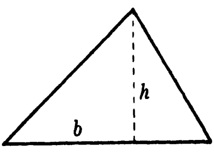

FIG. 2.

4. Area A of triangle in terms of its sides a, b, and c.

$$A = \sqrt{s(s-a)(s-b)(s-c)},$$

where,

$$s = \frac{a+b+c}{2}. \quad \text{(Fig. 3.)}$$

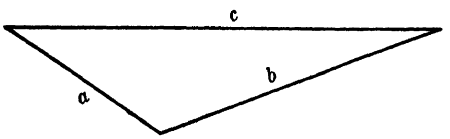

FIG. 3

5. Area A of a circle of radius r, or diameter D.

$$A = \pi r^2, \quad \text{or} \quad A = \tfrac{1}{4}\pi D^2.$$

6. Circumference C of a circle of diameter D or of radius r.

$$C = \pi D, \quad \text{or} \quad C = 2\pi r.$$

As an approximation which is sufficiently close for our purpose we may use 3.14159 as the value of π. For many practical purposes we may use $\pi = \frac{22}{7} = 3.14+$.

7. Area A of a trapezoid of base b and c and altitude h.

$$A = \tfrac{1}{2}(b + c)h. \quad \text{(Fig. 4.)}$$

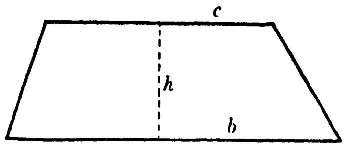

FIG. 4.

8. Length c of the hypotenuse of a right triangle of sides a and b.

$$c = \sqrt{a^2 + b^2}. \quad \text{(Fig. 5.)}$$

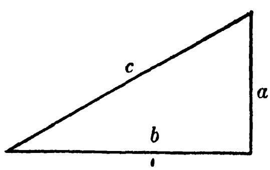

FIG. 5.

9. Volume V of a cube of edge a.

$$V = a^3. \quad \text{(Fig. 6.)}$$

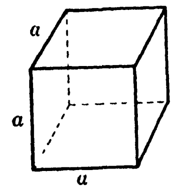

FIG. 6.

10. Volume V of a rectangular solid of length l, width w and altitude h.

$$V = lwh. \quad \text{(Fig. 7.)}$$

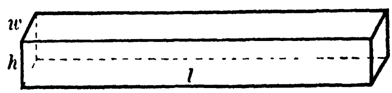

FIG. 7.

11. Volume V of a cylinder of altitude h, and radius of base r.

$$V = \pi r^2 h. \quad \text{(Fig. 8.)}$$

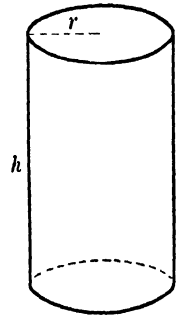

FIG. 8.

12. Volume V of a cone of altitude h and radius of base r.

$$V = \tfrac{1}{3}\pi r^2 h. \quad \text{(Fig. 9.)}$$

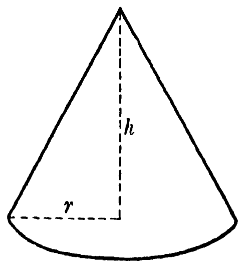

FIG. 9.

13. Volume V of a sphere of radius r, or diameter D.

$$V = \tfrac{4}{3}\pi r, \quad \text{or} \quad V = \tfrac{1}{6}\pi D^3. \quad \text{(Fig. 10.)}$$

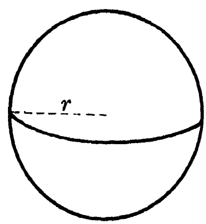

FIG. 10.

14. Surface S of a sphere of radius r, or diameter D.

$$S = 4\pi r^2, \quad \text{or} \quad S = \pi D^2.$$

Problems

1. The height of a cylinder is 5 feet greater than the radius. Express the volume as a function of the height; as a function of the radius.

2. The altitude of a right triangle is k feet less than the base b. Express the hypotenuse in terms of the base b.

3. How many cubic yards must be excavated in digging a ditch 300 rods long, 18 inches wide at the bottom, 6 feet wide at the top and 5 feet deep? How much water would be discharged by such a ditch in 3 hours' time if it flows 3 feet deep at the rate of 1.2 feet per second?

4. How much concrete is there in a circular silo whose walls are 9 inches thick, 14 feet outside diameter and 32 feet high?

5. What is the capacity of a silo of the dimensions of Ex. 4? How many cows will it maintain for 150 days? (One ton of silage occupies 50 cubic feet, and the daily ration per cow is 35 pounds.)

6. How much concrete will be required to build a water tank 6 feet long 3 feet wide and 2 feet high (all inside dimensions) if the walls and bottom are 8 inches thick? The proportions of the mixture are to be 1 : 2 : 3 (1 sack of cement, 2 cubic feet sand, 3 cubic feet gravel). It is figured that one cubic yard of concrete of the above proportions requires 7 sacks cement, 14 cubic feet of sand and 21 cubic feet of gravel.

What would be the total cost of material, if cement costs 90 cents per sack and sand and gravel \$2.00 each per cubic yard?

30. Graphical representation of functional relations. Functional relations may be represented graphically. This may be done whether the relation can be expressed by an algebraic equation or not. (See Art. 28.) Let $X'X$ and $Y'Y$ (Fig. 11) be two straight lines meeting at right angles. Let them be considered as two number scales having the point of intersection as the zero point of each. Let A be any point in the plane. From A drop perpendiculars to the two lines. Let x represent the distance to $Y'Y$, and y the distance to $X'X$. If A lies to the left of $Y'Y$, x is considered negative and if A lies below $X'X$, y is considered negative. It is evident that no matter where A lies in the plane there corresponds to it two and only two numbers and those numbers are the perpendiculars to $Y'Y$ and $X'X$ respectively.

The lines $X'X$ and $Y'Y$ are called the *coordinate axes,* and

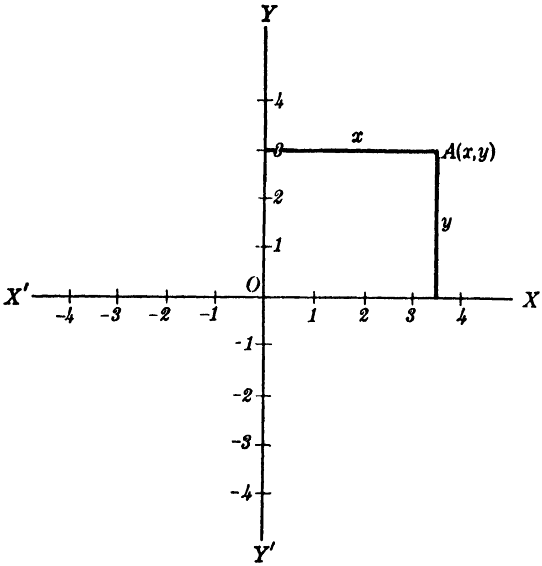

FIG. 11.

their point of intersection, O, is called the *origin.* The first line is called the X-axis and the second line is called the Y-axis. The distance from the point to the Y-axis is called the *abscissa* and the distance to the X-axis is called the *ordinate.* The two values are called the *coordinates* of the point. A customary notation is $A(x, y)$ which means the point A whose coordinates are x and y.

If we have any two numbers given it is evident that there is one and only one point in the plane having these numbers as its coordinates. The first number is the abscissa and the second number is the ordinate of the point. If for example we have the numbers 3 and -4, we measure from the origin in the positive direction a distance 3 on the X-axis and at this point we erect a perpendicular and measure downwards a distance 4. This gives us the point whose $x = 3$ and whose $y = -4$. The point may be represented by the symbol $(3, -4)$.

When a point is thus located, it is said to be plotted. In plotting points and representing graphically functional relations, it will be convenient to use coordinate paper. Then to represent a number the side of a square may be used as the unit of length. To plot a point, count off from the origin along the X-axis the number of divisions required to represent the abscissa and from this point count off the number of divisions parallel to the Y-axis required to represent the ordinate.

The change in a function can be represented on coordinate paper. As an example the change in the area of a square due to a change in the length of the sides can be represented in the following way: Let A be the area and l the length of the side. Now construct a table showing the area for different values of l.

$l =$	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3	4	5
$A =$	$\frac{1}{4}$	1	$2\frac{1}{4}$	4	9	16	25

Draw coordinate axes on paper and plot the points $(\frac{1}{2}, \frac{1}{4})$, $(1, 1)$, $(\frac{3}{2}, 2\frac{1}{4})$, $(2, 4)$ and so on. (Fig. 12.) Connect the points by a smooth curve. From the table of values we see that the area increases more rapidly than the side. This fact is also shown by the upward bending of the curve.

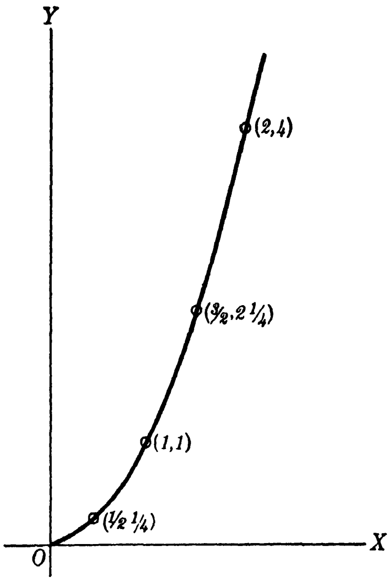

FIG. 12.

By this method any function may be represented on coordinate paper. This representation of a function is called the *graph* of the function. The graph of the function $f(x)$ contains all the points whose coordinates are $x, f(x)$ and no other points.

As an example let us obtain the graph of $x + 3$ for values of x between -5 and $+3$. Let $f(x) = x + 3$. Any value of x with the corresponding value of $f(x)$ determines a point whose ordinate is $f(x)$. Now, assuming values of x and computing the corresponding values of $f(x)$, we obtain the following table of values:

$x =$	-5	-4	-3	-2	-1	0	1	2	3
$f(x) =$	-2	-1	0	1	2	3	4	5	6

The corresponding points, $(-5, -2)$, $(-4, -1)$, . . . , are plotted in Fig. 13. It is seen that the curve connecting these points in order is a straight line. This shows that the function $x + 3$ increases at a uniform rate as x increases.

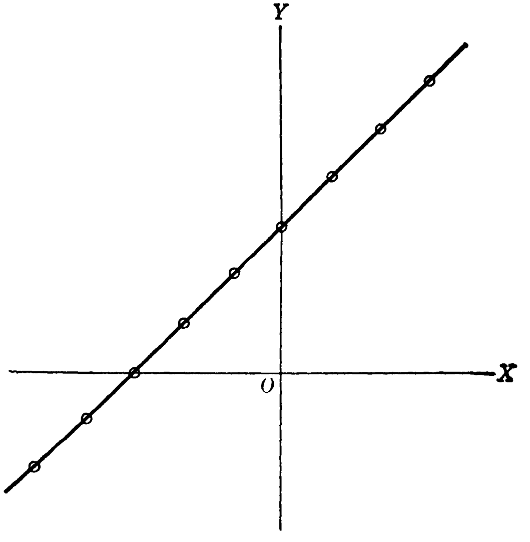

FIG. 13.

Exercises and Problems

1. Plot the points (3, 4), (5, −6), (−2, 3), (−3, −4).

2. Draw the triangle having for vertices the points (0, 0), (3, 2), (−3, 3).

3. Draw the quadrilateral having for vertices the points (2, 2), (6, 5), (5, −1), (−1, −5).

4. What is the abscissa of a point on the Y-axis? The ordinate of a point on the X-axis?

5. Find the distance between the points (1, 2) and (4, 6).

6. Draw a curve showing the change in the volume of a cube as the length of the edge 1 changes from 0 to 5.

7. One side of a rectangle is l, the other side is $l + 2$. Show by a graph the change in the area as l changes from 0 to 5.

8. Show by a graph the change in volume of a sphere as the diameter d changes.

Graph the following functions on coordinate paper.

9. $2x + 3$.

10. $3x - 2$.

11. x^2.

12. $x^2 + 1$.

13. $2x^2 + x$.

14. $x^2 + x + 1$.

15. $x - x^2$.

16. x^3.

31. Statistical Data. In Art. 30 it was shown how to graph a functional relation where the relation can be expressed by an algebraic equation. As stated before there are many functional relations that can not be expressed by an algebraic equation. As an example there is a relation between the weight of a calf

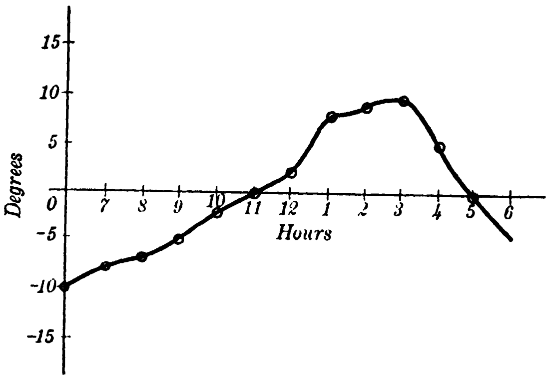

FIG. 14.—Time-Temperature Graph.

and its age, but we can not express this relation by an algebraic equation. Such relations may be exhibited by means of a graph as will now be shown.

Example. On a winter's day the thermometer was read at 6 a.m. and every hour afterward until 6 p.m. The readings were $-10°$, $-8°$, $-7°$, $-5°$, $-2°$, $0°$, $2°$, $8°$, $9°$, $10°$, $5°$, $0°$, $-4°$. Make a graph showing the relation between temperature and time.

Choose two lines at right angles as axes, Fig. 14. Time in hours is measured on the horizontal axis. The temperature in degrees is measured vertically upward and downward. Thus for 10 a.m. we count 4 spaces to the right and 2 spaces down locating a point. In a similar way we locate points for all of the data. By joining these points in order the graph is obtained.

From this temperature curve we may obtain much information, e.g.: When was the temperature changing most rapidly? When was it warmest? When coldest? When was the change a rise? When a fall?

Exercises

1. The daily gain in weight of a calf in pounds for period of one hundred days is given in the following table:

Age in days	0	100	200	300	400	500
Daily gain in pounds	3.2	2.8	2.55	2.3	2.16	2
Age in days................	600	700	800	1000	1100	1200
Daily gain in pounds......	1.9	1.8	1.7	1.57	1.5	1.47

Draw a curve showing this information. Plot days on the horizontal scale and pounds on the vertical scale.

2. The Statistical Abstract for 1915 gives the following figures for the values of exports and imports of merchandise for the years 1900–1915.

Year	Exports	Imports	Year	Exports	Imports
1900	$1,394,483,082	$849,941,184	1908	$1,860,773,346	$1,194,341,792
1901	1,487,764,991	823,172,165	1909	1,663,011,104	1,311,920,224
1902	1,381,719,401	903,320,948	1910	1,744,984,720	1,556,947,430
1903	1,420,141,679	1,025,719,237	1911	2,049,320,199	1,527,226,105
1904	1,460,827,271	991,087,371	1912	2,204,322,409	1,653,264,934
1905	1,518,561,666	1,117,513,071	1913	2,465,884,149	1,813,008,234
1906	1,743,864,500	1,226,562,446	1914	2,364,579,148	1,893,925,657
1907	1,880,851,078	1,434,421,425	1915	2,768,589,340	1,674,169,740

Make a graphical representation of these statistics.

3. The Year Book, Department of Agriculture, gives the following South Dakota Farm prices of corn and hay for the years 1899–1919:

Year	Corn	Hay	Year	Corn	Hay	Year	Corn	Hay
1899	$0 26	$3.10	1906	$0.29	$4.50	1913	$0.56	$6.50
1900	0.29	3.95	1907	0.46	5.50	1914	0.50	5.70
1901	0.45	4.49	1908	0.50	4.10	1915	0.49	5.30
1902	0.41	4.15	1909	0.50	5.10	1916	0.77	5 40
1903	0.35	4.63	1910	0.40	7.10	1917	1.20	10.60
1904	0.36	4.29	1911	0.53	8.50	1918	1.10	10 00
1905	0.31	4.02	1912	0.37	6.10	1919	1.19	13.50

Make a graph showing the price of corn also a graph showing the price of hay.

4. Plot a graph of the attendance of students at your college or university for the years 1910–1930.

5. Using the data below, plot a curve using years as abscissa and price of corn as ordinates. Do you notice any regularity in the number of years elapsing between successive high prices? Successive low prices? Draw like graphs for the other crops listed.

AVERAGE FARM PRICE DECEMBER FIRST

Data from the Year Book of the Department of Agriculture

Year	Corn	Wheat	Oats	Barley	Rye	Potatoes	Hay, Dollars per Ton
1870	$49.4	$94.4	$39.0	$79 1	$73.2	$65.0	12.47
1871	43 4	114.5	36 2	75 8	71.1	53.9	14.30
1872	35.3	111 4	29.9	68 6	67.6	53.5	12 94
1873	44 2	106 9	34 6	86.7	70.3	65.2	12.53
1874	58 4	86 3	47 1	86.0	77.4	61 5	11.94
1875	36 7	89.5	32 0	74.1	67 1	34.4	10.78
1876	34 0	97.0	32 4	63.0	61.4	61 9	8.97
1877	34 8	105 7	28.4	62.5	57.6	43.7	8.37
1878	31 7	77.6	24 6	57.9	52.5	58.7	7.20
1879	37.5	110 8	33.1	58 9	65.6	43 6	9.32
1880	39 6	95.1	36.0	66.6	75.6	48.3	11.65
1881	63 6	119.2	46.4	82 3	93.3	91.0	11.82
1882	48.5	88 4	37.5	62.9	61.5	55.7	9 73
1883	42.4	91.1	32.7	58.7	58.1	42.2	8.19
1884	35.7	64 5	27.7	48.7	51.9	39.6	8.17
1885	32 8	77.1	28.5	56.3	57.9	44.7	8.71
1886	36 6	68 7	29.8	53 6	53.8	46.7	8.46
1887	44 4	68.1	30 4	51.9	54.5	68.2	9 97
1888	34.1	92 6	27.8	59.0	58 8	40.2	8.76
1889	28 3	69 8	22.9	41.6	42.3	35 4	7.04
1890	50 6	83 8	42.4	62 7	62.9	75.8	7 87
1891	40.6	83 9	31 5	52.4	77.4	35.8	8.12
1892	39.4	62.4	31.7	47.5	54.2	66.1	8 20
1893	36.5	53.8	29.4	41.1	51.3	59.4	8.68
1894	45.7	49.1	32.4	44.2	50.1	53.6	8.54
1895	25.3	50.9	19.9	33.7	44.0	26.6	8.35
1896	21.5	72 6	18.7	32.3	40.9	28.6	6.55
1897	26.3	80.8	21.2	37.7	44.7	54.7	6.62
1898	28.7	58.2	25.5	41.3	46.3	41.4	6.00
1899	30 3	58.4	24.9	40 3	51.0	39.0	7.27
1900	35.7	61.9	25.8	40.9	51.2	43.1	8.89
1901	60.5	62.4	39.9	45.2	55.7	76.7	10.01

.AVERAGE FARM PRICE DECEMBER FIRST—*Continued*

Year	Corn	Wheat	Oats	Barley	Rye	Potatoes	Hay, Dollars per Ton
1902	$40.3	$63.0	$30.7	$45.9	$50.8	$47.1	9.06
1903	42.5	69.5	34.1	45.6	54.5	61.4	9.07
1904	44.1	92.4	31.3	42.0	68.8	45.3	8.72
1905	41.2	74.8	29.1	40.5	61.1	61.7	8.52
1906	39.9	66.7	31.7	41.5	58.9	51.1	10.37
1907	51.6	87.4	44.3	66.6	73 1	61.8	11.68
1908	60.6	92.8	47.2	55.4	73.6	70.6	8.98
1909	57.9	98 6	40.2	54.0	71.8	54.1	10.50
1910	48.0	88.3	34.4	57.8	71.5	55.7	12.14
1911	61.8	87.4	45.0	86 9	83.2	79 9	14.29
1912	48.7	76.0	31.9	50.5	66.3	50.5	11.79
1913	69.1	79.9	39.2	53 7	63.4	68.7	12.43
1914	64.4	98.6	43.8	54 3	86.5	48 9	11.12
1915	57.5	92.0	36.1	51.7	83.9	61.6	10.70
1916	88.9	160 3	52.4	88.2	122.1	146.1	10.59
1917	127.9	200.8	66.6	113.7	166.0	122 8	17.09
1918	136.5	204.2	70.9	91.7	151.6	119 3	20.13
1919	134.5	214.9	70.4	120.6	133.2	159.5	20.08
1920	67.0	143.7	46.0	71.3	126.8	114 5	17.76
1921	42.3	92.7	30.3	42.2	70.2	111 1	12.13
1922	65.8	100.7	39.4	52.5	68.5	58.1	12.56
1923	72.6	92.3	41.4	54.1	65.0	78 1	14.13
1924	98.2	129.9	47 7	74.1	106 5	62 5	13.77
1925	67.4	141.6	38.0	58 9	78.2	186.8	13 94
1926	64.4	119.9	39.8	57.4	83.5	141.7	14.09

CHAPTER VI

SYSTEMS OF LINEAR EQUATIONS

32. Graphs of Linear Equations. In Art. 30 the graphical representation of functional relations was discussed and the graphs of some functions were given.

Review Questions

1. What is meant by (*a*) coordinate axes, (*b*) abscissa, (*c*) ordinate, (*d*) coordinates of a point?

2. What is meant by the origin? What are its coordinates?

3. Locate points represented by $(-3, 4)$, $(5, -3)$, $(-2, -3)$, $(3, 2)$.

4. What is the plane figure having the points $(3, 2)$, $-3, 2)$, $(-3, -2)$, $(3, -2)$ for its vertices?

An equation of the form $Ax + By + C = 0$ (1) is called a *linear equation*. If $B \neq 0$, this equation may be thrown into the form

$$y = -\frac{Ax}{B} - \frac{C}{B}. \tag{2}$$

In (2) we may assume A, B and C as fixed and assign values to x and compute the corresponding values of y. This will give any number of pairs of values which may be plotted as coordinates of points. Equation (2) expresses y as a function of x, and the graph of this function is called the graph of equation (1).

It may be easily shown that the graph of all equations of the form of (1) is a straight line.

It is because of this fact that such equations are called linear equations. When A or B is zero, the graph is a line parallel to the X-axis or to the Y-axis respectively. Thus, the equation $y - 3 = 0$ gives a line parallel to the X-axis, and 3 units above it. And the equation

$x - 2 = 0$ gives a line parallel to the Y-axis, and 2 units to the right of that axis.

Exercises and Problems

Obtain the graphs of the following equations:

1. $x + 2y - 6 = 0$.

Solution. Solve the equation for y, thus getting $y = 3 - \frac{x}{2}$. This expresses y as a function of x. Now, assigning values to x and computing the corresponding values of y, we obtain the following table of values:

$x =$	-4	-2	0	2	4	6	8	10	12
$y =$	5	4	3	2	1	0	-1	-2	-3

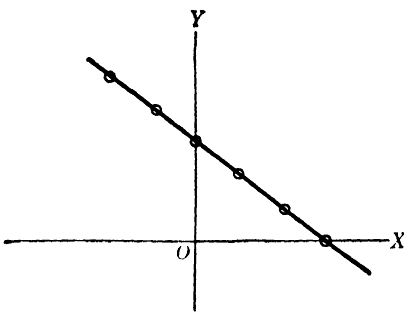

FIG. 15.

Plotting the points $(-4, 5)$, $(-2, 4)$, . . . we obtain the graph (Fig. 15), of the above equation. We might have plotted only two of the above points and connected them by a straight line, thus obtaining the required graph. Why? Thus in graphing any other linear equation, we need only to locate two points and connect them by a straight line.

2. $2x - 3y - 6 = 0$.

3. $4x - 6y + 6 = 0$.

4. $3x + 2y - 4 = 0$.

5. $4x - 5y = 0$

6. $4x - 5y - 10 = 0$.

7. $2y - 3 = 0$.

8. $3x - 4 = 0$.

9. Construct the graph of the equation $F = \frac{9}{5}C + 32$, taking the values of C along the horizontal and the corresponding values of F along the vertical axis.

10. Where does the graph $3x - 2y - 6 = 0$ cut the X-axis? The Y-axis? The abscissa of the point where the line intersects the X-axis is called the X-intercept and the ordinate of the point where it cuts the Y-axis is called the Y-intercept. What is the ordinate for the X-intercept? The abscissa for the Y-intercept?

11. Find the intercepts of the following:

$$a.\ 3x - 2y - 12 = 0.$$
$$b.\ 5x + 2y - 4 = 0.$$
$$c.\ 2x + 3y = 0.$$

12. Graph the equations $2x - y - 4 = 0$ and $x + y - 2 = 0$ using the same coordinate axes for both graphs. Do the two lines have a point in common? What are its coordinates? Do these coordinates satisfy both equations?

13. Graph $x - 2y - 4 = 0$ and $x - 2y - 8 = 0$. Do these lines have a point in common?

14. Graph $x - 2y - 4 = 0$ and $2x - 4y - 8 = 0$. Do these lines have a point in common?

33. Graphical Solution. In Art. 32 it was stated that the graph of a linear equation in two unknowns, x and y, is a straight line. The equation of this line will be satisfied by any number of pairs of values for x and y and these values will be the coordinates of the points on the graph.

Now assume that we have a second linear equation and that its graph is drawn using the same coordinate axes. This equation too will be satisfied by any number of pairs of values for x and y and these pairs of values will be the coordinates of the points on its graph.

Further assume that these two graphs intersect in some point P. Since this point lies on both graphs, its coordinates will satisfy both equations. In the solution of a system of linear equations in two unknowns x and y we are seeking a pair of values for x and y which will satisfy both equations simultaneously. The coordinates of this point then is the solution of the system.

Example. Solve graphically the system of equations

$$x - y + 1 = 0; \quad (2)\ 2x + y - 7 = 0. \qquad (1)$$

The graphs of equations (1) and (2) are numbered (1) and (2) in Figure 16. They intersect in the point whose coordinates are (2, 3), and consequently $x = 2$, $y = 3$ is the solution of the system.

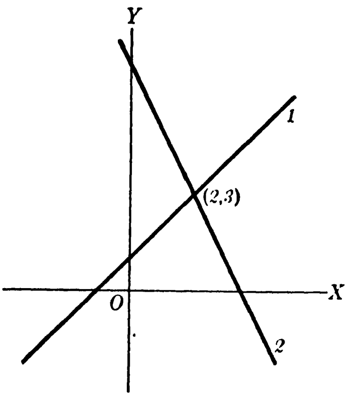

FIG. 16.

The graphs of two equations may be parallel lines. Then the lines have no point in common and their equations have no solution. Such equations are said to be *incompatible* or *inconsistent*. (See Ex. 13, Art. 32.)

Again the graphs of two equations may be coincident. Then the lines have an indefinitely large number of points in common and their equations do not have a unique solution. The two equations of the system are in this case *equivalent* or *dependent*. (See Ex. 14, Art. 32.)

Exercises

Find the solutions of the following systems of equations by plotting their graphs.

1. $2x + y = 4,$
$3x + 2y = 10.$

2. $6x - 5y = 14,$
$7x + 2y = 32.$

3. $2x + 3y = 10,$
$5x + 3y = 7.$

4. $4x + 6y = 8,$
$2x + 3y = 6.$

5. $3x + y = 19,$
$2x - y = 1.$

6. $7x - 3y = 26,$
$2x + 11y = 43.$

34. Algebraic solution: Two simple equations in two unknowns may be solved simultaneously for the two values of the unknowns by the process of elimination as is illustrated below.

Example. Solve the equations

$$x - y = 4, \tag{1}$$

$$x - 4y = -14. \tag{2}$$

Solution. First Method.

From (1) we have

$$x = 4 + y. \tag{3}$$

Substituting this value for x in (2), we find

$$4 + y - 4y = -14, \tag{4}$$

or

$$-3y = -18, \quad y = 6.$$

Substituting 6 for y in (1), we find

$$x - 6 = 4, \quad \text{or} \quad x = 10.$$

Hence the required values for x and y are 10 and 6 respectively.

This method is known as elimination by substitution.

Solution. Second Method.

From (1) subtract (2) and we get

$$3y = 18, \quad y = 6. \tag{3}$$

Multiplying (1) by 4 and (2) by -1, the two equations become

$$4x - 4y = 16, \tag{4}$$

$$-x + 4y = 14. \tag{5}$$

Adding (4) and (5), we get

$$3x = 30, \quad x = 10.$$

Hence the required solution is $x = 10$, $y = 6$.

This method is known as elimination by addition and subtraction.

Exercises and Problems

1. $3x - 4y = 26,$
$x - 8y - 22 = 0.$

2. $x + \frac{y}{3} = 11,$
$\frac{x}{3} + 3y = 21.$

3. $\frac{x + y}{2} - \frac{x - y}{3} = 8,$
$\frac{x + y}{3} + \frac{x - y}{4} = 11.$

4. $y + 1 = 3x,$
$5x + 9 = 3y.$

5. $\frac{4}{x} - \frac{3}{y} = \frac{14}{5},$
$\frac{2}{x} + \frac{5}{y} = \frac{25}{3}.$
$\left(\text{Hint: Solve first for } \frac{1}{x} \text{ and } \frac{1}{y}.\right)$

6. $x + y = m + n,$
$mx - ny = m^2 - n^2.$

7 $ax + by = 2ab,$
$bx + ay = a^2 + b^2.$

8. A rectangular field is 35 rods longer than it is wide. The length of the fence around it is 310 rods. Find the dimensions of the field.

9. A man has $25,000 at interest. For one part he receives 6% and for the other part 5%. His total income is $1,350. How is the money divided?

10. What quantities of two liquids, one 95% alcohol and the other 20% alcohol, must be used to give a 20 gallon mixture of 50% alcohol?

35. Solution of three linear equations in three unknowns. The process of solving three linear equations in three unknowns may be illustrated by the following example:

Solve the equations

$$3x + 2y - z = 4, \quad (1)$$

$$5x - 3y + 2z = 5, \quad (2)$$

$$6x - 4y + 3z = 7, \quad (3)$$

for x, y, and z.

Solution. Eliminate z between (1) and (2). This may be done by multiplying (1) by 2 and adding the result to (2), we have,

$$6x + 4y - 2z = 8, \qquad (4)$$

$$5x - 3y + 2z = 5, \qquad (5)$$

$$11x + y = 13. \qquad (6)$$

Now eliminating z between (1) and (3), we have,

$$9x + 6y - 3z = 12, \qquad (7)$$

$$6x - 4y + 3z = 7, \qquad (8)$$

$$15x + 2y = 19. \qquad (9)$$

Now solve (6) and (9) for x and y as illustrated in Art. 34. Multiply (6) by -2 and add the result to (9) and we obtain,

$$7x = 7,$$

$$x = 1.$$

Substituting $x = 1$ in (6), we have,

$$y = 2.$$

Substituting $x = 1$, $y = 2$ in (1) and solving for z, we have,

$$z = 3.$$

Hence the solution of equations (1), (2), and (3) is

$$x = 1, \quad y = 2, \quad z = 3.$$

Exercises

Solve for x, y, and z:

1. $2x - 4y + 5z = 18,$
$5x + 3y - 4z = 5,$
$x + 2y + 3z = 19.$

2. $x + y = 1,$
$y + z = 2,$
$z + x = 4.$

3. Make up 100 pounds of an ice cream mixture which will contain 12% fat and 10% milk solids, not fat. The following ingredients are used:

Sugar,	14 pounds.
Gelatine,	$\frac{1}{2}$ pound.
Flavoring,	$\frac{1}{2}$ pound.
Condensed milk,	8 pounds.
Cream, whole milk and skim milk powder.	

The composition of the products are:

Condensed milk,	9% fat and	20% solids.
Cream,	30% fat and	6.3% solids.
Whole milk,	3% fat and	8.73% solids.
Skim milk powder,		100% solids.

Solution.

Let x = no. of pounds of cream,
y = no. of pounds of milk,
z = no. of pounds of skim milk powder.

Then,

$$0.30x + 0.03y + 0.09(8) = 12,$$
$$0.063x + 0.0873y + 0.020(8) + z = 10,$$
$$x + y + z + 14 + 8 + \tfrac{1}{2} + \tfrac{1}{2} = 100.$$

Or

(1) $10x + y = 376,$

(2) $630x + 873y + 10000z = 84000,$

(3) $x + y + z = 77.$

(4) $9370x + 9127y = 686000,$ (3) − (2)

(5) $10x + y = 376.$

Multiply (5) by 937 and subtract it from (4). We have,

(6) $8190y = 333688,$

$y = 40.74$ pounds of milk.

$$(7)\ 10x = 376 - 40.74 = 335.26,$$
$$x = 33.53 \text{ pounds of cream.}$$
$$z = 77 - 40.74 - 33.53,$$
$$z = 2.73 \text{ pounds of milk powder.}$$

Check:

Cream,	33.53
Milk,	40.74
Milk powder,	2.73
Sugar,	14.00
Gelatine,	.50
Flavor,	.50
Condensed milk,	8.00
Total,	100.00 pounds

$$.3(33.53) + .03(40.74) + .09(8) = 12.0012$$
$$.063(33.53 + .0873(40.74) + .2(8) + 2.73 = 9.9990.$$

Suppose skim milk powder is not added. We will have three equations in two unknowns which may not be solved. We would have,

$$(1)\ 10x + y = 376,$$
$$(2)\ 630x + 873y = 84000,$$
$$(3)\ x + y = 77.$$

Any two of the above equations may be solved for x and y but these values of x and y will not satisfy the other equation. Hence, the mixture is impossible, without adding skim milk or some other ingredient to make the balance.

4. Make up 100 pounds of ice cream mixture which will have 12% fat and 10% milk solids. The following ingredients are used: 14 pounds of sugar, $\frac{1}{2}$ pound of gelatine, $\frac{1}{2}$ pound of flavoring, 16 pounds of condensed milk, whole milk, cream and skim milk powder. The composition of the products are: condensed milk, 9% fat and 20% solids; cream, 34% fat and 5.95% solids; whole milk, 4% fat and 8.75% solids; skim milk powder, 100% solids. Find the proper amounts of cream, whole milk and skim milk powder and check the results.

36. Slope of a straight line. Given a line AB, Fig. 17. Take any point P on the line and through P draw a line PQ, toward the right, parallel to the X-axis and at Q erect a perpendicular to PQ intersecting AB in R. QR is defined as the rise of the line as the point R moves along the line from P toward the right, and PQ is known as the run. *The rise divided by the corresponding run is defined as the slope of the line* AB.

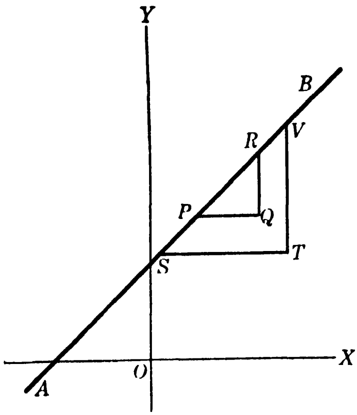

FIG. 17.

It is evident that one line can have but one slope, for if we take any other point S on AB and draw through it a line parallel to the X-axis, say ST, and erect at T a perpendicular TV, we get the triangles PQR and STV which are similar. Therefore the slope of $AB = \frac{QR}{PQ} = \frac{TV}{ST}$ (a constant value).

Problem. Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$. Express the slope of the line joining these points in terms of the coordinates of the points, Fig. 18.

Solution. Drop perpendiculars to the axes as shown. Then,

$$\text{Slope of } PQ = \frac{AQ}{PA} = \frac{y_2 - y_1}{x_2 - x_1}. \qquad (1)$$

Thus the slope of a line between two points is equal to the difference of the ordinates of the points divided by the difference of their abscissas subtracted in the same order.

In Fig. 18 (a), the slope is positive since both AQ and PA

are positive, but in Fig. 18 (*b*) the slope is negative for AQ is negative and PA is positive.

In Fig. 18 (*a*) AQ is a rise but in Fig. 18 (*b*) AQ is a fall.

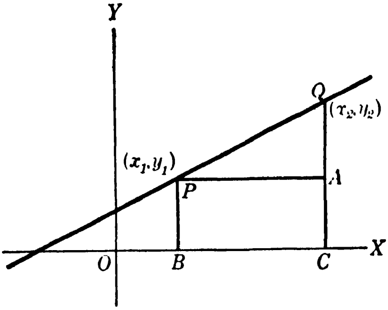

FIG. 18*a*.

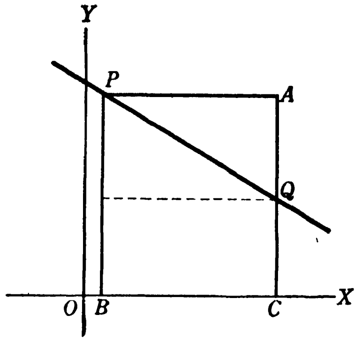

FIG. 18*b*

We observe that a rise gives us a positive slope, while a fall gives us a negative slope.

Exercises

1. Construct a line through (2, 3) whose slope is $\frac{3}{5}$. (Hint: A slope $\frac{3}{5}$ means a rise of 3 and a run of 5. Therefore begin at (2, 3), rise 3 units and run 5 units to the right. Connect the final point with (2, 3). The resulting line will have the slope $\frac{3}{5}$.)

2. Construct a line through (1, −2) having $\frac{2}{3}$ for its slope, also, one having $-\frac{2}{3}$ for its slope.

For each of the following pairs of points:

(*a*) Plot the points,

(*b*) Draw the straight line through them,

(*c*) Find the slope of the line.

3. (1, 2) and (3, 5).

4. (3, 2) and (−3, −5).

5. (−2, 3) and (2, −2).

6. (−3, −4) and (−2, −3).

7. (6, 7) and (−3, 2).

8. (0, 5) and 2, 0).

37. Distance between two points, $P(x_1, y_1)$ and $Q(x_2, y_2)$ in terms of the coordinates of the points. In Fig. 18 we see that

$$PQ = \sqrt{(PA)^2 + (AQ)^2}.$$

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}, \qquad (2)$$

since $\qquad PA = x_2 - x_1 \quad \text{and} \quad AQ = (y_2 - y_1).$

Example. Find the distance between the points (5, 6) and (1, 3).

Solution. $PQ = \sqrt{(5 - 1)^2 + (6 - 3)^2} = \sqrt{16 + 9} = 5.$

Exercises

1. Find the distance between the pairs of point in Exercises 3 to 8, Art. 36.

2. Find the distance from the origin to the point (a, b).

3. Prove that the triangle having for its vertices the points $(-1, 2)$, $(4, -3)$, $(5, 3)$ is an isosceles triangle.

4. Find the lengths of the sides of the triangle having the points $(2, 1)$, $(5, 5)$ and $(-5, 0)$ for its vertices.

38. Equation of a straight line. Up to this time we have had certain equations given to find the graphs of these equations. Our problem now is to find the equation when the graph is given. We must find an algebraic expression for the relation existing between the x-distance and the y-distance of a point which will hold for all points on the line.

For example, if a point is located anywhere on the y-axis, its x-coordinate is always zero. The algebraic statement for this fact is the equation $x = 0$, hence this is the equation of the Y-axis, for it is the one statement that is true for all points on the Y-axis, and for no other points.

What is the equation of the X-axis.

As another example let us find the equation of the line parallel to the X-axis and 2 units above it?

In this case y will always be 2, regardless of the value of x. The algebraic statement of this fact is the equation $y = 2$, or $y - 2 = 0$, and this is the required equation.

What is the equation of the line that is always the same distance from each of the coordinate axes?

39. Problem. To derive the equation of a straight line in terms of the coordinates of two given points on the line. Let LM be the line determined by the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and let $R(x, y)$ be any other point on LM. Since R, P, Q are all on the same line LM, the slopes of RP and PQ are equal. Hence by (1) Art. 36, the required equation is

$$\frac{y - y_1}{x - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, \qquad (3)$$

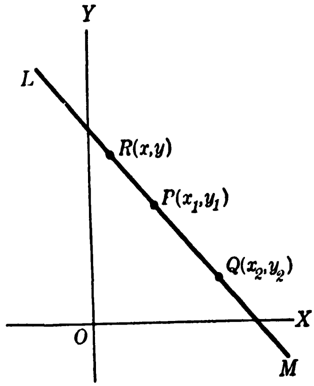

FIG. 19.

which may also be written in the form

$$y - y_1 = \frac{y_1 - y_2}{x_1 - x_2}(x - x_1). \qquad (4)$$

Either (3) or (4) is known as the *two-point form* of the equation of the straight line.

If one of the given points, say (x_1, y_1), is the origin $(0, 0)$ equation (4) takes the form

$$y = \frac{y_2}{x_2}x, \qquad (5)$$

which is the equation of the line through the origin and another given point.

Exercises

1. Find the equation of the line determined by the points (1, 2) and (5, 4).

Solution. Construct the line determined by these points and take any other point R on the line having (x, y) for its coordinates. Then, the slopes of PQ and QR are equal, and we have,

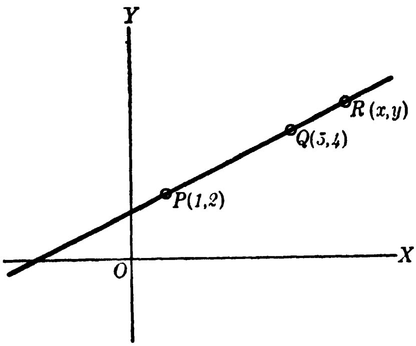

FIG. 20.

$$\frac{4-2}{5-1} = \frac{y-4}{x-5},$$

or

$$\frac{1}{2} = \frac{y-4}{x-5},$$

from which $y = \frac{1}{2}x + \frac{3}{2}$. What is the slope of the straight line? Does the equation show this slope? The equation could be gotten by substituting the coordinates of the points in equation (4). This would give us

$$y - 4 = \frac{4-2}{5-1}(x-5),$$

or

$$y - 4 = \tfrac{1}{2}(x-5),$$

or

$$y = \tfrac{1}{2}x + \tfrac{3}{2},$$

which is the same result as obtained above.

2. Find the equations of the straight lines determined by the following points. Reduce each equation to the form showing its slope.

(*a*) (3, 4) and (−2, 2). (*c*) (2, 3) and (−2, 4).
(*b*) (3, 2) and (5, 6). (*d*) (−3, 5) and (2, 3).

3. Find the equations of the sides of a triangle whose vertices are the points (4, 3), (2, −2), (−3, 4).

40. Problem. To derive the equation of a straight line in terms of its slope and the coordinates of a given point on the line. Let $P(x_1, y_1)$ be the given point and m the given slope. And let $R(x, y)$ be any other point on the line. From Fig. 21 we see that the slope of the line is $\frac{y - y_1}{x - x_1}$. But the slope of the line is given as m. Hence we may write,

$$\frac{y - y_1}{x - x_1} = m, \qquad (6)$$

or

$$y - y_1 = m(x - x_1). \qquad (7)$$

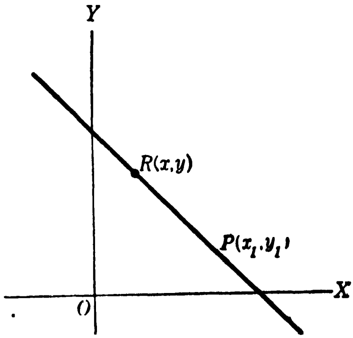

FIG. 21.

Equation (7) is known as the *slope* and *one-point form* of the equation of the line.

Exercises

1. Find the equation of the line which passes through the point (3, 2) and has the slope 3.

Solution. Substituting direct in equation (7) (3, 2) for (x_1, y_1) and 3 for m, we get, $y - 2 = 3(x - 3)$ or $y = 3x - 7$ as the required equation. Does the equation of this line show its slope?

2. Find the equations of the lines passing through the following points and having the given slopes.

(*a*) Through (2, 3) with slope $\frac{3}{4}$.
(*b*) Through (−3, 4) with slope −2.
(*c*) Through (5, −3) with slope $-\frac{2}{3}$.

3. What are the slopes of the lines whose equations are:

(*a*) $2x - 3y + 6 = 0$? Ans. $m = \frac{2}{3}$.

(*b*) $ax + by + c = 0$? Ans. $m = \frac{-a}{b}$.

4. Find the equation of each of the straight lines described below.

(*a*) A line whose X-intercept is 3 and whose slope is $\frac{2}{3}$.
(*b*) A line whose Y-intercept is k and whose slope is m.

Answer (*b*); $y = mx + k$. This equation is known as the *slope Y-intercept* form of the equation of a line.

41. Parallel lines. If two straight lines are parallel, their slopes are equal.

Draw two parallel lines and select any two points on each line. (See Fig. 22.) The slopes are respectively, $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$. The triangles ABC and DEF are similar, since their corresponding sides are parallel. Hence $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and the two slopes are equal. Therefore, if two lines are parallel, their slopes are equal.

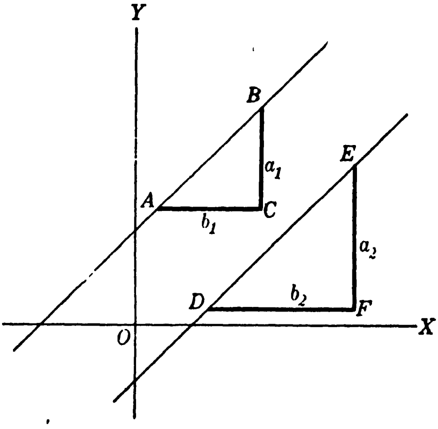

FIG. 22.

And conversely, *if the slopes of two lines are equal, the lines are parallel.*

Example. Find the equation of the line which passes through the point (1, 2) and is parallel to the line $3x - y - 7 = 0$.

Solution. The equation of the given line may be written in the form, $y = 3x - 7$, which shows that its slope is 3. Since the line, whose equation we are seeking, is parallel to the given line it will likewise have 3 for its slope. Hence, substituting in equation (7) Art. 40, we get

$$y - 2 = 3(x - 1),$$

or

$$3x - y - 1 = 0.$$

42. Perpendicular lines. *If two straight lines are perpendicular to each other, the slope of one is the negative reciprocal of the slope of the other.*

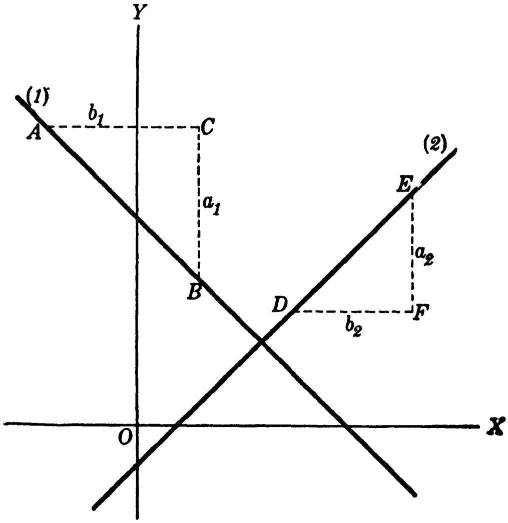

FIG. 23.

Draw two perpendicular lines as shown in Fig. 23. The slope of the one line is $m_1 = \frac{-a_1}{b_1}$. (Why negative?) The other slope is

$$m_2 = \frac{a_2}{b_2}.$$

The triangles ABC and EDF are similar, since their corresponding sides are perpendicular to each other. Hence we have,

$$\frac{a_2}{b_2} = \frac{b_1}{a_1} = \frac{1}{\frac{a_1}{b_1}} = -\left[\frac{1}{\frac{-a_1}{b_1}}\right]$$

or

$$m_2 = -\frac{1}{m_1}.$$

Therefore, if two lines are perpendicular, the slope of one is the negative reciprocal of the slope of the other.

And conversely, *if the slope of one line is the negative reciprocal of the slope of another, the lines are perpendicular to each other.*

Example. Show that the lines (1) $3x - y + 6 = 0$ and (2) $2x + 6y - 5 = 0$ are perpendicular to each other.

Writing the above equations in the slope Y-intercept form. (See Ex. 4 (*b*) Art. 40), we get,

$$y = 3x + 6, \text{ and} \tag{1}$$

$$y = \frac{-1}{3}x + \frac{5}{6}. \tag{2}$$

We observe that their respective slopes are 3 and $-\frac{1}{3}$. The lines are therefore perpendicular.

Exercises on Chapter VI

1. Write the equation of the line which shall pass through the intersection of $x + y + 1 = 0$ and $x - 3y + 8 = 0$, and have a slope equal to 4.

2. Find the equations of the lines satisfying the following conditions:

(*a*) Passing through (2, 3) and with slope $= -4$.
(*b*) Having the X-intercept $= 4$, Y-intercept $= -5$.
(*c*) Slope $= -3$, X-intercept $= 8$.

3. Prove by means of slopes that (0, −2), (4, 2), (0, 6), (−4, 2) are the vertices of a rectangle.

4. What are the equations of the sides of the figure in example 3?

5. Find the equation of the straight line passing through the point of intersection of $2x + 5y - 8 = 0$ and $2x - y + 4 = 0$ and perpendicular to the line $5x - 10y = 0$.

6. Show that the points (2, 4), (−1, 0), (5, 8) are on the same straight line.

7. Show that the points (−1, 2), (4, −3), (5, 3) are the vertices of an isosceles triangle.

8. Prove that the diagonals of a square are equal and perpendicular to each other.

9. Find the equation of the line which passes through (2, −1) and is:

(*a*) Parallel to $3x + 2y + 3 = 0$,

(*b*) Perpendicular to $3x + 2y + 3 = 0$.

CHAPTER VII

QUADRATIC EQUATIONS

43. Typical form. We may regard the equation

$$Ax^2 + Bx + C = 0 \tag{1}$$

as the typical form of every quadratic equation in a single unknown x, for every quadratic equation can be thrown into the form (1) by the proper rearrangement of its terms. The coefficients A, B, and C represent numbers which are in no way dependent upon the unknown number x and A is not zero, for if it were equation (1) would become $Bx + C = 0$ which is not a quadratic equation but a linear equation.

The function $Ax^2 + Bx + c(A \neq 0)$ is the typical quadratic function.

Exercises

Arrange the following equations in the typical form. What are the values of A, B, and C?

1. $x^2 + (3x - 5)^2 + 2x - 5 = 0.$

Expanding and collecting terms, we get,

$$10x^2 - 28x + 20 = 0,$$

or

$$5x^2 - 14x + 10 = 0,$$

and

$$A = 5, B = -14, C = 10.$$

2. $3x(x - 1) = x^2 - 2x - 3.$

3. $\dfrac{1}{x} - \dfrac{1}{x + 1} = 2.$

4. $(z + 2)^3 - (z - 3)^3 + 3 = (z - 2)^2$.

5. $(x + m)^2 + (x - m)^2 = 4mx + 3x^2$.

Solution of Ex. 5: We get,

$$x^2 + 2mx + m^2 + x^2 - 2mx + m^2 - 4mx - 3x^2 = 0,$$

and combining terms,

$$-x^2 - 4mx + 2m^2 = 0,$$

or

$$x^2 + 4mx - 2m^2 = 0.$$

This is of form (1) and $A = 1$, $B = 4m$, $C = -2m^2$.

6. $4m^2x^2 + 3k^2x^2 - 8mx + 3x - m + k = 0$.

7. $x^2 + (mx + b)^2 = r^2 - mx$.

44. Solution of the quadratic equation. The quadratic formula. A quadratic equation may be solved by the process known as "completing the square."

As an example, solve $9x^2 + 3x = 2$.

Solution. Write the equation in the form,

$$x^2 + \tfrac{1}{3}x = \tfrac{2}{9}. \tag{1}$$

Add $(\frac{1}{2}\cdot\frac{1}{3})^2 = \frac{1}{36}$ to both members, and the left hand member is a perfect square. That is,

$$x^2 + \tfrac{1}{3}x + \tfrac{1}{36} = \tfrac{2}{9} + \tfrac{1}{36} = \tfrac{9}{36} = \tfrac{1}{4}, \tag{2}$$

or

$$(x + \tfrac{1}{6})^2 = \tfrac{1}{4}. \tag{3}$$

Extract the square root of both members.

$$x + \tfrac{1}{6} = \pm\tfrac{1}{2}, \tag{4}$$

$$x = -\tfrac{1}{6} \pm \tfrac{1}{2},$$

$$x = \tfrac{1}{3} \text{ or } -\tfrac{2}{3}.$$

Both of these values of x satisfy the original equation, as may be seen by substituting them for x in the original equation.

Apply the method of "completing the square" to the typical form,

$$Ax^2 + Bx + C = 0. \tag{1}$$

Transpose C and divide through by A,

$$x^2 + \frac{B}{A}x = -\frac{C}{A}. \tag{2}$$

Add $\left(\frac{1}{2}\cdot\frac{B}{A}\right)^2$ to both members,

$$x^2 + \frac{B}{A}x + \frac{B^2}{4A^2} = \frac{B^2}{4A^2} - \frac{C}{A} = \frac{B^2 - 4AC}{4A^2}, \tag{3}$$

or

$$\left(x + \frac{B}{2A}\right)^2 = \frac{B^2 - 4AC}{4A^2}. \tag{4}$$

Extracting the square root of both members,

$$x + \frac{B}{2A} = \pm\frac{1}{2A}\sqrt{B^2 - 4AC}, \tag{6}$$

$$x = \frac{-B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4AC},$$

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

The roots, then, of the typical form (1) are

$$x_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

and

$$x_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

which could be verified by substitution.

We may therefore use the expression,

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A},$$

as the formula for the solution of any quadratic equation.

As an example, solve $3x^2 + 7x - 6 = 0$. In this equation, $A = 3$, $B = 7$, $C = -6$, and substituting these values of A, B and C in the formula, we get,

$$x = \frac{-7 \pm \sqrt{49 - 4\cdot 3(-6)}}{2\cdot 3} = \frac{-7 \pm 11}{6},$$

or $$x_1 = \frac{-7 + 11}{6} = \frac{2}{3},$$

and $$x_2 = \frac{-7 - 11}{6} = -3.$$

Our solutions then are $\frac{2}{3}$ and -3.

As another example, solve $x^2 - x - 1 = 0$.

By formula, $$x_1 = \frac{1 + \sqrt{5}}{2},$$

$$x_2 = \frac{1 - \sqrt{5}}{2}.$$

Here the quantity under the radical is not a perfect square and we say the *solutions are irrational.* We will now define *rational* and *irrational* numbers.

A *rational number* is defined as one that can be expressed as the *quotient of two integers.* An *irrational number* is one that can not be thus expressed.

Thus 15, $\frac{1}{2}$, $\frac{2}{3}$ are rational numbers; $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $1 + \sqrt{5}$, $\frac{1 - \sqrt{5}}{2}$ are *irrational numbers.*

Exercises

Solve the following equations by formula and check the results:

1. $6x^2 - 11x + 4 = 0.$
2. $5x^2 - 3x - 14 = 0.$
3. $14x^2 + 11x - 15 = 0.$
4. $2x^2 - 5x + 2 = 0.$
5. $3x^2 + 8x - 3 = 0.$
6. $7y^2 + 9y - 10 = 0.$
7. $x^2 + x - 1 = 0.$
8. $x^2 + 2x - 1 = 0.$
9. $2x^2 + 3x - 9 = 0.$
10. $7x^2 - 32 = -2x.$
11. $2x^2 - x - 2 = 0.$
12. $2x^2 - 3x - 2 = 0.$
13. $\dfrac{x+3}{2x-7} - \dfrac{2x-1}{x-3} = 0.$
14. $x^2 - 2ax + 3x - 6a = 0.$
15. $\dfrac{w + \frac{1}{w}}{w - \frac{1}{w}} + \dfrac{1 + \frac{1}{w}}{1 - \frac{1}{w}} = \dfrac{13}{4}.$
16. $x^2 + lx + m = 0.$
17. $(2x - 3)^2 = 8x.$
18. $\dfrac{2x}{x+2} + \dfrac{x+2}{2x} = 2.$

45. Classification of numbers. Algebraic numbers are divided into two classes, *real numbers* and *imaginary numbers*.

Real numbers are of two kinds, *rational* and *irrational* (see Art. 44 for definition of rational and irrational numbers).

In order to care for the square root of a negative number, we introduce the symbol $\sqrt{-1} = i$ and define it as the *imaginary unit* just as 1 is defined as the real unit. Then any number of the form ai, where a is real, is defined as a *pure imaginary*; and any number of the form $a + bi$, where a and b are real is defined as a *complex number*.

For example, $\sqrt{-25} = 5\sqrt{-1} = 5i,$

and $\sqrt{-37} = \sqrt{37}\sqrt{-1} = \sqrt{37}i$

also, $3 + \sqrt{-37} = 3 + \sqrt{37}i.$

Imaginary numbers occur in the solution of certain quadratic equations.

As an example, solve $2x^2 - 3x + 4 = 0$.

By formula:

$$x_1 = \frac{3 + \sqrt{(-3)^2 - 4\cdot 2\cdot 4}}{4} = \frac{3 + \sqrt{-23}}{4},$$

$$= \frac{3 + \sqrt{23}i}{4},$$

and $$x_2 = \frac{3 - \sqrt{23}i}{4}.$$

Here, both roots are of the form $a + bi$ and are complex.

We can interpret the number $\sqrt{17}$ as the length of the hypotenuse of a right triangle whose sides are 4 and 1, but we can not interpret in an elementary way the number $\sqrt{-17}$ or $\sqrt{17}i$. However, the new number $\sqrt{-1}$ is of great importance in studying the physical world, particularly in the theory of alternating currents in electricity.

Exercises

Solve for x:

1. $2x^2 - 5x + 4 = 0$.

2. $x^2 - x + 1 = 0$.

3. $3x^2 - x + 2 = 0$.

4. $7x^2 - 3x + 1 = 0$.

46. Character of the roots of the quadratic. Discriminant. We have shown in Art. 44 that the solutions of the quadratic equation, $Ax^2 + Bx + C = 0$, are given by the formula,

$$\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

The expression $B^2 - 4AC$ which appears under the radical sign is called the *discriminant* of the equation. An inspection of the value of the discriminant is sufficient to determine the character of the roots. It is easily observed that the following statements are true:

I. When $B^2 - 4AC$ is negative, the roots are imaginary.

II. When $B^2 - 4AC = 0$, the roots are real and equal.

III. When $B^2 - 4AC$ is positive, the roots are real and unequal.

IV. When $B^2 - 4AC$ is positive and a perfect square, the roots are real, unequal and rational.

V. When $B^2 - 4AC$ is positive and not a perfect square, the roots are real and unequal and irrational.

Why is the expression $B^2 - 4AC$ called the discriminant?

Exercises

Without solving, determine the character of the roots of the following equations:

1. $2x^2 - 7x + 3 = 0$.

Solution of example 1.

Here $A = 2, B = -7, C = 3$.

Then $B^2 - 4AC = (-7)^2 - 4 \cdot 2 \cdot 3 = 49 - 24 = 25$, which is positive. Therefore by III, the roots are real and unequal.

Also, since 25 is a perfect square, we have from IV that the roots are rational.

2. $3x^2 + 2x + 1 = 0$.

3. $2x^2 - 4x + 3 = 0$.

4. $x^2 + 6x - 8 = 0$.

5. $4x^2 + 4x + 1 = 0$.

6. $x^2 + x = -1$.

7. $3x^2 - x - 10 = 0$.

8. $x^2 + x = 1$.

9. $4x^2 + 16x + 7 = 0$.

10. For what values of k will the roots of the quadratic $k^2y^2 + 5y + 1 = 0$, be equal?

Solution of Ex. 10.

Here $A = k^2, B = 5, C = 1$, and $B^2 - 4AC = (5)^2 - 4k^2 = 25 - 4k^2$.

According to II, the roots will be equal when k is so determined that $25 - 4k^2 = 0$ or $4k^2 = 25$, or $k = \pm \frac{5}{2}$.

11. For what value (or values) of m will the solutions of the following be equal?

(a) $y^2 + 12y + 8m = 0.$ (c) $(m + 1)y^2 + my + m + 1 = 0.$
(b) $(2z + m)^2 = 8z.$ (d) $a^2(mx + 1) + b^2x^2 = a^2b^2.$

47. The sum and product of the roots. The two roots of the typical quadratic equation are

$$x_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \quad \text{and} \quad x_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A}.$$

The sum of these roots is $-\frac{B}{A}$, and their product is $\frac{C}{A}$, which is easily obtained by adding and multiplying them together, respectively.

Summing up, we have,

$$x_1 + x_2 = -\frac{B}{A} \tag{1}$$

and

$$x_1x_2 = \frac{C}{A}. \tag{2}$$

Thus, by means of (1) and (2) above, we can find the sum and product of the roots without solving the equation. Thus, in the equation, $2x^2 - 5x + 3 = 0$, the sum of the roots is $\frac{5}{2}$, and the product is $\frac{3}{2}$.

Exercises

What is the sum and product of the solutions of each of the following equations?

1. $3x^2 + 6x - 1 = 0.$
2. $5x^2 - 4x + 2 = 0.$
3. $x^2 + \frac{1}{2}x + \frac{1}{7} = 0.$
4. $x^2 - 10x + 13 = 0.$
5. $m^2x^2 - m(a - b)x - ab = 0.$
6. $acx^2 - bcx + adx - bd = 0.$
7. $4a + ax^2 = 2x + 2a^2x.$
8. $x^2 - 2ax + a^2 + b^2 = 0.$

48. Graphical solution of a quadratic equation. In order to solve graphically the equation $x^2 - 4x + 3 = 0$, we let $y = x^2 - 4x + 3$ and compute a table of values as follows:

$x =$	-3	-2	-1	0	1	2	3	4	5	6	7
$y =$	24	15	8	3	0	-1	0	3	8	15	24

Plotting the points $(-2, 15)$, $(-1, 8)$ . . . from the table and drawing a smooth curve through them we get the curve in Fig. 24. The graph crosses the X-axis at 1 and 3; hence, for

FIG. 24.

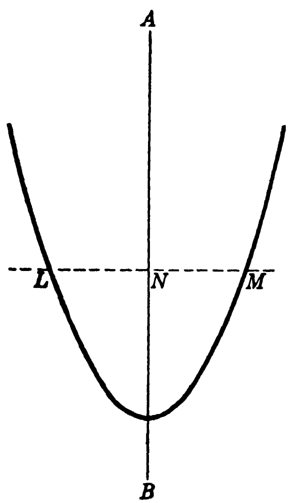

FIG. 25.

these values of x the function $x^2 - 4x + 3$ is zero. That is to say, 1 and 3 are the solutions of the equation $x^2 - 4x + 3 = 0$. These solutions are represented graphically by the abscissas of the points where the graph crosses the X-axis.

Were we to graph the function $Ax^2 + Bx + C$ where A is positive and not zero, we would get a curve having the same general shape as the curve in Fig. 25. This curve is called a parabola. If the graph crosses the X-axis, the X-intercepts give the real solutions of the equation $Ax^2 + Bx + C = 0$. If the curve has no point in common with the X-axis, the roots are imaginary. If the curve touches the X-axis, the roots are real and equal.

We have just stated above that the graph of the general quadratic function, $Ax^2 + Bx + C$, is called a parabola and is similar in shape to Fig. 25. We note that the parabola is symmetrical with respect to a certain line. The curve in Fig. 25 is symmetrical with respect to the line AB. This line AB is called the axis of the parabola. If we draw any line LM perpendicular to the axis AB intersecting the parabola in L and M and the axis in N we find that $LN = MN$. Then, what do we mean by the parabola being symmetrical with respect to its axis?

The curve in Fig. 24 is a parabola and we notice that it is symmetrical with respect to the line parallel to the Y-axis and two units to the right. What are the coordinates of the lowest point on this curve?

Exercises

Construct the graphs of the functions in the following equations and determine the roots if they are real. Determine the axis of symmetry of each of the curves. What are the coordinates of the lowest point on each curve?

1. $x^2 - 2x - 3 = 0$.
2. $4x^2 - 12x + 9 = 0$.
3. $x^2 - 2x + 5 = 0$.
4. $x^2 - 9x + 14 = 0$.
5. $x^2 + 2x - 1 = 0$.
6. $x^2 - 2x - 1 = 0$.
7. $x^2 + 4x + 3 = 0$.
8. $x^2 + x + 1 = 0$.
9. $x^2 + 4x + 6 = 0$.
10. $x^2 + 2x + 2 = 0$.

49. Minimum value of a quadratic function. We have just shown in Art. 48 that the graph of a quadratic function is a parabola symmetrical with respect to a certain vertical line called the axis of the parabola. (See Fig. 25.) We notice that the axis intersects the parabola in a single point B and that this point B is the lowest point on the curve. *Such a point is called a minimum point and the ordinate of such a point is defined as the minimum value of the quadratic function,* $Ax^2 + Bx + C$. In figure 24 the coordinates of the lowest point on the graph are $(2, -1)$ and -1 is the minimum value of the quadratic function $x^2 - 4x + 3$.

Consider again the equation, (1) $y = x^2 - 4x + 3$. The graph of this equation is Fig. 24. From the table of values we see that to a given value of y there corresponds two values of x. When $y = 3$, $x = 0$ and 4. When $y = 0$, $x = 1$ and 3. When $y = -1$, $x = 2$. We observe, then, that to every value of y there corresponds two values of x and that as y decreases the two corresponding values of x approach each other and finally for a certain value of y the two corresponding values of x are equal. In the above example the value of y, which causes the two values of x to be equal, is -1. But this value of y is the minimum value of the function, $x^2 - 4x + 3$.

Then, to determine the minimum value of the function, $x^2 - 4x + 3$, we must determine the value of y which will make equation (1) have equal values for x. Equation (1) may be written (2) $x^2 - 4x + 3 - y = 0$. Now, the roots of (2) will be equal when the discriminant equals zero.

We have,

$$(-4)^2 - 4(3 - y) = 0, \tag{3}$$

or

$$y = -1, \tag{4}$$

which is the minimum value of the function.

Example. Find the minimum value of the function,

$$x^2 + 3x + 4.$$

Solution.

Let

$$y = x^2 + 3x + 4. \tag{1}$$

Then $x^2 + 3x + (4 - y) = 0.$ (2)

Setting the discriminant equal to zero, we get,

$$9 - 4(4 - y) = 0, \text{ or} \quad (3)$$

$$y = \tfrac{7}{4}, \quad (4)$$

which is the minimum value of $x^2 + 3x + 4$.

Let us now find the minimum value of the typical quadratic function, $Ax^2 + Bx + C$.

Let $y = Ax^2 + Bx + C.$ (1)

Then $Ax^2 + Bx + (C - y) = 0.$ (2)

Setting the discriminant equal to zero, we get,

$$B^2 - 4A(C - y) = 0, \quad (3)$$

and $$y = \frac{4AC - B^2}{4A}, \quad (4)$$

which is the minimum value of the quadratic, $Ax^2 + Bx + C$.

Thus far in the discussion of the quadratic, $Ax^2 + Bx + C$, we have assumed that "A" was a positive number. Now, if "A" were a negative number, the graph would not be similar to Fig. 25, but would have the same general shape as Fig. 26. Fig. 26 is a parabola also, but here the point B is a maximum and not a minimum as in Fig. 25. *The expression,* $\frac{4AC - B^2}{4A}$, *gives us the minimum or maximum value of the quadratic according as "A" is positive or negative.*

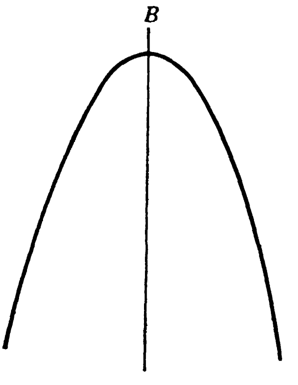

FIG. 26.

Thus the maximum value of $-2x^2 + 3x + 5$ is

$$\frac{4(-2)5 - 3^3}{4(-2)} = \frac{-40 - 9}{-8} = \frac{49}{8} = 6\tfrac{1}{8}.$$

Exercises

Find the maximum or minimum values of the following:

1. $x^2 - 6x + 10$.
2. $2x^2 - x - 3$.
3. $-3x^2 + 2x - 10$.
4. $1 - x - 2x^2$.
5. $x^2 + 2x + 2$.
6. $2 + 2x - x^2$.
7. $mx^2 + nx + k$.
8. $-5x^2 + x + 1$.

Exercises on Chapter VII

1. Solve:

(*a*) $12x^2 + x - 1 = 0$. (*b*) $\dfrac{3y - 6}{y + 2} = y - 2$.

(*c*) $x^2 - 4x + 1 = 0$. (*d*) $x^2 - 5.3x + 2.1 = 0$.

2. Find by the graphical method the approximate values of the roots of the equations

(*a*) $x^2 - 4x - 13 = 0$, (*b*) $x^2 + 2x - 13 = 0$.

3. Find the sum and product of the roots of the following equations:

(*a*) $3x^2 = 5 - 2x$, (*b*) $2x^2 + 5x + 3 = 0$,
(*c*) $(mx + 2)^2 = 4x$.

4. Determine the character of the roots of the following:

(*a*) $3x^2 + x + 1 = 0$, (*b*) $2x^2 - 5x + 1 = 0$,
(*c*) $16x^2 + 8x + 1 = 0$.

5. Find the number of acres in the largest rectangular field that can be inclosed by a mile of fence.

Solution.

Let x equal length of the field.

Then, $160 - x =$ width of field and A (the area) $= x(160 - x) = 160x - x^2$ or $A = -x^2 + 160x$. Thus the maximum area is given by

$$A = \frac{4(-1).0 - (160)^2}{4(-1)} = \frac{(160)^2}{4} = 6400 \text{ square rods,}$$

or 40 acres. When A equals 6400, we find that x equals 80 rods and the field is in the shape of a square.

6. Divide 20 into two parts such that the sum of their squares shall be a minimum.

Solution:

Let $x =$ one part.

Then, $20 - x =$ other part and S (the sum of their squares) $= x^2 + (20 - x)^2$ or $S = 2x^2 - 40x + 400$, and the minimum value of S is given by

$$S = \frac{4 \cdot 2 \cdot 400 - (-40)^2}{4 \cdot 2} = 200.$$

When $S = 200$, we have,

$$200 = 2x^2 - 40x + 400,$$

or

$$x^2 - 20x + 100 = 0,$$

$$x = 10, \text{ one part,}$$

and

$$20 - x = 10, \text{ other part.}$$

7. A window is to be constructed in the shape of a rectangle surmounted by a semicircle. Find the dimensions that will admit the maximum amount of light, if its perimeter is to be 48 feet.

8. A rectangular piece of ground is to be fenced off and divided into four equal parts by fences parallel to one of the sides. What should the dimensions be in order that as much ground as possible may be enclosed by 200 rods of fence?

9. A rectangular field is to be fenced off along the bank of a straight river, using 160 rods of fence. If no fence is needed along the river, what is the shape of the field in order that the enclosed area shall be the greatest possible?

10. A park is 150 rods long and 90 rods wide. It is decided to double the area of the park, still keeping it rectangular, by adding strips of equal width to one end and one side. Find the width of the strips.

11. A farmer starts cutting grain around a field 120 rods long and 80 rods wide. How wide a strip must he cut to make 10 acres?

12. A rectangular piece of ground is to be fenced off in the corner of a rectangular field and divided into four equal lots by fences parallel to one of the sides. What should the dimensions be in order that as much ground as possible may be enclosed by 200 rods of fence, the fences of the given field being used for two sides of the required field?

13. Build a water tank to hold 100 cubic feet. The length of the base is to be twice the width. Find the dimensions that will make the cost a minimum.

Solution. The cost will be a minimum when the surface is a minimum.

Let $x =$ width of the base.

Then, $2x =$ length of the base.

Let $y =$ depth.

Then,

$$2x^2y = 100 \text{ (volume)}, \qquad (1)$$

$$2x^2 + 6xy = S \text{ (surface)}. \qquad (2)$$

Substituting (1) in (2), we obtain,

$$S = 2x^2 + \frac{300}{x} \qquad (3)$$

$$= \frac{2x^3 + 300}{x}.$$

Our problem is to determine a value of x that will make S a minimum. This may be done by giving x values and computing the corresponding values of S.

$x =$	1	2	3	4	$4\frac{1}{4}$	$4\frac{1}{2}$	5	6	7
$S =$	302	158	118	107	106.7	107.16	110	122	$140\frac{6}{7}$

We notice that when $x = 4\frac{1}{4}$, $S = 106.7$ and this value is approximately the least value S can take. Hence the dimensions are (approx.)

$$4\tfrac{1}{4} \text{ feet, } 8\tfrac{1}{2} \text{ feet and } 2\tfrac{4}{5} \text{ feet.}$$

This problem illustrates another method for obtaining a minimum value of a function.

14. A covered box is to hold 200 cubic feet. The length of the base is to be two times the width. Find the dimensions that will make the cost a minimum.

CHAPTER VIII

EXPONENTS, RADICALS, BINOMIAL EXPANSION AND LOGARITHMS

50. Definition of a number. Laws of exponents. *Any number N may be defined as some other number a (a fixed number) raised to the nth power.* Thus we may write

$$N = a^n. \tag{1}$$

In (1) N is the number, a is defined as the base of the system of numbers and n is the exponent or the power to which a, the base must be raised to produce the number. For example, $1000 = 10^3$. Here 1000 is the number, 10 is the base and 3 is the power to which 10 must be raised to produce 1000.

By a^n, we mean the product of $a \cdot a \cdot a \dots$ to n factors, by a^4, we mean $a \cdot a \cdot a \cdot a$.

The laws of exponents are as follows:

I. $a^m \cdot a^n = a^{m+n}$. *To multiply numbers having the same base, we add their exponents.* Thus, $5^2 \cdot 5^4 = 5^6$.

II. $a^m \div a^n = a^{m-n}$. *To divide numbers having the same base, we subtract the exponent of the divisor from the exponent of the dividend.* Thus, $5^5 \div 5^3 = 5^{5-3} = 5^2$.

III. $(a^m)^n = a^{mn}$. Thus, $(5^3)^2 = 5^{3 \cdot 2} = 5^6$.

IV. $(ab)^m = a^m b^m$. Thus, $(3 \cdot 4)^3 = 3^3 \cdot 4^3$.

V. $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$. Thus, $\left(\frac{2}{3}\right)^3 = \frac{2^3}{3^3}$.

The above formulas apply not only when m and n are positive integers, but in all cases.

For example: $3^{2/5} \cdot 5^{-1/3} = 3^{2/5 - 1/3} = 3^{1/15}$.

By $a^{p/q}$ we mean the qth root of a^p. That is,

VI. $a^{p/q} = \sqrt[q]{a^p}$. Thus, $3^{2/5} = \sqrt[5]{3^2}$ and $5^{1/3} = \sqrt[3]{5}$.

VII. $a^\circ = 1$. For $a^\circ \cdot a^n = a^{\circ+n} = a^n$, and $a^\circ = \dfrac{a^n}{a^n} = 1$

VIII. $a^{-n} = \dfrac{1}{a^n}\cdot$ For $a^{-n} \cdot a^n = a^{-n+n} = a^\circ = 1$, and $a^{-n} = \dfrac{1}{a^n}\cdot$

Thus, $7^{-2/3} = \dfrac{1}{7^{2/3}} = \dfrac{1}{\sqrt[3]{7^2}} = \dfrac{1}{\sqrt[3]{49}}\cdot$

Exercises

Simplify the following indicated operations:

1. $x^3 \cdot x^5 \cdot x^{1/2}$.
2. $(x^2y^3)^4$.
3. $a^7 \div a^3$.
4. $(\frac{2}{3})^5 \div (\frac{2}{3})^3$.
5. $(a^{-1/2})^3$.
6. $(m^{1/3} + n^{1/3})m^{1/3}n^{1/3}$.
7. $(8a^3b^6)^{1/3}$.

Write each of the following with a radical sign and simplify:

8. $(16)^{1/4}$.
9. $(27)^{2/3}$.
10. $(3)^{2/3}$.
11. $x^{1/3}y^{1/3}$.
12. $\left(\dfrac{8a^6}{27b^9}\right)^{1/3}$.

Write the following in a form such that negative exponents do not appear and reduce to simplest form:

13. $12a^{-2/3}$.
14. $\dfrac{1}{(a+b)^{-2}}\cdot$
15. $a^2b^{-3}c^{-2}$.
16. $\dfrac{1}{2a^{-2}b^{-3}}\cdot$
17. $\dfrac{1}{a^{-2}+b^{-2}}\cdot$
18. $\left(\dfrac{2}{3}\right)^{-2}\left(\dfrac{225}{16}\right)^{-1/2}$.
19. $\dfrac{1}{a^{-3}} + \dfrac{1}{b^{-3}}\cdot$
20. $(8x^{-3}y^{-6})^{1/3}$.
21. $2^{-2} - 2^{-3}$.
22. $(a^2 + b^2)^\circ$.

Change the following into expressions without radical signs or negative exponents:

23. $\sqrt{b}$.

24. $\sqrt[3]{x^4}$.

25. $\sqrt{a^2b^4c^8}$.

26. $\sqrt[3]{x^{-6}y^{-2}}$.

27. $\sqrt[3]{(a+b)^6}$.

28. $\sqrt{(x+y)^{-4}}$.

29. $\sqrt[3]{(a^3)^{-2}}$.

30. $\sqrt[3]{a}\cdot\sqrt[5]{b}$.

31. $\sqrt[3]{\sqrt{a^4}}$.

32. $\sqrt[4]{(x+y)^{-3}}$.

33. $\sqrt{9(x+y)^3}$.

34. $\sqrt[3]{a^6b^3c^{-3}}$.

Solve the equation:

35. $y^{-2/3} = 9$.

Solution. $$\frac{1}{y^{2/3}} = 9.$$

$$\frac{1}{y} = 9^{3/2}, \quad \text{or} \quad y = \frac{1}{9^{3/2}}.$$

But $$9^{3/2} = \sqrt{9^3} = 27.$$

Therefore, $$y = \tfrac{1}{27}.$$

Solve the following for x:

36. $x^{1/3} = 2$.

37. $x^{-1/3} = 4$.

38. $\frac{1}{2}x^{-1/3} = 3$.

39. $x^{2/3} = 4$.

Multiply the following:

40. $a^{2/3} - a^{1/3}b^{1/3} + b^{2/3}$ by $a^{1/3} + b^{1/3}$.

41. $\sqrt{a^3} + \sqrt{b^3}$ by $a^{3/2} - b^{3/2}$.

Divide the following:

42. $x^{3/5} + b^{3/4}$ by $x^{1/5} + b^{1/4}$.

43. $16x^2 - 81y^4$ by $2x^{1/2} - 3y$.

44. $\sqrt[4]{a^3} - \sqrt[5]{b^6}$ by $a^{1/4} - b^{2/5}$.

51. Radicals. Simplification of radicals. *An indicated root of a number is called a radical.* Thus the expression $\sqrt[n]{a}$ is a radical. The quantity a under the radical sign is known as the radicand; n the number which indicates the root of the radicand is known as the index of the root.

For the purpose of computation it is often convenient to be able to change the form of radicals. A few examples will illustrate the processes:

Example 1. Simplify $\sqrt{32}$.

Solution. $\sqrt{32} = \sqrt{16 \cdot 2} = \sqrt{16}\,\sqrt{2} = 4\sqrt{2}$.

Example 2. Simplify $\sqrt[3]{128}$.

Solution. $\sqrt[3]{128} = \sqrt[3]{64 \cdot 2} = \sqrt[3]{64}\,\sqrt[3]{2} = 4\sqrt[3]{2}$.

Example 3. Simplify $\sqrt{\frac{32}{27}}$.

Solution. $\sqrt{\frac{32}{27}} = \frac{\sqrt{32}}{\sqrt{27}} = \frac{4\sqrt{2}}{3\sqrt{3}}$.

Example 4. Simplify $\sqrt{20} + 8\sqrt{45} - \sqrt{5}$.

Solution.

$$\sqrt{20} + 8\sqrt{45} - \sqrt{5} = 2\sqrt{5} + 24\sqrt{5} - \sqrt{5} = 25\sqrt{5}.$$

In example 4 we reduced each radical to the same radicand and then added terms.

Example 5. Simplify $\sqrt[3]{27x^5y^3z^4}$.

Solution. $\sqrt[3]{27x^5y^3z^4} = \sqrt[3]{27x^3y^3z^3x^2z}$

$$= \sqrt[3]{27x^3y^3z^3}\,\sqrt[3]{x^2z} = 3xyz\,\sqrt[3]{x^2z}.$$

Exercises

Simplify the following radicals:

1. $\sqrt{75}$.
2. $\sqrt[3]{81}$.
3. $7\sqrt{147}$.
4. $\sqrt[4]{81}$.
5. $5\sqrt[3]{32}$.
6. $\sqrt{m^6 + m^3n^2}$.
7. $\sqrt[3]{(x + y)^4}$.
8. $\left(\frac{1}{x^3} + \frac{1}{y^3}\right)^{1/3}$.
9. $\sqrt{4a^2b^3c}$.
10. $\sqrt{(x + y)^2(x - y)^3}$.
11. $\sqrt[6]{a^3x^3}$.

Solution. $\sqrt[6]{a^3x^3} = (a^3x^3)^{1/6}$
$= a^{3/6}x^{3/6} = a^{1/2}x^{1/2} = \sqrt{ax}$.

12. $\sqrt[4]{4x^2y^2}$.
13. $\sqrt[6]{216x^3y^6}$.
14. $\sqrt[6]{8}$.
15. $\sqrt[4]{9}$.
16. $\sqrt{3} - 2\sqrt{3} + 11\sqrt{3}$.
17. $3\sqrt[3]{28} - \sqrt[3]{63} + 4\sqrt[3]{175}$.
18. $\sqrt[3]{81} + 5\sqrt[3]{24} - \sqrt[3]{375}$.
19. $3\sqrt{b^3} + 4\sqrt{a^2bc^4} + \sqrt{4b^5c^2}$.
20. $\dfrac{1 + \sqrt{2}}{\sqrt{5}}$.

Solution. $$\frac{1 + \sqrt{2}}{\sqrt{5}} = \frac{1 + \sqrt{2}}{\sqrt{5}} \cdot \frac{\sqrt{5}}{\sqrt{5}} = \frac{\sqrt{5} + \sqrt{10}}{5}.$$

21. $\dfrac{3}{\sqrt{3} - \sqrt{2}}$.

Solution. $$\frac{3}{\sqrt{3} - \sqrt{2}} = \frac{3}{\sqrt{3} - \sqrt{2}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}}$$

$$= \frac{3\sqrt{3} + 3\sqrt{2}}{(\sqrt{3} - \sqrt{2})(\sqrt{3} + \sqrt{2})} = \frac{3\sqrt{3} + 3\sqrt{2}}{3 - 2} = 3\sqrt{3} + 3\sqrt{2}.$$

In examples 20 and 21 we have multiplied both the numerator and the denominator by the same expression. This expression was chosen so as to free the denominators of radicals. This process is known as the rationalization of the denominator.

22. $\dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$.

23. $\dfrac{3}{\sqrt{6} + 3}$.

24. $\dfrac{2}{2 - \sqrt{3}}$.

25. $\dfrac{2 + \sqrt{3}}{\sqrt{2} - \sqrt{3}}$.

26. $\sqrt{80} \div \sqrt{5}$.

27. $\sqrt[3]{135} \div \sqrt[3]{5}$.

28. $\sqrt[3]{a^2bc^2} \div \sqrt[3]{ac^2}$.

Solution. $\sqrt[3]{a^2bc^2} \div \sqrt[3]{ac^2}$

$$= \sqrt[3]{\frac{a^2bc^2}{ac^2}} = \sqrt[3]{ab}.$$

To divide radicals having the same index, divide the radicand of the numerator by the radicand of the denominator. If the radicals do not have the same index, reduce them to radicals having the same index and then divide.

29. $\sqrt[3]{25} \div \sqrt{5}$.

Solution. $\sqrt[3]{25} = (25)^{1/3} = (25)^{2/6} = \sqrt[6]{(25)^2} = \sqrt[6]{625},$

$\sqrt{5} = (5)^{1/2} = (5)^{3/6} = \sqrt[6]{(5)^3} = \sqrt[6]{125},$

and $\sqrt[3]{25} \div \sqrt{5} = \sqrt[6]{625} \div \sqrt[6]{125} = \sqrt[6]{5}.$

30. $6\sqrt{150} \div 5\sqrt{45}$.

31. $\sqrt[3]{a^2n} \div \sqrt[3]{an^2}$.

32. $(\sqrt{5} + 2\sqrt{3})^2$

33. $\sqrt{2} \div \sqrt[3]{2}$.

34. $\sqrt[4]{9} \div \sqrt[3]{3}$.

35. $\sqrt{\frac{3}{5}} \div \sqrt{\frac{5}{6}}$.

36. $\sqrt{\dfrac{n^2}{n+1}} \div \sqrt{\dfrac{n^3}{n-1}}$.

37. $\sqrt{\dfrac{n-1}{n+1}} \div \sqrt{\dfrac{n+1}{n-1}}$.

52. Binomial expansion; positive integral exponents. By multiplication we find:

$$(a + b)^2 = a^2 + 2ab + b^2,$$
$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

From the above expansion we observe the following properties:

(1) *The first term of the expansion is the first term of the binomial raised to the same power as that of the binomial.*

(2) *The exponents of a decrease by unity from term to term while the exponents of b increase by unity.*

(3) *The coefficient of the second term of the expansion is equal to the exponent of the binomial.*

(4) *If in any term the coefficient be multiplied by the exponent of a and divided by the exponent of b increased by unity, we get the coefficient of the next term.*

The question now arises: Do the four properties stated above hold for the expansion of $(a + b)^n$, for all positive integral values of n? By actual multiplication we see that these properties do hold for all positive integral values of n up to $n = 4$, and we assume that they hold for all positive integral values of n. This gives us the expansion.

$$\begin{aligned}(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 \\ + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \dots \\ + \frac{n(n-1)\dots(n-r+2)}{2\cdot 3\cdot 4\dots(r-1)}a^{n-r+1}b^{r-1} \\ + \dots + b^n. \qquad (1)\end{aligned}$$

Expansion (1) is known as the *binomial expansion* or *binomial theorem.* We have assumed that it is true for all positive integral values of n. This fact may be proven by the process known as mathematical induction but that is beyond the scope of this text.

In the expansion of $(a + b)^n$, the rth term is

$$\frac{n(n-1)(n-2)\dots(n-r+2)}{2\cdot 3\cdot 4\dots(r-1)}a^{n-r+1}b^{r-1}. \qquad (2)$$

Exercises

Expand:

1. $(2 - 3x)^5$.

Solution. Here $a = 2$, $b = -3x$, $n = 5$.

Then, $$(2 - 3x)^5 = (2)^5 + 5(2)^4(-3x) + 10(2)^3(-3x)^2 + 10(2)^2(-3x)^3 + 5(2)(-3x)^4 + (-3x)^5$$
$$= 32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5.$$

2. $(a + b)^7$.
3. $(a - b)^6$.
4. $(2 + a)^4$.
5. $(2x - 5)^5$.
6. $(2x + y)^4$.
7. $\left(\frac{a}{2} + 3\right)^5$.
8. $(a + \sqrt{c})^3$.
9. $(x + y + z)^3$.
(Hint: Consider $x + y$ as representing one number.)

10. Find the fourth term of $(a + 3b)^8$.

Solution. The rth term is given by the expression

$$\frac{n(n-1)(n-2)\ldots(n-r+2)}{2\cdot 3\cdot 4\ldots(r-1)}a^{n-r+1}b^{r-1}.$$

Here, $n = 8$, $r = 4$, $a = a$, $b = 3b$, $n - r + 2 = 6$, $n - r + 1 = 5$, $r - 1 = 3$. Substituting these values in the above expression, we have, $\frac{8\cdot 7\cdot 6}{2\cdot 3}a^5(3b)^3 = 1512a^5b^3$.

11. Find the 13th term of $(2x + y)^{18}$.
12. Find the middle term of $(x^2 + 2y)^8$.
13. Find the 9th term of $(3 - 2y)^{13}$.
14. Use the binomial theorem to find $(1.1)^{15}$, correct to four significant figures. (Hint: Write $(1.1)^{15}$ as $(1 + .1)^{15}$.)
15. Find $(1.01)^{10}$ correct to 5 significant figures.

53. Logarithms. Definition. In Article 50 a number N was defined by the equation, (1) $N = a^n$, where a was defined as the base of the system of numbers and n as the power to which the base must be raised to produce the number N. We there assumed that the definition held for all positive and negative values of n both integral and fractional. It could be shown that it also holds for irrational values of n, and we now assume this without proof. That is, we now give a meaning to such numbers as $a^{\sqrt{2}}$, $a^{\sqrt{3}}$, where $a > 0$.

If $a^n = N (a > 0, a \neq 1)$ then n is said to be the *logarithm* of N to the base a, and this is written $n = \log_a N$.

The two equations $$a^n = N \tag{2}$$
and $$n = \log_a N \tag{3}$$
thus mean the same thing; and the terms exponent and logarithm are equivalent.

We assume that the laws of exponents given in Article 50 which apply to rational exponents are also valid when irrational exponents are involved.

Exercises

1. $\log_5 25 = ?$ $\log_{10} 100 = ?$ $\log_2 16 = ?$

2. $\log_2 \frac{1}{4} = ?$ $\log_a a = ?$ $\log_{16} 4 = ?$

3. Fill out the following table:

Base	Number	Logarithm
	1000	3
5	....	4
2	$\frac{1}{32}$	
	32	−5
7	343	
6	....	3

54. Properties of logarithms.

1. The logarithm of a product equals the sum of the logarithms of its factors.

Let $\log N = n$ and $\log M = m$, then $a^n = N$, $a^m = M$, (definition of logarithm) and $NM = a^{n+m}$ (1. Art. 50).

Hence $\log_a NM = n + m.$

That is, $\log_a NM = \log_a N + \log_a M.$

This property is true for any number of factors in the product.

Example. $\log_{10} 105 = \log_{10} 3 + \log_{10} 5 + \log_{10} 7.$

2. The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor. The proof of this property is left as an exercise for the student.

Example. $\log_{10} \frac{125}{73} = \log_{10} 125 - \log_{10} 73.$

3. The logarithm of N^n equals n times the logarithm of N. This property is true for any value of the exponent n, whether positive or negative, integer or fraction. The proof is left for the student.

Example. $\log_{10} (153)^3 = 3 \log_{10} 153.$

Exercises

1. With 10 as base, $\log 2 = 0.30103$, $\log 3 = 0.47712$, $\log 5 = 0.69897$. Find log 4, log 6, log 8, log 9, log 12, log 15, log 20.

Ans.: 0.60206, 0.77815, 0.90309, 0.95424, 1.07918, 1.17609, 1.30103.

2. From the results of Ex. 1 above find $\log (\frac{9}{5})$, $\log (\frac{15}{8})$, log 144.

55. Common logarithms. Characteristic and mantissa. Any positive number (except 0 and 1) may be used as a base for a logarithmic system. Logarithms with 10 as a base are called *common logarithms.* This is the system used for all ordinary calculations.

From the table

$10^3 = 1000$	$10^{-1} = .1$
$10^2 = 100$	$10^{-2} = .01$
$10^1 = 10$	$10^{-3} = .001$
$10^0 = 1$	

it is evident that the logarithm of an integral power of 10 is an integer, either positive or negative. The logarithms of numbers between 1 and 10 are between 0 and 1, logarithms of numbers between 10 and 100 are between 1 and 2, and so on. For example, log 7 = 0.84510, log 70 = 1.84510, log 700 = 2.84510, log 7000 = 3.84510.

The integral part of a logarithm is called the *characteristic*; the decimal part is called the *mantissa.*

(*A*) **Law of the characteristic.** From the above examples, we observe that log 7 has a characteristic 0, log 70 has 1, log 700 has 2, and log 7000 has 3. From this we see that the *characteristic of a logarithm of a whole number is one less than the number of digits in the number. We also observe from the table that the characteristics of the logarithms of numbers less than* 1 *are negative and equal to the number of places which the first significant figure occupies to the right of the decimal point.* Thus

$$\log 0.00325 = -3 + .51188$$

In such cases the characteristic is negative and the mantissa is positive. It is customary in case of negative characteristics to write

$$\log 0.00325 = \bar{3}.51188$$

or

$$\log 0.00325 = 7.51188-10.$$

(*B*) **Law of the mantissa.** *The mantissa is the same for any sequence of digits and does not depend upon the position of the decimal point.*

For example,

$$\begin{aligned} \log 3256 &= 3.51268 \\ \log 325.6 &= 2.51268 \\ \log 32.56 &= 1.51268 \\ \log 3.256 &= 0.51268 \end{aligned}$$

56. Use of tables. In Table I in the back of this book five-place logarithms are given. The mantissas of the logarithms of all integers from 1 to 9999 are recorded correct to five decimal places. The methods by which such a table can be made will not be discussed here as it is beyond the scope of this text. In order to use the tables intelligently we must know how to read from the tables the logarithm of a given number, and also the number having a given logarithm.

Examples

1. Find the logarithm of 2354. Read down the column headed N for the first three significant figures, then at the top of the table for the fourth figure. In the row with 235 and the column with 4 is found 37181.

Hence, $$\log 2354 = 3.37181.$$

2. Find the logarithm of 32.625. This number has more than four significant figures, so we must obtain its logarithm by the process known as *interpolation*. As in example 1, we find that the mantissas of 32620 and 32630 are 51348 and 51362, respectively. The difference between these two mantissas is 14. Since 32625 is five tenths of the interval from 32620 to 32630, we add to 51348

$$0.5 \times 14 = 7.$$

Hence, $$\log 32.625 = 1.51355.$$

3. Find the number whose logarithm is 1.78147. The mantissa 78147 is found in the table and is in the column headed by 6 and opposite the digits 604 in the column headed by N. Thus the digits corresponding to mantissa 78147 are 6046.

Hence, $$\log 60.46 = 1.78147.$$

4. Find the number whose logarithm is $\bar{2}.62029$. The mantissa 62029 is not found in the table, but it lies between the two adjacent mantissas 62024 and 62034. The mantissa 62024 corresponds to the number 4171 and 62034 corresponds to 4172. The mantissa 62029 is $\frac{5}{10}$ of the interval from 62024 to 62034. Thus the number whose mantissa is 62029 is $41710 + \frac{5}{10} \times 10 = 41715$.

Hence, $\log 0.041715 = \bar{2}.62029.$

5. Find the value of $N = \dfrac{3.26 \times 72.65}{2.72}$ to five significant figures.

Solution.

$$\log N = \log 3.26 + \log 72.65 - \log 2.72$$

$$\begin{aligned}
\log 3.26 &= 0.51322 \\
\log 72.65 &= 1.86124 \\
\hline
\log (3.26)(72.65) &= 2.37446 \\
\log 2.72 &= 0.43457 \\
\hline
\log N &= 1.93989 \\
N &= 87.074.
\end{aligned}$$

6. Find the value of

$$N = \frac{\sqrt[3]{0.345}\sqrt{7.5}}{\sqrt{52.3}}.$$

Solution.

$$\log N = \tfrac{1}{3}\log 0.345 + \tfrac{1}{2}\log 7.5 - \tfrac{1}{2}\log 52.3$$

$$\begin{aligned}
\log 0.345 &= \bar{1}.53782 = 29.53782 - 30 \\
\log 7.5 &= 0.87506 \\
\log 52.3 &= 1.71850 \\
\tfrac{1}{3}\log 0.345 &= 9.84594 - 10 \\
\tfrac{1}{2}\log 7.5 &= 0.43753 \\
\hline
& 10.28347 - 10 \\
\tfrac{1}{2}\log 52.3 &= 0.85925 \\
\hline
\log N &= 9.42422 - 10 \\
N &= 0.26559
\end{aligned}$$

Why did we write $\log 0.345 = 29.53782 - 30$?

7. Find the value of $N = \sqrt[5]{0.235}$.

8. Find the value of $N = \dfrac{78.54 \times 9.67}{8.269}$.

9. Find the value of $N = \dfrac{(104.6)^{1/2} \times (0.2536)^{1/3}}{(5.87)^{1/2}}$.

10. Find the value of $S = P(1 + i)^n$, when $P = 235$, $i = .06$, $n = 7$.

11. Find the value of $\dfrac{0.07}{(1.07)^{11} - 1}$.

CHAPTER IX

PROGRESSIONS

57. Arithmetical progressions. *An arithmetical progression is a succession of numbers so related that each one is obtained by adding a fixed number to the preceding number.*

The numbers forming the progression are called its *terms.* The fixed amount which must be added to any term to get the next term is called the *common difference.*

Thus, 1, 3, 5, 7, 9, . . . is an arithmetical progression, having 2 for its common difference.

58. Elements of an arithmetical progression. Let a represent the first term, d the common difference, n the number of terms, l the nth or last term, s the sum of the terms. The five numbers a, d, n, l and s are called elements of the arithmetical progression.

59. Relations among the elements. If a is the first term and d the common difference, the progression is a, $(a + d)$, $(a + 2d)$, $(a + 3d)$, . . . $(a + (n - 1)d)$. It is evident that the nth or last term is

$$l = a + (n - 1)d. \tag{1}$$

Since s denotes the sum of the progression, we may write,

$$s = a + (a + d) + (a + 2d) + \dots (l - 2d) + (l - d) + l, \tag{2}$$

or $$s = l + (l - d) + (l - 2d) + \dots (a + 2d) + (a + d) + a. \qquad (3)$$

By adding (2) and (3) we get,

$$2s = (a + l) + (a + l) + \dots (a + l) + (a + l) + (a + l) = n(a + l).$$

Hence, $s = \frac{n}{2}(a + l).$ (4)

Equations (1) and (4) are the two relations among the five elements that always exist. If we know any three of these elements, we may find the other two by using (1) and (4).

60. Arithmetic means. The first and last terms of an arithmetical progression are called the *extremes*, and the remaining terms in between are called the *arithmetical means*. By the aid of (1) any number of means may be inserted between any two numbers.

Exercises

Find l and s for the following arithmetical progressions:

1. 3, 5, 7, 9, ... to 15 terms.

Solution. $l = a + (n - 1)d.$

Here, $a = 3, d = 2, n = 15.$

Then, $l = 3 + 14 \cdot 2 = 31.$

And $s = \frac{15}{2}(3 + 31) = 15 \times 17 = 255.$

2. 5, 2, -1, -4, to 12 terms.

3. $\frac{2}{3}$, $\frac{7}{12}$, $\frac{1}{2}$, to 10 terms.

4. 2, 9, 16, 23, to 9 terms.

5. Given $d = 4$, $n = 15$, $l = 59$; find a and s.

6. Given $a = 12$, $l = -64$, $s = -520$; find n and d.

7. Insert 5 arithmetical means between 2 and 14.

8. Insert 11 arithmetical means between 3 and 7.

61. Geometrical progression. *A geometric progression is a succession of numbers so related that the ratio of each one to the preceding one is a fixed number, called the ratio.* Thus 2, 6, 18, 54, . . . is a geometrical progression having three for its ratio.

62. Elements of a geometrical progression. Let a represent the first term, r the ratio, n the number of terms, l the nth, or last, term and s the sum of the terms. The numbers a, r, n, l, and s are called the elements of the geometrical progression.

63. Relations among the elements. If a is the first term and r the ratio, the progression is $a, ar, ar^2, ar^3, \ldots ar^{n-1}$.

It is evident that the nth or last term is

$$l = ar^{n-1}. \tag{5}$$

Since s denotes the sum of the progression, we may write,

$$s = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}. \tag{6}$$

Then, $$sr = ar + ar^2 + ar^3 + ar^4 + \ldots ar^{n-1} + ar^n. \tag{7}$$

Subtracting (6) from (7), we have $sr - s = ar^n - a$.

Hence, $$s = \frac{a(r^n - 1)}{r - 1}. \tag{8}$$

Equations (5) and (8) are the two relations among the five elements that always exist. If we know any three of these elements we may find the other two by using (5) and (8).

The first and last term of a geometrical progression are called the *extremes*, and the remaining terms in between are called the *geometrical means*. By the aid of (5), any number of means may be inserted between two numbers.

Exercises

1. Given $a = 2$, $r = 3$, $n = 8$; find l and s.

2. Given $a = 3$, $r = 2$, $n = 10$; find l and s.

3. Given $s = 242$, $a = 2$, $n = 5$; find r and l.

4. Insert 4 geometrical means between 3 and 96.

5. The first term of a geometrical progression is 3, and the last term 81. If there are four terms in the progression, find the ratio and the sum of the terms.

6. An employer hires a clerk for five years at a beginning salary of $500 per year with either a raise of $100 each year after the first, or a raise of $25 every six months after the first half year. Which is the better proposition for the clerk?

7. Find the sum of the progression

$$1 + (1 + i) + (1 + i)^2 + (1 + i)^3 + \ldots (1 + i)^{n-1}.$$

Ans. $\dfrac{(1 + i)^n - 1}{i}$.

8. Find the sum of the progression

$$(1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3} + \ldots (1 + i)^{-n}.$$

Ans. $\dfrac{1 - (1 + i)^{-n}}{i}$.

9. By the use of logarithms find the value of $\dfrac{(1 + i)^n - 1}{i}$, when $i = .06$ and $n = 8$.

Solution. We have $\dfrac{(1.06)^8 - 1}{.06}$

$$\log (1.06) = 0.02531$$
$$\log (1.06)^8 = 0.20248$$
$$(1.06)^8 = 1.59396$$
$$(1.06)^8 - 1 = 0.59396$$
$$\frac{(1.06)^8 - 1}{.06} = \frac{0.59396}{.06} = 9.899.$$

10. Find the value of $255\left[\dfrac{(1.07)^{10} - 1}{.07}\right]$.

CHAPTER X

INTEREST, ANNUITIES, SINKING FUND

64. Simple interest. Simple interest at any rate is most readily computed by the application of the principle of aliquot parts.

If we consider a year as composed of 12 months of 30 days each (360 days),

At 6%, the interest on $1 for 1 year is $0.06,

At 6%, the interest on $1 for 2 mo. (60 days) is $0.01,

At 6%, the interest on $1 for 6 days is $0.001.

That is, *to find the interest on any sum of money at* 6% *for* 6 *days, point off three places in the principal sum; and for* 60 *days, point off two places in the principal sum.*

The interest on $1357 for 6 days at 6% is $1.357 and the interest on $1357 for 60 days is $13.57.

Illustrations:

Find the interest on:

1. $385.60 for 32 days at 6%.

$0.3856 = int. for 6 days

$1.9280 = int. for 30 days (5·6 days)

.1285 = int. for 2 days ($\frac{1}{3}$·6 days)

$2.0565 or $2.06 = int. for 32 days.

2. $435.00 for 115 days at 6%.

$4.350 = int. for 60 days
2.175 = int. for 30 days ($\frac{1}{2}\cdot$60 days)
1.450 = int. for 20 days ($\frac{1}{3}\cdot$60 days)
.362 = int. for 5 days ($\frac{1}{4}\cdot$20 days)

$8.337 = int. for 115 days.

3. $520.00 for 93 days at 8%.

$5.20 = int. for 60 days at 6%
2.60 = int. for 30 days at 6% ($\frac{1}{2}\cdot$60 days)
.26 = int. for 3 days at 6% ($\frac{1}{10}\cdot$30 days)

$8.06 = int. for 93 days at 6%
2.69 = int. for 93 days at 2% ($\frac{1}{3}\cdot$6%)

$10.75 = int. for 93 days at 8%.

4. $285.50 for 78 days at 5%.

$2.855 = int. for 60 days at 6%
.714 = int. for 15 days at 6% ($\frac{1}{4}\cdot$60 days)
.143 = int. for 3 days at 6% ($\frac{1}{5}\cdot$15 days)

$3.712 = int. for 78 days at 6%.
.619 = int. for 78 days at 1%

$3.093 = int. for 78 days at 5%.

5. $275.00 from March 3, 1928 to January 2, 1929 at 7%.

1929	—	1	—	2
1928	—	3	—	3
		9 mo.		29 days

$ 2.75	= int. for	60 days (2 mo.) at 6%
$11.00	= int. for	240 days (8 mo.) at 6%
1.375	= int. for	30 days (1 mo.) at 6%
.917	= int. for	20 days at 6% ($\frac{1}{3}\cdot 60$ days)
.275	= int. for	6 days at 6%
.137	= int. for	3 days at 6% ($\frac{1}{2}\cdot 6$ days)
$13.704	= int. for	9 mo. 29 days at 6%
2.284	= int. for	9 mo. 29 days at 1%
$15.988	= int. for	9 mo. 29 days at 7%.

Exercises

1. Find the interest at 6% on:

$825 for 50 days,
$753.40 for 70 days,
$365.50 for 97 days,
$847.60 for 125 days.

2. Solve 1, if the rate is 7%.

3. Find the interest at 8% on:

$425 for 38 days,
$575 for 68 days,
$545 for 90 days,
$750 for 115 days,
$800 for 100 days,
$250 for 83 days.

Find the interest at 6% on the following:

4. $756.50 from Feb. 20, 1928 to Sept. 15, 1928,

5. $3756.40 from March 1, 1927 to July 10, 1928,

6. $5250.00 from April 10, 1928 to March 5, 1929.

7. A note for $350 was given July 7, 1927. What was the interest at 7% due Sept. 5, 1928? (Ans. $28.45)

8. Find the interest on 4, 5, 6 at 8%.

65. Compound interest. Simple interest is calculated on the original principal only, and is proportional to the time. *If the interest, when due, is added to the principal, and the interest for the next period is calculated on the principal thus increased and this process is continued with each succeeding accumulation of interest, the interest is said to be compound.* Interest may be computed annually, semi-annually, quarterly, or at some other regular interval. That is, interest is converted into principal at these regular intervals.

66. Compound interest formulas. Let P be the principal, i the rate of interest, and S the amount to which P will accumulate in n years. The interest for one year will be Pi, and the amount at the end of the year will be $P + Pi = P(1 + i)$. This is the principal for the second year, and the interest for the second year will be $P(1 + i)i$. The amount at the end of the second year will be

$$P(1 + i) + P(1 + i)i = P(1 + i)^2.$$

By similar reasoning we find that the amount at the end of the third year is $P(1 + i)^3$, and in general the amount at the end of n years is $P(1 + i)^n$. We thus have the formula

$$S = P(1 + i)^n. \tag{1}$$

In equation (1) i is the annual rate of interest and the formula is used when the interest is converted into principal annually. If the interest were converted into principal m times per year, we would replace i in the formula by $\frac{i}{m}$ and n by mn. That is we would find the compound amount at $\frac{i}{m}$ per cent per period for mn periods.

Example. Find the compound amount of \$100 for 15 years at 6% converted semi-annually. The amount would be $\$100(1.03)^{30}$.

Then in general, if interest is at rate j converted m times per year, formula (1) is replaced by

$$S = P\left(1 + \frac{j}{m}\right)^{mn}. \tag{2}$$

Exercises

1. Find the amount of \$250 at 6% interest converted annually for 5 years.

2. Solve example 1 if the interest is converted semi-annually.

3. How long will it take \$100 to double itself at 6% interest converted annually?

Solution. Here $P = \$100$ and $S = \$200$, since it is to be double the value of P. We then have, $200 = 100(1.06)^n$. Our problem now is to find n. Taking logarithms of both sides of the above equation, we have,

$$\log 200 = \log 100 + n \log (1.06),$$

and solving for n we get,

$$n = \frac{\log 200 - \log 100}{\log 1.06} = \frac{2.3010 - 2.0000}{.0253} = 11.9 \text{ years.}$$

NOTE. This is the time required for any principal to double itself at 6%.

4. How long will it require \$75 to double itself at 5% interest converted annually?

5. How long will it take any principal to double itself at i% converted annually?

6. What principal will amount to \$1000 in 6 years at 6% converted annually?

7. A father wishes to have \$2000 to give his son on his 21st birthday. What sum should he deposit at his birth in a savings bank paying 5% interest converted annually?

67. Annuities. *Any series of equal payments, made at equal intervals of time, is known as an annuity.* The word annuity implies yearly payments, but in a broader sense the term annuity is used to describe any series of equal payments made at equal intervals of time. Unless otherwise designated, the payments are understood to be made at the end of the interval of time and to continue for a specified number of periods. The dividends from an investment, income from rented property, and insurance premiums are some examples of an annuity.

68. Amount of an annuity. *The sum to which the entire number of payments accumulate is called the amount of the annuity.* We now find the amount of an annuity of one dollar per annum. The symbol $s_{\overline{n}|}$, is universally used to represent the amount of an annuity of 1 per annum, payable annually for n years at rate i per annum. The first payment made at the end of the first year will be at interest for $n - 1$ years and its compound amount will be $(1 + i)^{n-1}$. (See (1) Art. 66.) The second payment made at the end of the second year will accumulate to $(1 + i)^{n-2}$ and the third payment made at the end of the third year will accumulate to $(1 + i)^{n-3}$ and so on. The last payment will be a cash payment of 1 and will draw no interest. We then have

$$s_{\overline{n}|} = (1 + i)^{n-1} + (1 + i)^{n-2} + (1 + i)^{n-3} + \ldots + (1 + i) + 1 = 1 + (1 + i) + (1 + i)^2 + \ldots + (1 + i)^{n-2} + (1 + i)^{n-1}.$$

This is a geometrical progression of n terms, having 1 for first term and $(1 + i)$ for ratio.

The sum of this series is $\frac{(1 + i)^n - 1}{i}$. (See ex. 7, Art. 63.)

Hence,
$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}. \tag{1}$$

If the annual payment is R and if K represents the amount, we have,

$$K = Rs_{\overline{n}|} = R\frac{(1+i)^n - 1}{i}. \tag{2}$$

Formulas (1) and (2) are true where the interest is converted once a year. Now, if the interest is converted m times a year, we replace $(1+i)$ by $\left(1+\frac{j}{m}\right)^m$ and i by $\left(1+\frac{j}{m}\right)^m - 1$ in (1) and (2) and get,

$$s_{\overline{n}|} = \frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^m - 1}, \tag{3}$$

and

$$K = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{\left(1+\frac{j}{m}\right)^m - 1}. \tag{4}$$

Exercises

1. Find the amount of an annuity of \$200 per annum for 15 years at 5% interest.

Solution. From equation (2) above we have,

$$K = \$200s_{\overline{n}|} = 200\frac{(1.05)^{15} - 1}{.05}$$

$$= 4000((1.05)^{15} - 1)$$

$$\log 1.05 = 0.02119$$

$$15 \log 1.05 = 0.31785$$

$$(1.05)^{15} = 2.0790$$

$$(1.05)^{15} - 1 = 1.0790$$

and $$K = 4000 \times 1.0790 = \$4316.00.$$

NOTE. If the number of conversions is not specified in a problem, it will be understood that the interest is converted annually.

2. If in example 1 the interest is converted semi-annually, find the amount of the annuity.

Solution. From equation (4) above we have,

$$K = 200\frac{(1.025)^{30} - 1}{(1.025)^{2} - 1}.$$

$$\log 1.025 = 0.01072$$

$$30 \log 1.025 = 0.32160$$

$$(1.025)^{30} = 2.0970$$

$$(1.025)^{30} - 1 = 1.0970$$

$$2 \log 1.025 = 0.02144$$

$$(1.025)^{2} = 1.0506$$

$$(1.025)^{2} - 1 = 0.0506$$

and

$$K = \frac{200 \times 1.0970}{.0506} = \$4,336.$$

3. The annual rent of a house is $500. Find the amount of this annuity for 20 years at 5%.

4. A man deposits in a savings bank at the end of each year $400. What will be the amount of his savings at the end of 16 years, if the bank pays 4% interest converted semi-annually?

5. What sum must be deposited in a savings bank at the end of each year to amount to $5000 at the end of 10 years, if the bank pays 4% interest?

Solution. Here we have the amount of an annuity to find the annual deposit.

From (2) we have, $5000 = R\frac{(1.04)^{10} - 1}{.04}.$

Solving for R we get, $R = \dfrac{.04(5000)}{(1.04)^{10} - 1}$

$$= \frac{\$200}{0.4801} = \$416.58.$$

6. What sum must be deposited in a savings bank paying 5% interest, converted semi-annually, to provide for the payment of a debt of $5000 due in 6 years?

7. A man gives a mortgage on his farm for $3000, which is to be paid in 5 years. How much money must he deposit at the end of each year in a savings bank, paying 4% interest, to care for the debt when due? The interest on the mortgage is 6%. What will be his total yearly outlay to care for this debt?

69. Amount of an annuity, where the annual payment, R is payable in p equal installments. The amount of an annuity of 1 per annum, payable in p equal installments at equal intervals during the year, will be denoted by the symbol, $s_{\overline{n|}}^{(p)}$. If the interest is converted yearly and i is the rate, $s_{\overline{n|}}^{(p)}$ can be expressed in terms of n, i, and p as follows. At the end of the pth part of a year, $\frac{1}{p}$ is paid. This sum will remain at interest for $\left(n - \frac{1}{p}\right)$ years and will amount to

$$\frac{1}{p}(1 + i)^{n-1/p}.$$

The second installment of $\frac{1}{p}$ will be at interest for $\left(n - \frac{2}{p}\right)$ years and will amount to $\frac{1}{p}(1 + i)^{n-2/p}$, and so on until np installments are paid. The last installment will be paid at the end of n years and will draw no interest. Adding all these installments beginning with the last one, we have

$$s_{\overline{n|}}^{(p)} = \frac{1}{p} + \frac{1}{p}(1+i)^{1/p} + \frac{1}{p}(1+i)^{2/p} + \ldots \frac{1}{p}(1+i)^{n-2/p}$$

$$+\frac{1}{p}(1+i)^{n-1/p}. \tag{1}$$

This is a geometrical progression of np terms having $\frac{1}{p}$ for first term and $(1+i)^{1/p}$ as the ratio.

Hence,
$$s_{\overline{n|}}^{(p)} = \frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]}. \tag{2}$$

and
$$K = Rs_{\overline{n|}}^{(p)} = R\frac{(1+i)^n - 1}{p[(1+i)^{1/p} - 1]}. \tag{3}$$

If the interest is converted m times a year, we substitute $\left(1 + \frac{j}{m}\right)^m$ for $(1 + i)$ and (3) becomes

$$K = R\frac{\left(1+\frac{j}{m}\right)^{mn} - 1}{p\left[\left(1+\frac{j}{m}\right)^{m/p} - 1\right]}. \tag{4}$$

If the number of conversion periods is equal to the number of installments per year, i.e., $m = p$, equation (4) takes a simpler form. Then,

$$K = \frac{R\left[\left(1+\frac{j}{p}\right)^{np} - 1\right]}{p\left[\left(1+\frac{j}{p}\right)^{p/p} - 1\right]} = \frac{R}{p}\frac{\left(1+\frac{j}{p}\right)^{np} - 1}{\frac{j}{p}}. \tag{5}$$

Equation (5) is the same as equation (2) Art. 68, $\frac{R}{p}$ being the periodic payment for np periods at rate $\frac{j}{p}$ per period.

Exercises and Problems

1. Find the amount of an annuity of \$400 per year paid in four quarterly installments of \$100 for 6 years if the rate of interest is 6%.

Solution. Here $R = \$400$, $i = .06$, $p = 4$, $n = 6$, and using (3), Art. 69, we get,

$$K = \frac{400[(1.06)^6 - 1]}{4[(1.06)^{1/4} - 1]} = 100\frac{(1.06)^6 - 1}{(1.06)^{1/4} - 1}.$$

$$\begin{aligned} \log 1.06 &= 0.02531 \\ \log (1.06)^{1/4} &= 0.00633 \\ (1.06)^{1/4} &= 1.01467 \\ (1.06)^{1/4} - 1 &= 0.01467 \\ \log (1.06)^6 &= 0.15186 \\ (1.06)^6 &= 1.41860 \\ (1.06)^6 - 1 &= 0.41860 \end{aligned}$$

Hence, $$K = \frac{100(0.41860)}{0.01467} = \$2853.$$

2. If in Ex. 1, the interest were converted semi-annually, what would be the amount of the annuity?

Solution. Here $R = 400$, $p = 4$, $n = 6$, $m = 2$ and $j = .06$. Using equation (4), Art. 69, we get,

$$K = \frac{400[(1.03)^{12} - 1]}{4[(1.03)^{1/2} - 1]} = 100\frac{(1.03)^{12} - 1}{(1.03)^{1/2} - 1}.$$

$$\begin{aligned} \log 1.03 &= 0.01284 \\ \log (1.03)^{1/2} &= 0.00642 \end{aligned}$$

$$(1.03)^{1/2} = 1.01488$$
$$(1.03)^{1/2} - 1 = 0.01488$$
$$\log (1.03)^{12} = 0.15408$$
$$(1.03)^{12} = 1.42587$$
$$(1.03)^{12} - 1 = 0.42587$$

Hence, $$K = \frac{100(0.42587)}{0.01488} = \$2862.$$

3. What would be the amount of the annuity defined in Ex. 1, if the interest were converted quarterly?

Solution. Here $R = 400$, $p = 4$, $n = 6$, $m = 4$, and $j = .06$. Since $m = p$, we use (5), Art. 69, and

$$K = \frac{100(1.015)^{24} - 1}{.015}.$$

$$\log 1.015 = 0.00647$$
$$\log (1.015)^{24} = 0.15528$$
$$(1.015)^{24} = 1.42980$$
$$(1.015)^{24} - 1 = 0.42980$$

Hence, $$K = \frac{100(0.42980)}{.015} = \$2865.$$

NOTE. In solving Examples 1, 2, 3 above, 5 place logarithms were used. Had 7 place interest and annuity tables been used the results would have been \$2852.15, \$2859.53 and \$2863.35 respectively, which are correct to the nearest cent. But ordinarily 5 place logarithms will give results which are accurate enough. Should the student desire complete interest and annuity tables, he is referred to "Tables of Compound Interest Functions and Logarithms of Compound Interest Functions," by James W. Glover and Harry C. Carver, published by George Wahr, Ann Arbor, Michigan.

4. Find the amount of an annuity of \$400 per year, payable in two semi-annual installments of \$200 for 8 years, if the rate of interest is 4% converted quarterly.

5. A man pays into a Building and Loan Association \$25 at the end of each month for 10 years. If the association pays 6% interest and computes its interest at the end of each six months, what will he have to his credit at the end of the 10 years?

6. In purchasing a house priced at \$6000, a man pays \$3000 down and gives a five-year mortgage for the balance. In order to meet the mortgage when due he deposits in a 5% savings bank at the end of each month a portion of his monthly salary. Find the monthly deposit. (Hint: Use equation (3), Art. 69 and solve for $\frac{R}{12}$ as R was solved for in Ex. 5, Art. 68.)

70. Present value. We may need to find the value of a sum of money at some time before it is due. *By the present value of a sum S, due in n years, we mean the principal that will at a given rate amount to S in n years.* This problem is solved by equation (1) Art 66. Solving this equation for P, we get

$$P = \frac{S}{(1+i)^n} = Sv^n, \text{ where } v = \frac{1}{1+i}. \tag{1}$$

Example. Find the present value of a note of \$200 due in 5 years if money is worth 5% interest.

Solution. Here we have,

$$P = \frac{\$200}{(1.05)^5} = \frac{\$200}{1.2763} = \$156.70.$$

71. Present value of an annuity. *By the present value of an annuity we mean the sum of the present values of all the payments.* The present value of an annuity of 1 per annum is represented by the symbol $a_{\overline{n}|}$. We now find the present value of an annuity of 1 per annum for n years at rate i per annum. The present value of the first payment made at the end of the first year will be

$$\frac{1}{1+i} = (1+i)^{-1}.$$

The present value of the second payment made at the end of the second year will be $(1 + i)^{-2}$ and the third payment made at the end of the third year will have for its present value $(1 + i)^{-3}$ and so on. The last payment made at the end of n years will have $(1 + i)^{-n}$ for its present value. We have then,

$$a_{\overline{n|}} = (1 + i)^{-1} + (1 + i)^{-2} + (1 + i)^{-3} + \ldots (1 + i)^{-n}. \quad (1)$$

This is a geometrical progression of n terms, having $(1 + i)^{-1}$ for first term and $(1 + i)^{-1}$ for ratio. The sum of this series is

$$a_{\overline{n|}} = \frac{(1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1}$$

$$= \frac{(1 + i)^{-n} - 1}{1 - (1 + i)}$$

$$= \frac{(1 + i)^{-n} - 1}{-i} = \frac{1 - (1 + i)^{-n}}{i}. \quad (2)$$

If the annual payment is R and A represents the present value, we have,

$$A = Ra_{\overline{n|}} = R\frac{1 - (1 + i)^{-n}}{i}. \quad (3)$$

If the interest is converted m times a year, we substitute $\left(1 + \frac{j}{m}\right)^m$ for $(1 + i)$ and $\left(1 + \frac{j}{m}\right)^m - 1$ for i in (3) and get,

$$A = R\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{\left(1 + \frac{j}{m}\right)^m - 1}. \quad (4)$$

72. Present value of an annuity, where the annual payment R is payable in p equal installments. The present value of an annuity of 1 per annum payable in p equal installments will be denoted by $a_{\overline{n}|}^{(p)}$. If the interest is converted yearly and i is the rate, $a_{\overline{n}|}^{(p)}$ can be expressed in terms of n, i, and p as follows. The first payment will be made at the end of the pth part of the year and its present value will be $\frac{1}{p}(1 + i)^{-1/p}$. Similarly, the present value of the second payment will be $\frac{1}{p}(1 + i)^{-2/p}$ and so on. The present value of the last payment will be $\frac{1}{p}(1 + i)^{-n}$. Adding the present values of all these payments we get,

$$a_{\overline{n}|}^{(p)} = \frac{1}{p}(1 + i)^{-1/p} + \frac{1}{p}(1 + i)^{-2/p} + \ldots \frac{1}{p}(1 + i)^{-n}. \quad (1)$$

This is a geometrical progression of np terms having $\frac{1}{p}(1 + i)^{-1/p}$ for first term and $(1 + i)^{-1/p}$ for ratio.

Hence,
$$a_{\overline{n}|}^{(p)} = \frac{1 - (1 + i)^{-n}}{p[(1 + i)^{1/p} - 1]}. \quad (2)$$

And
$$A = R\frac{1 - (1 + i)^{-n}}{p[(1 + i)^{1/p} - 1]}. \quad (3)$$

If the interest is converted m times per year, (3) becomes

$$A = R\frac{1 - \left(1 + \frac{j}{m}\right)^{-mn}}{p\left[\left(1 + \frac{j}{m}\right)^{m/p} - 1\right]}. \quad (4)$$

When $m = p$, (see (5) Art. 69), (4) takes the form,

$$A = R\frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{p\left[\left(1 + \frac{j}{p}\right)^{p/p} - 1\right]} = \frac{R}{p}\,\frac{1 - \left(1 + \frac{j}{p}\right)^{-np}}{\frac{j}{p}}. \qquad (5)$$

which is similar to (3) Art. 71, $\frac{R}{p}$ being the periodic payment for np periods at rate $\frac{j}{p}$ per period.

Exercises and Problems

1. What is the present value of an annuity of $200, payable at the end of each year, for 12 years, if money is worth 6%?

Solution. $A = 200\,\frac{1 - (1.06)^{-12}}{.06}.$

$$\log 1.06 = 0.02531$$

$$\log (1.06)^{-12} = -(0.30372) = 9.69628 - 10$$

$$(1.06)^{-12} = 0.49691$$

$$1 - (1.06)^{-12} = 0.50309$$

$$A = \frac{200(0.50309)}{.06} = \$1676.97.$$

2. Find the cost of an annuity of $500, to run 20 years and payable at the end of the year if money is worth 6%, converted semi-annually.

3. A man purchased a house, paying $5000 down and $500 at the end of each year for 8 years. What would be the equivalent price if he paid all in cash at the time of purchase, money being worth 8%?

4. Find the cost of an annuity of $100 payable at the end of each month and to run for 10 years, if money is worth 4%. (Hint: Use equation (3), Art. 72.)

5. A house is purchased for \$10,000 and it is arranged that \$5000 cash be paid and the balance in 10 equal annual installments, including interest at 6%. Find the annual payment.

Solution. \$10,000 − \$5000 = \$5000, balance to be paid in 10 equal annual payments including interest. Here we have the present value of an annuity and are required to find the annual payment. Substituting in equation (3), Art. 71, we get,

$$5000 = R\frac{1-(1.06)^{-10}}{.06},$$

and
$$R = \frac{.06(5000)}{1-(1.06)^{-10}} = \frac{.06(5000)}{0.44160} = \$679.34.$$

6. A piece of property is offered for sale for \$500 cash and \$1000, at the end of each year for 5 years, or for \$5000 cash. Which is the better plan for the buyer if money is worth 5%?

7. A man wishes to provide an income for old age. He assumes that at the age of 25 years he will have 35 years of productive activity ahead of him, and that he can save \$300 per year during that time. This accumulation at 5% compound interest at age of 60 will purchase what annual payments for 20 years, if money is worth 5%?

Ans. \$2174.40.

73. Sinking funds. When an obligation becomes due at some future date, it is usually desirable to provide for the payments by accumulating a fund by periodic contributions, together with interest earnings. *Such an accumulated fund is called a sinking fund.*

Example. A debt of \$8000, bearing 8% interest is due in 4 years. A sinking fund is to be accumulated at 6%. What sum must be deposited in the sinking fund at the end of each year to care for the principal when due? Ans. \$1828.73.

The amount in the sinking fund at any particular time may be shown by the following schedule known as an *accumulation schedule*:

ACCUMULATION SCHEDULE

Years	Annual Deposit	Interest on Fund	Total Annual Increment	Value of Fund at End of Each Year
1	1828.73	....	....	1828.73
2	1828.73	109 72	1938.45	3767.18
3	1828 73	226.03	2054 76	5821.94
4	1828 73	349 32	2178 05	7999.99

We notice that at the end of the fourth year the value of the fund is $7999.99 or one cent less than the amount of the debt. This would have been avoided had we used the nearest mill instead of the nearest cent in our computations.

DEPRECIATION SCHEDULE

Age in Years	Book Value at End of Year	Annual Payment to Sinking Fund to Cover Depreciation	Interest on Depreciation Allowance	Total in Sinking Fund
0	235 00			
1	211.44	23 56	0.00	23.56
2	186 70	23.56	1.18	48 30
3	160.72	23 56	2.42	74.28
4	133 45	23 56	3 71	101.55
5	104.81	23.56	5.08	130.19
6	74.74	23.56	6 51	160.26
7	43.17	23 56	8 01	191.83
8	10.02	23.56	9.59	224.98

Example. A farmer pays $235 for a binder. The best estimates show that it will have a life of 8 years and a scrap value of $10. He wishes to create a sinking fund to provide for its depreciation. Assuming money worth 5%, what is the annual depreciation charge? Make a schedule showing the book value of the machine at the end of each year and the total amount in the sinking fund at any time.

Solution. The annual depreciation charge will equal the annual deposit required to accumulate in 8 years at 5% to \$225 (\$235 − \$10).

Using (2) Art. 68, we find the annual charge to be \$23.56.

We notice that the book value of the machine at the end of any year equals the original cost less the total amount in the sinking fund at that time.

74. Amortization. Instead of leaving the entire principal of a debt standing until the end to be cancelled by a sinking fund, we may consider any payment over what is needed to pay interest on the principal to be applied at once toward liquidation of the debt. As the debt is being paid off, a less and less amount goes towards the payment of interest, so that with a uniform payment per year, a greater amount goes towards the payment of principal. This method of extinguishing a debt is called the method of *amortization of principal.*

75. Amortization schedules.

Consider a debt of \$2000 bearing 6% interest. Suppose that it is desired to repay this in 8 equal annual installments, including interest.

Substituting in equation (3) Art 71, we get,

$$2000 = R\frac{1 - (1.06)^{-8}}{.06},$$

and

$$R = \frac{.06(2000)}{1 - (1.06)^{-8}} = \$322.07.$$

The interest for the first year will be \$120; hence \$202.07 of first payment would be used for the reduction of principal, leaving $1797.93 due on principal at the beginning of the second year. The interest on this amount is $107.88; hence, the principal is reduced by $214.19, leaving $1583.74 due on principal at the beginning of the third year, and so on. This process may be continued by means of the following schedule known as an *amortization schedule*:

Year	Principal at Beginning of Year	Interest at 6%	Principal Repaid
1	2000 00	120.00	202.07
2	1797 93	107 88	214.19
3	1583.74	95 02	227.05
4	1356 69	81 40	240.67
5	1116.02	66 96	255.11
6	860 91	51 65	270.42
7	590 49	35.43	286 64
8	303 85	18.23	303.84
	9609 63	576.57	1999.99

Such a schedule gives us the amount remaining due on the principal at the beginning of any year during the amortization period. The principal at the beginning of the last year should equal the last principal repaid and the sum of the principals repaid should equal the original principal. You will notice that there is a discrepancy in the above example of only one cent. This would have been avoided had we used the nearest mill instead of the nearest cent in our computations. As a further check we notice that the interest on the sum of all the principals outstanding is equal to the sum of all the interest paid. In the above example we see that the sum of the principals outstanding is $9609.63 and that the interest on this sum at 6% is $576.57.

The Federal Farm Loan Act provides for the lending of money to farmers at a reasonable rate of interest, with the privilege of amortizing the principal by equal annual payments over a long period of time. The maximum loan on a farm is for 40% of the appraised value. The rate is 5% and the usual time allowed is 30 years.

Example. A farmer buys a farm for $10,000. He has $6000 to pay down and secures a Federal farm loan for the balance to be amortized in 30 years at 5%.

Using equation (3), Art. 71 we find the annual payment to be $260.206 or $260.21.

The following table shows the progress of this loan for the first five years:

Year	Principal at Beginning of Year	Interest at 5%	Principal Repaid
1	4000.00	200 00	60 21
2	3939 79	196.99	63 22
3	3876 57	193.83	66.38
4	3810 19	190.51	69 70
5	3740.49	187.02	73.19

76. Interest and annuity tables. In the note of Art. 69 we referred to certain interest and annuity tables. Tables giving the values of $(1+i)^n$ $(1+i)^{-n}$, $\dfrac{(1+i)^n-1}{i}$, $\dfrac{1-(1+i)^{-n}}{i}$, and other interest functions for all integral values of n up to 200 and for different values of i have been computed accurately to seven decimal places. Time and space do not permit of the inclusion of such complete tables in this text. However, it seems advisable to spend a little time here in pointing out the use of such tables and their value as time saving devices. For this reason brief tables for $(1 + i)^n$, $(1 + i)^{-n}$, $\frac{(1 + i)^n - 1}{i}$ and $\frac{1 - (1 + i)^{-n}}{i}$ have been included.

In solving Ex. (1), Art. 68, we would have,

$$K = 200 s_{\overline{15}|} \text{ at } 5\%.$$

Here, $n = 15$, $i = .05$ and the tabular value of $s_{\overline{15}|}$ at 5% is 21.5785636.

Hence, $K = 200 \times 21.5785636 = \$4315.71$.

It would be interesting to discuss the methods used in constructing such tables but time and space do not permit of this discussion.

The student may now solve by tables, exercises 1, 2, 6, 7, Art. 66; exercises 1, 3, 4, 5, 6, 7, Art. 68; exercises 1, 2, 5, 6, 7, Art. 72.

Exercises and Problems

1. Find the annual payment that will be necessary to amortize in 10 years a debt of $2000, bearing interest at 8%. Construct a schedule.

2. The Federal Farm Loan Bank loaned a farmer $5000 at 5% interest, convertible semi-annually. The agreement was that the farmer should repay principal and interest in equal semi-annual installments covering a period of 15 years. Find the amount of each semi-annual payment.

3. At the age of 25 a young man resolves that, when he is 60 years of age, he will have $40,000 saved. If he invests his savings semi-annually at 6% interest, convertible semi-annually, what amount must he save semi-annually? If at age 60 he desired to have it paid back to him as an annual annuity payable at the end of each year, what would be his annual income over a period of 25 years if money at that time were worth 5% interest?

4. The beneficiary of a policy of insurance is offered a cash payment of $20,000 or an annuity of $1500 for 20 years certain, the first payment to be made one year hence. Allowing interest at 4% per annum, which is the better option, and how much better per annum?

5. A house is purchased for $15,000 and it is arranged that $5000 cash be paid, and the balance in 10 equal annual installments, including interest at 6%. Find the annual payment and construct a schedule.

6. Complete the farm loan schedule in Art. 75.

7. What would have been the equal payment if made semi-annually with interest at 5% semi-annually?

8. A tractor costs $1200. It is estimated that with proper care it will have a life of 8 years with a scrap value of $50 at the end of this time. Construct a depreciation schedule on a 4% interest basis.

CHAPTER XI

TRIGONOMETRIC FUNCTIONS

77. Meaning of trigonometry. The word trigonometry comes from two Greek words meaning *triangle* and *measurement.* This would suggest that the subject deals with the solution of the triangle. This is one of the important applications of trigonometry, but the subject is much broader than this for it is the basis of many important topics.

The development of trigonometry depends entirely upon six fundamental definitions which are called the trigonometric functions. These are sine, cosine, tangent, cotangent, secant and cosecant and will be defined in Art. 78.

Any triangle is composed of six parts, three sides and three angles. If any three parts are given, provided at least one of them is a side, geometry enables us to construct the triangle, and trigonometry enables us to compute the unknown parts from the numerical values of the known parts.

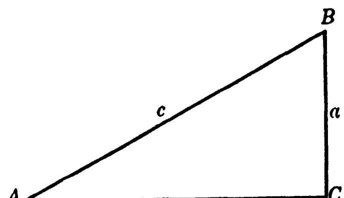

FIG. 27.

78. Trigonometric definitions. Consider the right-angled triangle ABC (Fig. 27). A and B are acute angles, C is the right angle, and a, b, c, are the sides opposite the respective angles. The six different ratios among the three sides are $\frac{a}{c}$, $\frac{b}{c}$, $\frac{a}{b}$, $\frac{b}{a}$, $\frac{c}{b}$, and $\frac{c}{a}$, and these are defined as the six *trigonometric functions of the angle A.* Thus we have,

$$\left.\begin{aligned}
\frac{a}{c} &= \frac{\text{side opposite } A}{\text{hypotenuse}} = \text{sine of } A, \text{ written } \sin A.\\
\frac{b}{c} &= \frac{\text{side adjacent } A}{\text{hypotenuse}} = \text{cosine of } A, \text{ written } \cos A.\\
\frac{a}{b} &= \frac{\text{side opposite } A}{\text{side adjacent } A} = \text{tangent of } A, \text{ written } \tan A.\\
\frac{b}{a} &= \frac{\text{side adjacent } A}{\text{side opposite } A} = \text{cotangent of } A, \text{ written } \cot A.\\
\frac{c}{b} &= \frac{\text{hypotenuse}}{\text{side adjacent } A} = \text{secant of } A, \text{ written } \sec A.\\
\frac{c}{a} &= \frac{\text{hypotenuse}}{\text{side opposite } A} = \text{cosecant of } A, \text{ written } \csc A.
\end{aligned}\right\} \quad (1)$$

Since the trigonometric functions of the angle A are ratios of the sides of a right triangle, it is evident that they are constant for any fixed angle and do not change value for different lengths of the sides of the triangle. (This follows from the definition of similar triangles.)

Applying the definitions to angle B, we may write

$$\left.\begin{aligned}
\sin B &= \frac{b}{c} = \cos A,\\
\cos B &= \frac{a}{c} = \sin A,\\
\tan B &= \frac{b}{a} = \cot A,\\
\cot B &= \frac{a}{b} = \tan A,\\
\sec B &= \frac{c}{a} = \csc A,\\
\csc B &= \frac{c}{b} = \sec A.
\end{aligned}\right\} \quad (2)$$

79. Co-functions and complementary angles. The cosine, cotangent and cosecant of an angle are co-functions of the sine, tangent and secant, respectively. Since in Fig. 27, A and B are complementary angles, $A + B = 90°$, it follows from (2) *that any function of an angle equals the co-function of the complement of that angle.* For example,

$$\sin 25° = \cos 65°, \quad \tan 29° = \cot 61°.$$

Exercises

Fill the blanks in the following with the proper co-function:

1. $\sin 75° = ?$

2. $\tan 18° 20' = ?$

3. $\cot 75° 18' = ?$

4. $\sec 19° 37' = ?$

5. $\csc 47° 29' = ?$

6. $\sec (90° - A) = ?$

7. $\tan 38° 15' = ?$

8. $\cos 72° 18' = ?$

9. Construct an acute angle A such that $\tan A = \frac{3}{4}$ and write the other trigonometric functions of the angle.

Solution. From the definition of the tangent, we know that A is an angle of a triangle having 3 for opposite side and 4 for adjacent side. The hypotenuse then is 5 (8, Art. 29). The functions are,

$$\sin A = \tfrac{3}{5}, \quad \cot A = \tfrac{4}{3},$$
$$\cos A = \tfrac{4}{5}, \quad \sec A = \tfrac{5}{4},$$
$$\tan A = \tfrac{3}{4}, \quad \csc A = \tfrac{5}{3}.$$

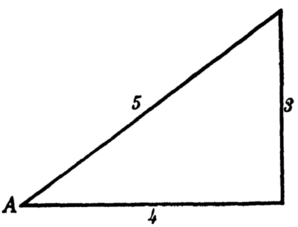

FIG. 28.

Construct the angle A in the following and write the other functions:

10. $\sin A = \frac{8}{17}$.

11. $\cos A = \frac{2}{3}$.

12. $\cot A = \frac{5}{4}$.

13. $\sec A = 3$.

14. $\csc A = 2$.

15. $\tan A = \frac{4}{3}$.

If in Fig. 27,

16. $\sin A = \frac{1}{5}$, $c = 15$, find a and b.

17. $\tan A = \frac{4}{3}$, $b = 24$, find a and c.

18. $\cos A = 0.325$, $b = 10$, find c.

80. Relations among the functions. From Fig. 27 we have,

$$a^2 + b^2 = c^2 \text{ (8, Art. 29)} \tag{3}$$

$$\frac{a^2}{c^2} + \frac{b^2}{c^2} = 1 \text{ (dividing (3) by } c^2) \tag{4}$$

But, $$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}.$$

Hence, $$\sin^2 A^* + \cos^2 A = 1. \tag{A}$$

$$\frac{a^2}{b^2} + 1 = \frac{c^2}{b^2} \text{ (dividing (3) by } b^2) \tag{5}$$

But, $$\tan A = \frac{a}{b} \quad \text{and} \quad \sec A = \frac{c}{b}.$$

Hence, $$\tan^2 A + 1 = \sec^2 A, \tag{B}$$

It is left for the student to show that,

$$\cot^2 A + 1 = \csc^2 A. \tag{C}$$

$$\tan A = \frac{1}{\cot A} = \frac{\sin A}{\cos A}. \tag{D}$$

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}. \tag{E}$$

$$\sec A = \frac{1}{\cos A}. \tag{F}$$

$$\csc A = \frac{1}{\sin A}. \tag{G}$$

The above relations are important and should be learned. They are known as *fundamental identities.*

* $\sin^2 A$, means $(\sin A)^2$.

Exercises

Making use of the fundamental identities verify the following identities:

1. $\dfrac{\tan A - 1}{\tan A + 1} = \dfrac{1 - \cot A}{1 + \cot A}.$

Verification: An identity may be verified by reducing the left-hand member to the form of the right, the right-hand member to the form of the left or both members to a common form. Thus,

$$\frac{\tan A - 1}{\tan A + 1} = \frac{\dfrac{1}{\cot A} - 1}{\dfrac{1}{\cot A} + 1}, \quad \text{by } (D)$$

$$= \frac{\dfrac{1 - \cot A}{\cot A}}{\dfrac{1 + \cot A}{\cot A}}$$

$$= \frac{1 - \cot A}{1 + \cot A}.$$

an $A + \cot A = \sec A \csc A.$

3. $\tan A \cos A = \sin A.$

4. $\dfrac{\sin A}{\csc A} + \dfrac{\cos A}{\sec A} = 1.$

5. $\cot A + \dfrac{\sin A}{1 + \cos A} = \csc A.$

6. $\dfrac{1 + \cot^2 A}{1 + \tan^2 A} = \cot^2 A.$

7. $\sin A \sec A \cot A = 1.$

8. $\sec A - \cos A = \sin A \tan A.$

81. Functions of 30°, 45°, 60°. From Fig. 29, we have,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\cos 45° = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2},$$

$$\tan 45° = 1, \quad \csc 45° = \frac{\sqrt{2}}{1} = \sqrt{2},$$

$$\cot 45° = 1, \quad \sec 45° = \frac{\sqrt{2}}{1} = \sqrt{2}.$$

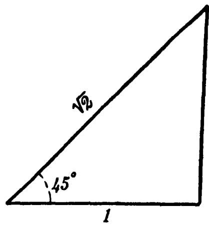

Fig. 29.

From Fig. 30, we have,

$$\sin 30° = \frac{1}{2} = \cos 60°,$$

$$\cos 30° = \frac{\sqrt{3}}{2} = \sin 60°,$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{1}{3}\sqrt{3} = \cot 60°,$$

$$\cot 30° = \frac{\sqrt{3}}{1} = \tan 60°,$$

$$\sec 30° = \frac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3} = \csc 60°,$$

$$\csc 30° = 2 = \sec 60°.$$

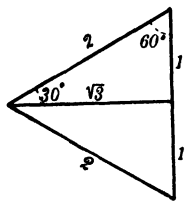

Fig. 30.

Exercises

Making use of the results of Art. 81, find the numerical values of the following:

1. $7 \cos 60° - 2 \sin 30° + 3 \cot 45°$. Ans. $5\frac{1}{2}$.

2. $6 \sin 60° (\sin 30° \tan 60° - \cot 60°)$. Ans. $\frac{3}{2}$.

3. $\left(\dfrac{\sin^2 60° - \cos^2 60°}{\tan^2 30°}\right)\left(\dfrac{\cot^2 45° + \tan^2 45°}{\cot^2 30°}\right)$. Ans. 1.

4. $\tan 45° \cot 30° + \sec 30° \cos 45°$. . Ans $\sqrt{3} + \frac{1}{3}\sqrt{6}$.

82. Line values of the functions. Fig. 31 is a circle having unity for its radius. DB and EC are perpendicular to AC and FG is perpendicular to AF. Then we may write,

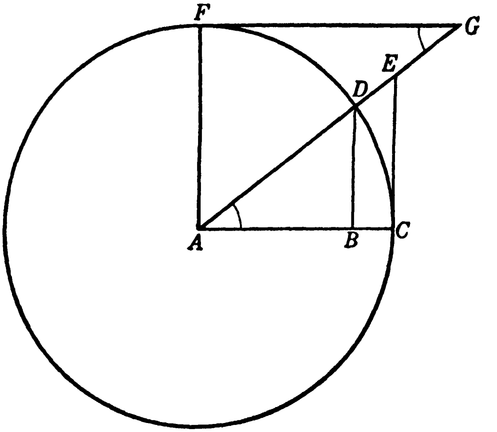

FIG. 31.

$$\sin A = \frac{BD}{AD} = BD.$$

since, AD is the unit.

$$\cos A = \frac{AB}{AD} = AB.$$

$$\tan A = \frac{CE}{AC} = CE.$$

$$\sec A = \frac{AE}{AC} = AE.$$

$$\cot A = \cot G = \frac{FG}{AF} = FG.$$

$$\csc A = \csc G = \frac{AG}{AF} = AG.$$

83. Variations of the functions. As A increases from 0° to 90°, it is easily seen from Fig. 31 that

$$\begin{aligned}
\sin A &\text{ varies from } 0 \text{ to } 1,\\
\cos A &\text{ varies from } 1 \text{ to } 0,\\
\tan A &\text{ varies from } 0 \text{ to } \infty,\\
\cot A &\text{ varies from } \infty \text{ to } 0,\\
\sec A &\text{ varies from } 1 \text{ to } \infty,\\
\csc A &\text{ varies from } \infty^{*} \text{ to } 1.
\end{aligned}$$

84. Natural trigonometric functions and logarithms of the trigonometric functions. In Table III in the back of this text, the values of the sine, cosine, tangent and cotangent are given correct to five decimal places, and in Table II, the logarithms of these functions are given. The method of using these tables differs very little from that employed in the use of Table I. A few exercises will illustrate the process.

Exercises

1. Find the value of sin 14° 35′. This value as found in the table is 0.25179.

2. Find the value of tan 35° 47′. This value is not given in the tables, but we find the values of tan 35° 45′ and tan 35° 50′ to be 0.71990 and 0.72211, respectively. The difference between these two values is 0.00221. Since 35° 47′ is two-fifths of the way from 35° 45′ to 35° 50′, we add to 0.71990

$$\tfrac{2}{5}\cdot 0.00211 = 0.00084.$$

Hence,
$$\tan 35^\circ\, 47' = 0.72074.$$

* ∞ is the symbol for infinity. It is evident that as A increases CE increases and when A becomes 90°, CE becomes larger than any finite value. We say then that $\tan 90^\circ = \infty$.

3. Find the value of cot 66° 38′. **When the angle is greater than 45° we must read up the page, reading the function *at the bottom of* the page and the angle on the right.** We find the values of cot 66° 35′ and cot 66° 40′ to be 0.43308 and 0.43136, respectively. The difference between these two values is 0.00172. Since 66° 38′ is three-fifths of the way from 66° 35′ to 66° 40′, we subtract from 0.43308

$$\tfrac{3}{5}\cdot 0.00172 = 0.00103.$$

Hence, $$\cot 66^\circ\, 38' = 0.43205.$$

4. Find the angle whose tangent is 0.41856. The angle is not found in these tables, but it lies between the angles 22° 40′ and 22° 45′, the values of whose tangents are 0.41763 and 0.41933, respectively. Now, 0.41856 is $\tfrac{93}{170}$ of the way from 0.41763 to 0.41933. Thus the angle, whose tangent is 0.41856, is

$$22^\circ\, 40' + \tfrac{93}{170}\cdot 5' = 22^\circ\, 43'.$$

Hence, $$\tan 22^\circ\, 43' = 0.41856.$$

5. Find log sin 43° 29′ 45″. We find log sin 43° 29′ and log sin 43° 30′ to be 9.83768 − 10 and 9.83781 − 10, respectively. The difference between these two values is 0.00013. Since 43° 29′ 45″ is three fourths of the way from 43° 29′ to 43° 30′, we add to 9.83768 − 10

$$\tfrac{3}{4}\cdot 0.00013 = 0.00010.$$

Hence, $$\log \sin 43^\circ\, 29'\, 45'' = 9.83778 - 10.$$

6. Find the angle the logarithm of whose cosine is 9.90504 − 10. The angle lies between 36° 31′ and 36° 32′, the logarithms of whose cosines are 9.90509 − 10 and 9.90499 − 10, respectively. Now, 9.90504 − 10 is $\tfrac{5}{10}$ of the way from 9.90509 − 10 to 9.90499 − 10.

Thus the angle, the logarithm of whose cosine is 9.90504 − 10, is

$$36^\circ\, 31' + \tfrac{5}{10}\cdot 60'' = 36^\circ\, 31'\, 30''.$$

Hence, $$\log \cos 36^\circ\, 31'\, 30'' = 9.90504 - 10.$$

7. Find the following:

(*a*) tan 38° 27′, (*b*) sin 75° 18′, (*c*) cot 5° 29′.

Ans. (*a*) 0.79421, (*b*) 0.96727, (*c*) 10.417.

8. Find the angle A when:

(*a*) $\sin A = 0.37820$, (*b*) $\cot A = 2.3424$.

Ans. (*a*) $A = 22° \, 13'$, (*b*) $A = 23° \, 7'$.

9. Find the following:

(*a*) $\log \cos 41° \, 28'$, (*b*) $\log \tan 76° \, 18' \, 40''$.

Ans. (*a*) $9.87468 - 10$, (*b*) 0.61333.

10. Find the angle A when:

(*a*) $\log \sin A = 9.32860 - 10$, (*b*) $\log \cot A = 9.36200 - 10$.

Ans. (*a*) $A = 12° \, 18' \, 17''$, (*b*) $A = 77° \, 2' \, 22''$.

CHAPTER XII

SOLUTION OF THE RIGHT ANGLE TRIANGLE

85. Formulas for the solution of a right triangle. If any two parts of a right triangle (at least one side) are known the following formulas are employed to obtain the other parts:

$$a^2 + b^2 = c^2, \qquad (1)$$

$$A + B = 90°, \qquad (2)$$

$$\sin A = \frac{a}{c} = \cos B, \qquad (3)$$

$$\cos A = \frac{b}{c} = \sin B, \qquad (4)$$

$$\tan A = \frac{a}{b} = \cot B. \qquad (5)$$

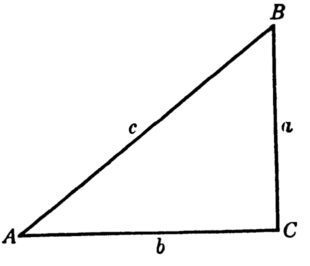

FIG. 32.

86. Applying the Formulas. Before attempting to solve any problem, a careful drawing should be made of the required triangle (enough parts will be given to completely construct it). The proper formulas should be chosen and an outline for the solution be made before making use of the tables. This will usually save much time. An exercise will illustrate the plan.

Illustrated Problem I. In a right triangle $A = 37° \, 50'$, $b = 15.6$. Find a and c.

Solution. Approximate construction.

(*a*) By trigonometric functions:

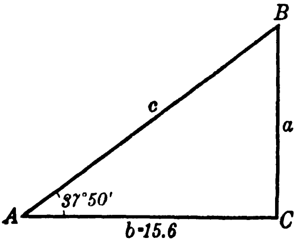

FIG. 33.

$$\tan A = \frac{a}{b}, \qquad \cos A = \frac{b}{c},$$

$$\therefore \quad a = b \tan A, \qquad c = \frac{b}{\cos A}.$$

$$a = 15.6 \times 0.77661, \; c = \frac{15.6}{0.78980},$$

$$B = 90 - A = 52° \, 10'.$$

$$a = 12.115, \qquad c = 19.75.$$

(*b*) By logarithms:

$$a = b \tan A, \qquad c = \frac{b}{\cos A},$$

$$B = 90 - A.$$

DATA AND RESULTS

A	37° 50′
b	15.6
B	52° 10′
a	12.115
c	19 751

LOGS

tan A	9.89020−10
b	1 19312
a	1.08332
b	11.19312−10
cos A	9.89752−10
c	1.29560

Illustrated Problem II. In a right triangle $a = 25.6$, $c = 31.3$. Find A, B and b.

Solution. Figure

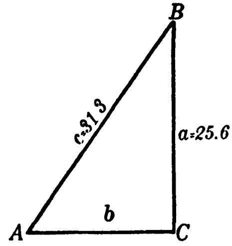

FIG. 34.

$$\sin A = \frac{a}{c}, \qquad \cos A = \frac{b}{c}.$$

$$\sin A = \frac{25.6}{31.3}, \qquad b = c \cos A.$$

$$\sin A = 0.81789, \qquad b = 31.3 \times 0.57548.$$

$$A = 54° \, 52', \qquad b = 18.012,$$

$$B = 90° - A = 35° \, 8'.$$

Exercises

Solve the first five exercises making use of the trigonometric functions. Use logarithms on the next three.

1. Given, $a = 17.5$, $A = 47° \, 10'$; Find b, c, and B.

2. Given, $a = 13.7$, $b = 35.3$; Find A, B, and c.

3. Given, $c = 340$, $B = 29° \, 30'$; Find A, b, and a.

4. Given, $b = 275$, $A = 52° \, 25'$; Find a, B, and c.

5. Given, $a = 37.5$, $b = 122$; Find A, B, and c.

6. Given, $a = 25.62$, $A = 33° \, 20'$; Find B, b, and c.

7. Given, $c = 67.7$, $A = 23° \, 30'$; Find a, b, and B.

8. Given, $a = 32.56$, $c = 42.82$; Find A, B, and b.

9. In measuring the width of a river, a line AB is measured 500 feet along one bank. A perpendicular to AB at A is erected which locates a point C upon the opposite bank, and the angle ABC is found to be $38° \, 10'$. Find the width of the stream. Ans. 393 feet.

10. Find the height of a tree which casts a horizontal shadow of 75.5 feet when the sun's angle of elevation is $57° \, 50'$. Ans. 120 feet.

NOTE. The angle which the line of sight to an object makes with a horizontal line in the same vertical plane is called an *angle of elevation* when the object is above the eye of the observer and an *angle of depression* when the object is below the eye of the observer.

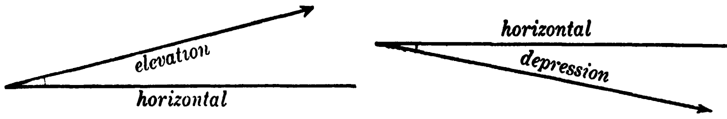

FIG. 35.

11. A building 30 feet wide and 50 feet long has a gable roof with a pitch (angle of elevation) of 35°. The rafters project 16 inches beyond the walls and the roof projects 16 inches beyond the ends. Find the length of the rafters and the number of squares of roofing required. (A square is 100 sq. ft.)

12. A line segment AB, has an angle of elevation of θ.* Find its horizontal and vertical projections.

Solution. The horizontal and vertical projections of AB are gotten by dropping the perpendiculars AC, BD, and AE, BF to the horizontal and vertical lines respectively. CD and EF are the required projections. We see then that the horizontal and vertical projections of a line segment are equal to the base and altitude of a right triangle of which the line segment is the hypotenuse. Hence,

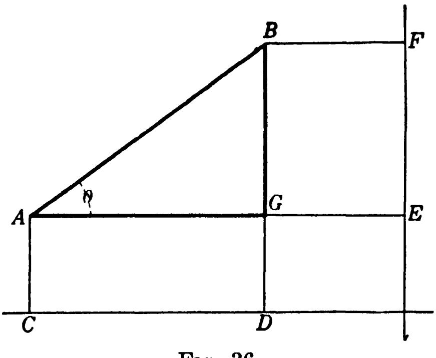

FIG. 36.

$$CD = AG = AB \cos \theta,$$

and

$$EF = GB = AB \sin \theta.$$

13. Find the projection of a line 560 feet long running N. 35° 20′ E. upon a line running East and West.

* θ is the Greek letter theta. Some of the other Greek letters that we shall use to denote angles are α, alpha; β, beta; γ, gamma; ϕ, phi.

14. A force of 250 lbs. making an angle of 36° 10′ with the horizontal acts upon a heavy body. Find the forces which tend to move the body horizontally and vertically. (These horizontal and vertical forces are called the horizontal and vertical components.)

15. Horizontal and vertical forces of 150 lbs. and 80 lbs., respectively, act upon a body. What is the resultant of these forces and what angle does the line of this resultant force make with the horizontal?

16. A flag pole 75 ft. high casts a shadow 122 ft. long. What is the angle of elevation of the sun at that time?

17. A telephone post is anchored to a stone buried in the ground by a stay wire which makes an angle of 63° with the horizontal. The tension in the wire is 500 lbs. Find the horizontal and vertical pull on the stone.

18. From a point A in a level plain the angle of elevation of the top of a hill is 38°. From a point B, 750 ft. closer to the hill the angle of elevation is 70°. How high is the top of the hill above the plain?

Ans. 818.9 ft.

CHAPTER XIII

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE SOLUTION OF THE OBLIQUE TRIANGLE

87. Trigonometric definitions. In Art. 78 the trigonometric functions for an acute angle were given. We shall now extend these definitions to include any angle. Coordinate axes (Art. 30, Fig. 11) will be employed in the location of the angle. Starting with the positive extremity of the X-axis and going in a counter-clockwise direction the coordinate axes divide the plane into four quadrants numbered I, II, III, IV (Fig. 37).

A positive angle is described when a radius OP is rotated about O, counter clockwise, from the initial position OX into a terminal position OP. Denoting this angle by θ, the coordinates of P by (x, y), and OP by r we have from Fig. 34 the following definitions:

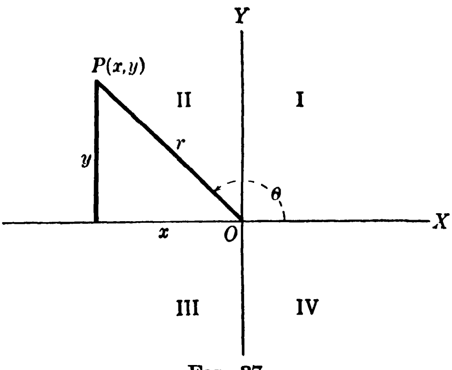

FIG. 37.

$$\sin\theta = \frac{y}{r}, \quad \csc\theta = \frac{r}{y},$$

$$\cos\theta = \frac{x}{r}, \quad \sec\theta = \frac{r}{x},$$

$$\tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}.$$

These definitions hold for an angle whose terminal side lies in any one of the four quadrants.

88. Laws of signs. The algebraic signs of the trigonometric functions for angles terminating in the respective quadrants are determined by the signs of x and y for that quadrant. The student may show that these signs are as indicated by the following diagram:

Quadrant	sin	cos	tan	cot	sec	csc
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

89. Functions of negative angles. A negative angle is described when a radius, OP is rotated about O, clockwise, from the initial position OX.

In Fig. 38 angle AOP_2 is equal to $-\theta$, angle AOP_1 is equal to θ.

$$r_2 = r_1, \quad x_2 = x_1, \quad y_2 = -y_1.$$

We may write,

$$\left.\begin{aligned}
\sin(-\theta) &= \frac{y_2}{r_2} = \frac{-y_1}{r_1} = -\sin\theta,\\
\cos(-\theta) &= \frac{x_2}{r_2} = \frac{x_1}{r_1} = \cos\theta,\\
\tan(-\theta) &= \frac{y_2}{x_2} = \frac{-y_1}{x_1} = -\tan\theta,\\
\cot(-\theta) &= \frac{x_2}{y_2} = \frac{x_1}{-y_1} = -\cot\theta.
\end{aligned}\right\} \qquad (1)$$

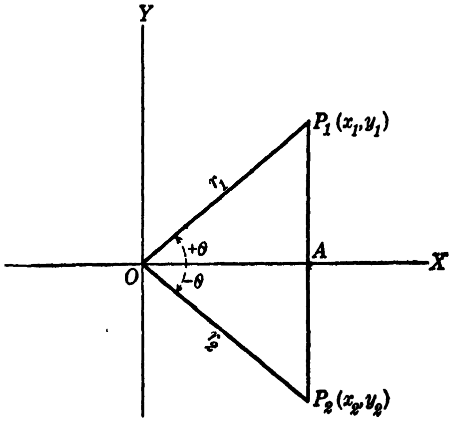

FIG. 38.

The above relations hold for angles whose terminal sides lie in any one of the four quadrants.

90. Functions of $180° - \theta$. **Supplementary angles.** In Fig. 39 triangle OA_2P_2 equals triangle OA_1P_1, and $x_2 = -x_1$, $y_2 = y_1$, $r_2 = r_1$. Hence,

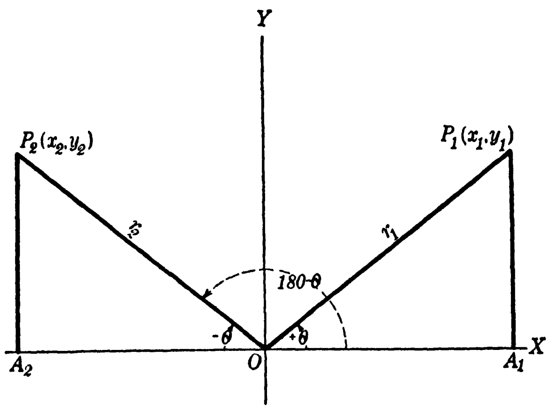

FIG. 39.

$$\left.\begin{aligned}
\sin(180-\theta) &= \frac{y_2}{r_2} = \frac{y_1}{r_1} = \sin\theta,\\
\cos(180-\theta) &= \frac{x_2}{r_2} = \frac{-x_1}{r_1} = -\cos\theta,\\
\tan(180-\theta) &= \frac{y_2}{x_2} = \frac{y_1}{-x_1} = -\tan\theta,\\
\cot(180-\theta) &= \frac{x_2}{y_2} = \frac{-x_1}{y_1} = -\cot\theta.
\end{aligned}\right\} \quad (2)$$

The student may show that the above relations hold when θ is an angle of the second quadrant.

Exercises

1. Show that the fundamental identities (Art. 80) hold for angles in any quadrant.

2. Writing $180° + \theta$ as $(180 - (-\theta))$ and making use of relations (2), Art. 90, and (1), Art. 89 show that

$$\left.\begin{aligned}
\sin(180+\theta) &= -\sin\theta,\\
\cos(180+\theta) &= -\cos\theta,\\
\tan(180+\theta) &= \tan\theta,\\
\cot(180+\theta) &= \cot\theta.
\end{aligned}\right\} \quad (3)$$

3. Make the proper drawings and show that the functions of $90° - \theta$ are equal to the co-functions of θ.

4. Write $90° + \theta$ as $(90 - (-\theta))$ and making use of (1), Art. 89 show that

$$\left.\begin{aligned}
\sin(90+\theta) &= \cos\theta,\\
\cos(90+\theta) &= -\sin\theta,\\
\tan(90+\theta) &= -\cot\theta,\\
\cot(90+\theta) &= -\tan\theta.
\end{aligned}\right\} \quad (4)$$

5. Fill the blanks with the proper function of the supplement of each angle:

(*a*) $\sin 110° = \sin 70°$
(*b*) $\tan 99° 18' =$
(*c*) $\tan (90° + \theta) =$
(*d*) $\sin 175° =$
(*e*) $\cot 109° 15' =$
(*f*) $\cos 135° =$
(*g*) $\cos (90° - \alpha) =$
(*h*) $\cot 120° =$

6. Draw figures and show:

(*a*) $\sin 70° = \cos 20° = \sin 110°$,
(*b*) $\sin 130° = \sin 50°$,
(*c*) $\sin 220° = - \sin 40°$,
(*d*) $\cos 190° = - \cos 10°$.

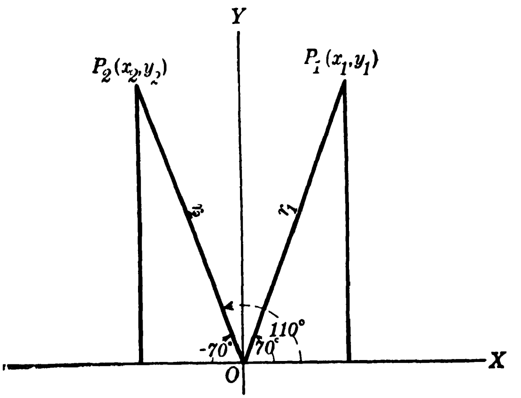

FIG. 40.

Solution (*a*). In Fig. 40

$$r_1 = r_2,\ x_2 = - x_1,\ y_2 = y_1.$$

Hence, $$\sin 70° = \frac{y_1}{r_1} = \frac{y_2}{r_2} = \sin 110°.$$

Also, $$\sin 70° = \cos 20°, \text{ (Art. 79)}$$

$$\sin 70° = \cos 20° = \sin 110°.$$

91. Functions of the sum of two angles. In Fig. 41, SR is perpendicular to OR and SN is perpendicular to OM, and $\Delta\, STR$ is similar to $\Delta\, OMR$.

We may write,

$$\sin(\theta + \phi) = \frac{NS}{OS} = \frac{MR + TS}{OS}.$$

But $$MR = OR \sin\theta = OS \cos\phi \sin\theta,$$

and $$TS = SR \cos\theta = OS \sin\phi \cos\theta.$$

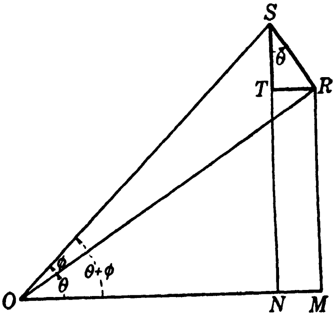

FIG. 41.

Hence, $\sin(\theta + \phi) = \sin\theta \cos\phi + \sin\phi \cos\theta.$ (5)

Also, $$\cos(\theta + \phi) = \frac{ON}{OS} = \frac{OM - TR}{OS}.$$

But $$OM = OR \cos\theta = OS \cos\phi \cos\theta,$$

and $$TR = SR \sin\theta = OS \sin\phi \sin\theta.$$

Hence, $\cos(\theta + \phi) = \cos\theta \cos\phi - \sin\theta \sin\phi.$ (6)

From (5) and (6) we have,

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\theta \cos\phi + \sin\phi \cos\theta}{\cos\theta \cos\phi - \sin\phi \sin\theta}.$$

If we divide both numerator and denominator of the above expression by $\cos\theta\cos\phi$, we get,

$$\tan(\theta+\phi) = \frac{\dfrac{\sin\theta}{\cos\theta}+\dfrac{\sin\phi}{\cos\phi}}{1-\dfrac{\sin\theta}{\cos\theta}\cdot\dfrac{\sin\phi}{\cos\phi}}.$$

Hence,
$$\tan(\theta+\phi) = \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}. \tag{7}$$

In Fig. 41, θ and ϕ are acute angles and their sum is also acute. However, relations (5), (6), and (7) hold for all angles of any magnitude, and we assume this without proof.

92. Functions of the difference of two angles. If we write $\sin(\theta-\phi)$ as $\sin(\theta+(-\phi))$ and substitute in (5), Art. 91, we get,

$$\sin(\theta-\phi) = \sin\theta\cos(-\phi)+\cos\theta\sin(-\phi).$$

But $\cos(-\phi)=\cos\phi$, $\sin(-\phi)=-\sin\phi$. ((1), Art. 89.)

Hence,

$$\sin(\theta-\phi) = \sin\theta\cos\phi-\cos\theta\sin\phi. \tag{8}$$

The student may show that,

$$\cos(\theta-\phi) = \cos\theta\cos\phi+\sin\theta\sin\phi, \tag{9}$$

and
$$\tan(\theta-\phi) = \frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}. \tag{10}$$

93. Functions of twice an angle. If we make $\phi=\theta$ and substitute in (5), Art. 91, we get,

$$\sin 2\theta = 2\sin\theta\cos\theta. \tag{11}$$

When $\phi = \theta$ (6) becomes,

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta, \tag{12}$$

$$= 1 - 2 \sin^2 \theta \qquad ((A), \text{ Art. } 80.)$$

$$= 2 \cos^2 \theta - 1. \qquad ((A), \text{ Art. } 80.)$$

The student may show that,

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \tag{13}$$

94. Half-angle formulas. In (11), (12), (13), Art. 93, let $\theta = \frac{x}{2}$, and we get,

$$\sin x = 2 \sin \frac{x}{2} \cos \frac{x}{2}. \tag{14}$$

$$\cos x = \cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \tag{15}$$

$$= 1 - 2 \sin^2 \frac{x}{2}$$

$$= 2 \cos^2 \frac{x}{2} - 1.$$

$$\tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}. \tag{16}$$

Solving the second, and third forms of (15) respectively for $\sin \frac{x}{2}$ and $\cos \frac{x}{2}$, we get,

$$\sin \frac{x}{2} = \pm \sqrt{\frac{1 - \cos x}{2}}, \tag{17}$$

and
$$\cos\frac{x}{2} = \pm\sqrt{\frac{1+\cos x}{2}}. \tag{18}$$

95. Sum and difference formulas. If we add (5) and (8), subtract (8) from (5), add (6) and (9), and subtract (9) from (6), respectively, we get,

$$\left.\begin{aligned} \sin(\theta+\phi) + \sin(\theta-\phi) &= 2\sin\theta\cos\phi,\\ \sin(\theta+\phi) - \sin(\theta-\phi) &= 2\cos\theta\sin\phi,\\ \cos(\theta+\phi) + \cos(\theta-\phi) &= 2\cos\theta\cos\phi,\\ \cos(\theta+\phi) - \cos(\theta-\phi) &= -2\sin\theta\sin\phi. \end{aligned}\right\} \tag{19}$$

Let $\theta + \phi = x$, $\theta - \phi = y$, then,

$$\theta = \frac{x+y}{2}, \qquad \phi = \frac{x-y}{2}.$$

Making these substitutions in (19), we have,

$$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}. \tag{20}$$

$$\sin x - \sin y = 2\cos\frac{x+y}{2}\sin\frac{x-y}{2}. \tag{21}$$

$$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}. \tag{22}$$

$$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}. \tag{23}$$

Exercises

1. Show that $\cos 75° = \frac{1}{4}(\sqrt{6} - \sqrt{2})$.

Solution.

$$\begin{aligned}\cos 75° &= \cos(45° + 30°)\\ &= \cos 45° \cos 30° - \sin 45° \sin 30°, \quad ((6), \text{Art. } 91)\\ &= \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}\sqrt{3} - \tfrac{1}{2}\sqrt{2}\cdot\tfrac{1}{2}, \quad (\text{Art. } 81)\\ &= \tfrac{1}{4}\sqrt{6} - \tfrac{1}{4}\sqrt{2}.\end{aligned}$$

2. Show that $\cos 15° = \sin 75° = \frac{1}{4}(\sqrt{6} + \sqrt{2})$.

3. Show that $\tan 15° = \tan(45° - 30°) = 2 - \sqrt{3}$.

4. Draw a figure and show that (5) and (6) hold when $(\theta + \phi)$ lies in the second quadrant.

5. If $\sin x = \frac{1}{3}$, find $\sin 2x$, also $\cos 2x$, and $\tan 2x$.

Solution. By the method of exercise 9, Art. 79, we find that if $\sin x = \frac{1}{3}$, $\cos x = \frac{2}{3}\sqrt{2}$, and $\tan x = \frac{1}{4}\sqrt{2}$.
Then by (11), Art. 93, we have,

$$\sin 2x = 2\cdot\tfrac{1}{3}\cdot\tfrac{2}{3}\sqrt{2} = \tfrac{4}{9}\sqrt{2}. \qquad \cos 2x = ? \qquad \tan 2x = ?$$

6. Find $\tan 43° 36'$, if $\tan 21° 48' = .4$.

7. Find $\sin 15°$, knowing $\cos 30° = \dfrac{\sqrt{3}}{2}$.

8. Show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.

Hint: Write $\sin 3\theta$ as $\sin(2\theta + \theta)$, and expand by (5), Art. 91, and apply (11), (12), Art. 93, and (A), Art. 80.)

9. Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

10. Show that $\sin 40° + \sin 10° = 2 \sin 25° \cos 15°$.

11. Show that $\sin 70° - \sin 40° = 2 \cos 55° \sin 15°$.

12. Show that $\dfrac{\sin 5x + \sin x}{\cos 5x + \cos x} = \tan 3x$.

13. Show that $\dfrac{\sin x + \sin y}{\sin x - \sin y} = \dfrac{\tan \frac{x+y}{2}}{\tan \frac{x-y}{2}}.$

96. Theorem of sines. The lettering in figures 42 and 43 is similar to that in figure 27. Draw a perpendicular p from

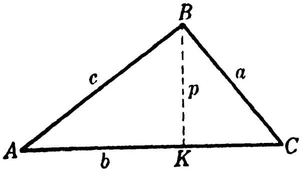

FIG. 42.

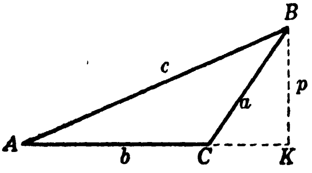

FIG. 43.

vertex B to opposite side b (opposite side produced in Fig. 43). Then, from the right triangles AKB and CKB, we get,

$$\sin A = \frac{p}{c}, \quad \sin C = \frac{p}{a}.$$

Dividing $\sin A$ by $\sin C$, we have

$$(a) \qquad \frac{\sin A}{\sin C} = \frac{a}{c}.$$

This is true for both Figures 42 and 43, for in Fig. 43, we have

$$\sin (180 - C) = \sin C.$$

Drawing a perpendicular from vertex C to side c, we would get, similarly,

$$(b) \qquad \frac{\sin A}{\sin B} = \frac{a}{b}.$$

Also, drawing a perpendicular from vertex A to side a, we would have,

$$(c) \qquad \frac{\sin C}{\sin B} = \frac{c}{b}.$$

The above results may be written in the form,

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}. \tag{24}$$

Theorem. *In any triangle the sines of the angles are proportional to the opposite sides.*

By observing (a), (b) and (c) it is easily seen that the Theorem of Sines may be used to solve a triangle when two sides and an angle opposite one of the sides are given, or when the angles and a side are given.

Example I. Given $a = 5.63$, $b = 42.3$ and $A = 25°\ 10'$; find B, C, and c.

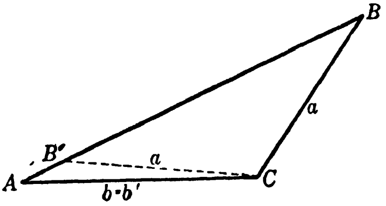

FIG. 44.

Solution. Construction, Formulas,

$$\sin B = \frac{b \sin A}{a},$$

$$C = 180 - (A + B),$$

$$c = \frac{a \sin C}{\sin A}.$$

Data and Results

a	35.6
b	42.3
A	25° 10′
B	30° 21′
C	124° 29′
c	69.005

Logs

b	1.62634
$\sin A$	9.62865−10
	11.25499−10
a	1.55145
$\sin B$	9.70354−10
a	1.55145
$\sin C$	9.91608−10
	11.46753−10
$\sin A$	9.62865−10
c	1.83888−10

This example admits of two solutions which is evident from the above construction, i.e., both triangles ABC and $AB'C$ are solutions.

Second solution ($AB'C$):

$$B' = 180° - B = 149° 39',$$
$$C = 180° - (A + B') = 5° 11',$$
$$c = \frac{a \sin C}{\sin A} = 7.567.$$

As a matter of fact, when two sides and an angle opposite one of the sides are given, the data may admit of two solutions, one solution or no solution, but these facts will come out in the solution and we need not generalize on them.

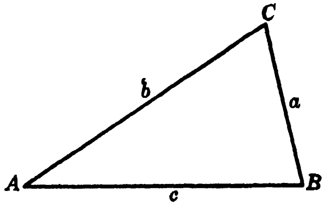

FIG. 45.

Example II. Given $a = 45.6$, $A = 35° 15'$, $B = 76° 10'$; find b, c, and C.

Solution. Construction, Formulas,

$$C = 180 - (A + B),$$

$$b = \frac{a \sin B}{\sin A},$$

$$c = \frac{a \sin C}{\sin A}.$$

DATA AND RESULTS

a	45.6
A	35° 15′
B	76° 10′
C	68° 35′
b	76.717
c	73.553

LOGS

a	1.65896
$\sin B$	9.98722−10
	11.64618−10
$\sin A$	9.76129−10
b	1.88489
$\sin C$	9 96893−10
a	1.65896
	11.62789−10
$\sin A$	9.76129−10
c	1.86660

Exercises

1. Make drawings to show that when two sides of a triangle and an angle opposite one of these sides are given there may be two solutions, one solution or no solution.

2. Given $a = 48.3$, $A = 48° 30'$, $B = 75° 15'$; find b, c.

3. Given $a = 149.5$, $b = 115.6$, $A = 71° 20'$; find B, C, c.

4. Given $a = 23.1$, $c = 16.5$, $C = 33° 10'$; find A, B, b.

5. Given $b = 125.6$, $B = 39° 45'$, $C = 105° 15'$; find A, a, c.

97. Theorem of cosines. In triangle ABC drop a perpendicular h from B to side b. From the right triangle BHC, we get,

$$(a)\quad a^2 = h^2 + \overline{HC}^2$$
$$= h^2 + (b - AH)^2$$
$$= h^2 + \overline{AH}^2 + b^2 - 2b\overline{AH}.$$

But from the right triangle ABH, we have,

$$(b)\quad h^2 + \overline{AH}^2 = c^2, \quad \text{and} \quad AH = c \cos A.$$

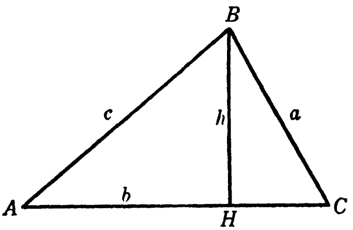

FIG. 46.

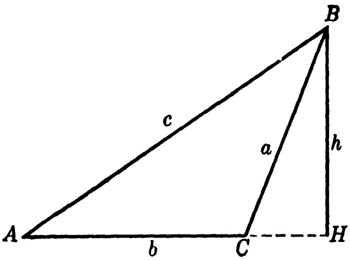

FIG. 47.

Substituting (b) in (a) we get,

$$a^2 = b^2 + c^2 - 2\,bc \cos A. \tag{25}$$

It may also be shown that,

$$b^2 = a^2 + c^2 - 2\,ac \cos B, \tag{26}$$

and

$$c^2 = a^2 + b^2 - 2\,ab \cos C. \tag{27}$$

Theorem. *In any triangle the square on any side is equal to the sum of the squares on the other two sides minus twice the product of the other two sides and the cosine of the included angle.*

It is evident that the Theorem of Cosines may be used to find the third side of a triangle when two sides and the included angle *are given. It may also be used to find the angles of a triangle when the three sides are given.*

Example I. Given $a = 37.5$, $b = 18.5$ and $C = 39° 45'$; find c.

Solution. From (27) we have,

$$c^2 = (37.5)^2 + (18.5)^2 - 2(37.5)(18.5)(0.7688)$$
$$= 681.79.$$
$$\therefore \quad c = 26.11.$$

Example II. Given $a = 42$, $b = 17$, $c = 53$; find the angles.

Solution. Solving (25) for $\cos A$, we get,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

and substituting for a, b, and c their values, we have,

$$\cos A = \frac{(17)^2 + (53)^2 - (42)^2}{2 \times 17 \times 53} = 0.7403.$$
$$\therefore \quad A = 42° 15'.$$

And by using (26) and (27) we could get angles B and C.

Exercises

1. In example II, find angles B and C.

2. Show that the Theorem of Pythagoras is a special case of the Cosine Theorem.

3. Making use of the Theorem of Sines, find angles A and B of Example I.

4. Given $b = 52.5$, $c = 43.4$, $A = 45° 20'$; find B, C, and a.

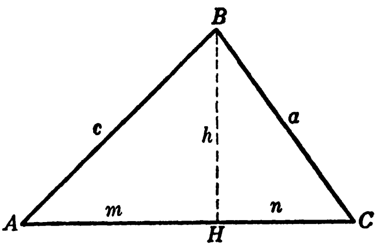

FIG. 48.

Solution. Construct the triangle and drop perpendicular h from B to side b. This gives us two right triangles, ABH and CBH. Let AH be represented by m and HC by n.

Formulas:

(1) $h = c \sin A$,

(2) $m = c \cos A$,

(3) $n = b - m$,

(4) $\tan C = \frac{h}{n}$,

(5) $B = 180 - (A + C)$,

(6) $a = \frac{c \sin A}{\sin C} = \frac{h}{\sin C}$.

Data and Results

b	52.5
c	43 4
A	45° 20′
h	30 866
m	30.509
n	21.991
C	54° 32′
B	80° 8′
a	37.898

Logs

c	1.63749
$\sin A$	9.85200−10
h	1.48949
$\cos A$	9 84694−10
c	1.63749
m	1.48443
h	1.48949
n	1.34225
$\tan C$	0.14724
h	11.48949−10
$\sin C$	9 91087−10
a	1.57862

Check: $\frac{a}{\sin A} = \frac{b}{\sin B}$; $a \sin B = b \sin A$.

$$\log a = 1.57862 \qquad \log b = 1.72016$$

$$\log \sin B = \underline{9.99353 - 10} \qquad \log \sin A = \underline{9.85200 - 10}$$

$$1.57215 \qquad 1.57216$$

5. Given $a = 296$, $c = 236$, $b = 75° 20'$; find A, C, and b. (Solve similar to exercise 4.)

6. Given $a = 385$, $b = 476$, $c = 225$; find angles A, B and C and check by the Sine Theorem.

98. Theorem of tangents.* From the Theorem of Sines, we have

$$(a) \qquad \frac{a}{b} = \frac{\sin A}{\sin B},$$

$$(b) \qquad \frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B},$$

adding 1 to both members of (a).

$$(c) \qquad \frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B},$$

subtracting 1 from both members of (a).

$$(d) \qquad \frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B},$$

dividing (b) by (c).

From (20) and (21), Art. 95, we have,

$$(e) \qquad \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\dfrac{A+B}{2}\cos\dfrac{A-B}{2}}{2\cos\dfrac{A+B}{2}\sin\dfrac{A-B}{2}},$$

$$= \tan\frac{A+B}{2}\cot\frac{A-B}{2},$$

$$= \frac{\tan\dfrac{A+B}{2}}{\tan\dfrac{A-B}{2}}.$$

* Art. 98 may be omitted from this course.

Combining (*d*) and (*e*), we get,

$$\frac{a+b}{a-b} = \frac{\tan \dfrac{A+B}{2}}{\tan \dfrac{A-B}{2}}. \tag{28}$$

We may show in a similar manner,

$$\frac{b+c}{b-c} = \frac{\tan \dfrac{B+C}{2}}{\tan \dfrac{B-C}{2}}, \tag{29}$$

and

$$\frac{a+c}{a-c} = \frac{\tan \dfrac{A+C}{2}}{\tan \dfrac{A-C}{2}}. \tag{30}$$

Theorem. *In any triangle the sum of two sides divided by their difference, is equal to the tangent of half the sum of the opposite angles divided by the tangent of half the difference of these angles.*

The Theorem of Tangents may be used to solve a triangle when two sides and the included angle are given. This will be illustrated by an example.

Example I. Given $a = 255$, $b = 182$, $C = 48° 20'$; find A, B and c.

Solution. Construction, Formulas,

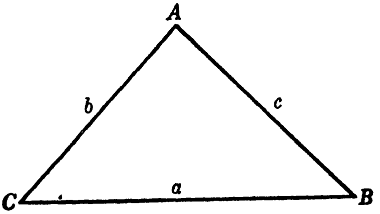

FIG. 49.

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B).$$

$$A + B = 180 - C.$$

$$c = \frac{a \sin C}{\sin A}.$$

DATA AND RESULTS		LOGS	
a	255	$a - b$	1.86332
b	182	$\tan \frac{1}{2}(A + B)$	0.34803
$a + b$	437		12.21135−10
$a - b$	73	$(a + b)$	2.64048
C	48° 20′	$\tan \frac{1}{2}(A - B)$	9.57087−10
$\frac{A + B}{2}$	65° 50′	a	2.40654
		$\sin C$	9.87334−10
$\frac{A - B}{2}$	20° 25′		12.27988−10
		$\sin A$	9.99907−10
A	86° 15′	c	2.28081
B	45° 25′		
c	190.9		

Check: $b \sin A = a \sin B$.

$$\log b = 2.26007 \qquad \log a = 2.40654$$

$$\log \sin A = \underline{9.99907 - 10} \qquad \log \sin B = \underline{9.85262 - 10}.$$

$$2.25914 \qquad 2.25916$$

99. Area of a triangle. We know that the *area of any triangle is equal to one half of any side multiplied by the altitude to that side* (Formula 3, Art. 29). Also, 4, Art. 29 gives us the area of a triangle when the three sides are given.

Formulas expressing the area of a triangle in terms of any three of its parts (not all three angles and no side) might be derived, but we prefer to have the student remember the above principle and work out each problem separately. A problem or two will illustrate the method of procedure.

Example I. Given $a = 25.6$, $b = 37.5$, $C = 42° 20'$; find the area of the triangle.

Solution. Construction,

Formulas,

(a) Area $= \frac{1}{2}bh,$

(b) $h = a \sin C,$

(c) Area $= \frac{1}{2}ab \sin C$

$= \frac{1}{2} \times 25.6 \times 37.5 \sin 42° 20'$

$= \frac{1}{2} \times 25.6 \times 37.5 \times 0.67344 = 323.25.$

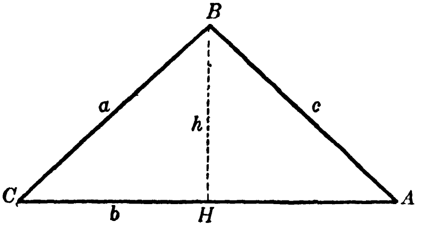

FIG. 50.

Example II. Given $a = 225$, $A = 45° 30'$, $B = 75° 10'$; find the area.

Solution. Construction, Formulas,

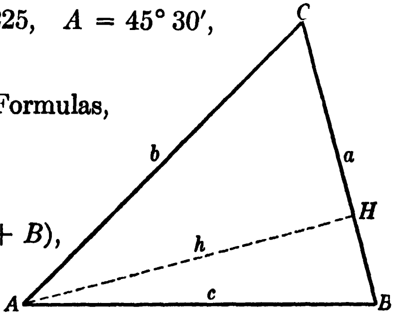

FIG. 51.

(a) Area $= \frac{1}{2}ah,$

(b) $C = 180 - (A + B),$

(c) $c = \dfrac{a \sin C}{\sin A},$

(d) $h = c \sin B = \dfrac{a \sin B \sin C}{\sin A},$

$$\text{Area} = \frac{1}{2}a^2 \frac{\sin B \sin C}{\sin A}$$

$$= \frac{1}{2} \frac{(225)^2 \sin 59° \, 20' \sin 75° \, 10'}{\sin 45° \, 30'}$$

$$= 29508.$$

Exercises

1. Given $a = 75$, $b = 38$, $A = 37°$; find the area of the triangle.

2. Given $a = 65$, $b = 75$, $c = 92$; find the area of the triangle.

3. Given $c = 492$, $a = 525$, $A = 76° \, 40'$; find the area of the triangle.

4. Given $A = 47° \, 20'$, $B = 75° \, 25'$, $c = 75.2$; find the area of the triangle.

100. Summary of methods of solving any triangle. We will divide the discussion up into four cases.

Case I. *Given two angles and a side.* The Sine Theorem will be applied in this case.

Case II. *Given two sides and an angle opposite one of the sides.* The Sine Theorem will be applied here.

Case III. *Given two sides and the included angle.* If only the third side is required, it may be obtained directly by using the Cosine Theorem. But if the other angles are also required, one of two methods may be used; we may apply the method of example 4, Art. 97, or we may use the Theorem of Tangents, Art. 98.

Case IV. *Given the three sides to find the angles.* The Cosine Theorem may be used in this case as illustrated in Example II, Art. 97.

Examples on Chapter XIII

1. Solve the following triangles for the unknown parts:

(1) $a = 372$, $b = 450$, $c = 525$; find the angles and the area.
(2) $a = 52$, $b = 75$, $C = 37° 10'$; find c, A, and B and area.
(3) $b = 62.8$, $a = 73.7$, $A = 35° 45'$; find c, B and C and the area.
(4) Given $A = 75° 25'$, $B = 37° 45'$, $c = 455$; find a, b, C and the area.

2. Given $b = 875$, $c = 458$, $A = 72° 20'$; find B, C and a, using the theorem of tangents.

3. Given a, b, A; write down the proper equations for obtaining the unknown parts, including the area.

4. In order to find the distance between two points, A and B, separated by a high hill, a point C was taken where both A and B could be seen. CA, CB and angle ACB were measured and found to be 2521 feet, 3623 feet and 70° 45′ respectively. Find the distance from A to B.

5. To determine the distance of a point A across a lake from a point B on the near shore, a line BC and the angles ABC and BCA were measured and found to be 2562 yd. 75°, and 62° 20′, respectively. Find the distance AB.

6. Two streets meet at an angle of 80° 10′. How much land is there in the triangular corner lot which fronts 425 feet on one street and 315 feet on the other?

7. From the top of a hill 650 feet high the angles of depression of the top and bottom of a tower are 52° and 65° respectively, what is the height of the tower?

8. Two forces of 200 lbs. and 175 lbs. act at an angle of 50° with each other. Find the resultant force and also the angle that the resultant makes with the 200 lb. force.

CHAPTER XIV

THE DERIVATIVE AND SOME APPLICATIONS

101. The meaning of a tangent to a curve. In Fig. 52 we have a curve C cut by a line l in the two points P and Q. Now assume that P is a fixed point and that Q moves along the curve towards P. As Q moves towards P the line l turns about P and approaches, in general, a limiting position (PT in the figure), and at the instant when Q coincides with P the line l coincides with PT. *The line PT is called the tangent to the curve C at the point P.*

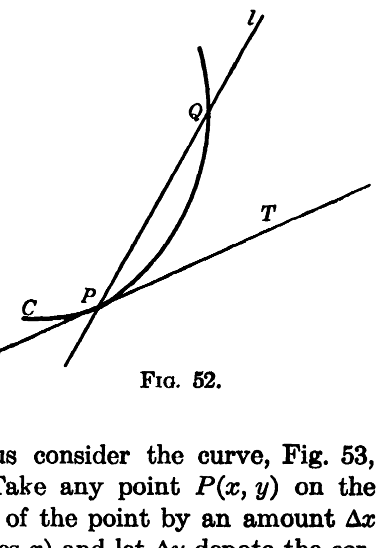

FIG. 52.

102. The derivative. Let us consider the curve, Fig. 53, whose equation is $y = f(x)$. Take any point $P(x, y)$ on the curve and increase the abscissa of the point by an amount Δx (read delta x, and not delta times x) and let Δy denote the corresponding increase of y. We notice that this gives us a second point $Q(x + \Delta x, y + \Delta y)$ on the curve. We note that y has changed by an amount Δy while x was changing by an amount Δx. The ratio $\dfrac{\Delta y}{\Delta x}$ is the average rate of change in y with respect to x within the interval Δx. We also observe that this ratio is the *slope of the chord PQ*. (See Art. 36.) If we now let Δx approach 0, the ratio $\frac{\Delta y}{\Delta x}$ generally approaches a fixed value *which is defined as the rate of change of y with respect to x at the point P*. It is also evident that as Δx approaches 0, the

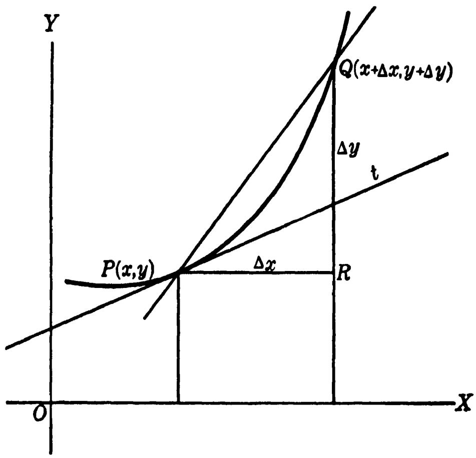

FIG. 53.

point Q approaches the point P, the chord PQ approaches the tangent t, and the ratio $\frac{\Delta y}{\Delta x}$ approaches as its value the slope of the tangent at the point P. *The limiting value of* $\frac{\Delta y}{\Delta x}$ *as* Δx *approaches* 0 *is defined as the derivative of y with respect to x at any point P(x, y).* The derivative is designated by the symbol $\frac{dy}{dx}$, and we write,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx}.$$

We shall now find $\frac{dy}{dx}$ for $y = x^2$. We have,

$$y = x^2. \tag{1}$$

$$y + \Delta y = (x + \Delta x)^2 = x^2 + 2x\Delta x + \Delta x^2. \tag{2}$$

Subtracting (1) from (2) we have,

$$\Delta y = 2x\Delta x + \Delta x^2. \tag{3}$$

Dividing (3) by Δx,

$$\frac{\Delta y}{\Delta x} = 2x + \Delta x,$$

and
$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (2x + \Delta x) = 2x. \tag{4}$$

Exercises

1. Find the slope of the curve $y = 3x^2 - 5x + 2$ at the point (2, 4).

Solution.

$$y = 3x^2 - 5x + 2. \tag{1}$$

$$y + \Delta y = 3(x + \Delta x)^2 - 5(x + \Delta x) + 2. \tag{2}$$

$$\Delta y = 6x\Delta x + 3\Delta x^2 - 5\Delta x. \tag{3}$$

$$\frac{\Delta y}{\Delta x} = 6x + 3\Delta x - 5. \tag{4}$$

$$\frac{dy}{dx} = 6x - 5. \tag{5}$$

The slope at any point (x, y) is $(6x - 5)$.

The slope at the point (2, 4) is obtained by substituting 2 for x in (5), which gives us 7. Hence the tangent to the curve, $y = 3x^2 - 5x + 2$, at the point (2, 4) has 7 for its slope.

2. Find the derivative of $y = \frac{1}{x}$.

Solution.

$$y = \frac{1}{x}. \tag{1}$$

$$y + \Delta y = \frac{1}{x + \Delta x}. \tag{2}$$

$$\Delta y = \frac{1}{x + \Delta x} - \frac{1}{x} = \frac{-\Delta x}{x(x + \Delta x)}. \tag{3}$$

$$\frac{\Delta y}{\Delta x} = \frac{-1}{x(x + \Delta x)}. \tag{4}$$

$$\frac{dy}{dx} = \frac{-1}{x^2}. \tag{5}$$

3. Find the derivative of $y = \sqrt{x}$.

Solution.

$$y = \sqrt{x}. \tag{1}$$

$$y + \Delta y = \sqrt{x + \Delta x}. \tag{2}$$

$$\Delta y = \sqrt{x + \Delta x} - \sqrt{x}. \tag{3}$$

$$= \frac{(\sqrt{x + \Delta x} - \sqrt{x})(\sqrt{x + \Delta x} + \sqrt{x})}{(\sqrt{x + \Delta x} + \sqrt{x})}$$

(See Ex. 21, page 78)

$$= \frac{(x + \Delta x) - x}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{\Delta x}{\sqrt{x + \Delta x} + \sqrt{x}}.$$

$$\frac{\Delta y}{\Delta x} = \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}. \tag{4}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}. \tag{5}$$

Find the slopes of the following curves at the points indicated.

4. $y = x^2 - 3x + 2$, at the point where $x = 3$. Trace the curve.

5. $y = 2x^3 + x^2 + x$, at $x = 2$.

6. $y = \dfrac{1}{x^2 + 1}$, at $x = 1$.

7. At what point does the curve $y = x^2 + 3x + 5$ have the slope 5? Ans. (1, 9).

8. At what point does the curve $y = x^2 - 4x + 10$ have the slope 0? Trace the curve and notice carefully its shape at the point where the slope is 0. (See Art. 48.) Ans. (2, 6).

9. If l is the length of the side of a square, the area A is given by $A = l^2$. If l is changing, find the rate at which A is changing when $l = 4$ ft.

Solution.

$$A = l^2. \tag{1}$$

$$A + \Delta A = (l + \Delta l)^2. \tag{2}$$

$$\Delta A = 2l\Delta l + \Delta l^2. \tag{3}$$

$$\frac{\Delta A}{\Delta l} = 2l + \Delta l. \tag{4}$$

$$\frac{dA}{dl} = 2l. \tag{5}$$

When $l = 4$, the rate of change is $\left.\dfrac{dA}{dl}\right]_{l=4} = 8$. That is to say, the rate of change of A with respect to l when $l = 4$, is 8 times the rate at which l is changing.

10. If the radius of a circle is changing, what is the rate of change of the area A when the radius is 3 feet? (See 5, Art. 29.)

103. Derivative of a constant. If $y = c$, then it does not matter what the values of x and Δx are; y will remain unchanged, and $\Delta y = 0$.

Hence, $$\frac{\Delta y}{\Delta x} = 0, \quad \text{and} \quad \frac{dy}{dx} = 0.$$

Thus, $$\frac{dc}{dx} = 0. \tag{I}$$

104. Derivative of a sum. If u and v are functions of x, then,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}. \qquad \text{(II)}$$

Proof. Let $y = u + v$.

$$y + \Delta y = u + \Delta u + v + \Delta v. \qquad (1)$$

$$\Delta y = \Delta u + \Delta v. \qquad (2)$$

$$\frac{\Delta y}{\Delta x} = \frac{\Delta u}{\Delta x} + \frac{\Delta v}{\Delta x}. \qquad (3)$$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}. \qquad (4)$$

105. Derivative of a product. If u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}. \qquad \text{(III)}$$

Proof. Let $y = u \cdot v$.

$$y + \Delta y = (u + \Delta u)(v + \Delta v) \qquad (1)$$

$$= uv + u\Delta v + v\Delta u + \Delta u\Delta v.$$

$$\Delta y = u\Delta v + v\Delta u + \Delta u\Delta v. \qquad (2)$$

$$\frac{\Delta y}{\Delta x} = u\frac{\Delta v}{\Delta x} + v\frac{\Delta u}{\Delta x} + \Delta u\frac{\Delta v}{\Delta x}. \qquad (3)$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}, \qquad (4)$$

since,
$$\lim_{\Delta x \to 0}\left(\Delta u \cdot \frac{\Delta v}{\Delta x}\right) = 0 \cdot \frac{dv}{dx} = 0.$$

If $u = c$ (constant), we have from (III),

$$\frac{d}{dx}(cv) = c\frac{dv}{dx}. \tag{III'}$$

106. Derivative of a quotient. If u and v are functions of x, then,

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{IV}$$

Proof. Let $y = \frac{u}{v}\cdot$

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}. \tag{1}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v} \tag{2}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}.$$

$$\frac{\Delta y}{\Delta x} = \frac{v\dfrac{\Delta u}{\Delta x} - u\dfrac{\Delta v}{\Delta x}}{v(v + \Delta v)}. \tag{3}$$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}. \tag{4}$$

If $u = c$ (constant), we have from (IV),

$$\frac{d}{dx}\left(\frac{c}{v}\right) = \frac{-c\dfrac{dv}{dx}}{v^2}. \tag{IV'}$$

107. Formulas stated in words.

I. *The derivative of a constant is* 0.

II. *The derivative of the sum of two functions is equal to the sum of their derivatives.*

III. *The derivative of the product of two functions is equal to the first function times the derivative of the second plus the second times the derivative of the first.*

III′. *The derivative of a constant times a variable is equal to the constant times the derivative of the variable.*

IV. *The derivative of the quotient of two functions is equal to the denominator times the derivative of the numerator minus the numerator times the derivative of the denominator, divided by the square of the denominator.*

IV′. *The derivative of a constant divided by a function is equal to minus the constant times the derivative of the function divided by the square of the function.*

108. Derivative of u^n. If $y = u^n$, where u is a function of x and n is a positive integer, then,

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}. \qquad \text{(V)}$$

Proof:

$$y + \Delta y = (u + \Delta u)^n$$

$$= u^n + nu^{n-1}\Delta u + \frac{n(n-1)}{2!}u^{n-2}\Delta u^2 + \ldots + \Delta u^n. \qquad (1)$$

(See Art. 52.)

$$\Delta y = nu^{n-1}\Delta u + \frac{n(n-1)}{2!}u^{n-2}\Delta u^2 + \ldots + \Delta u^n. \qquad (2)$$

$$\frac{\Delta y}{\Delta x} = nu^{n-1}\frac{\Delta u}{\Delta x} + \frac{(n-1)}{2!}u^{n-2}\Delta u\frac{\Delta u}{\Delta x} + \ldots + \Delta u^{n-1}\frac{\Delta u}{\Delta x}. \qquad (3)$$

$$\frac{dy}{dx} = nu^{n-1}\frac{du}{dx}, \qquad (4)$$

since $$\lim_{\Delta x \to 0}\left(\Delta u^{n-1}\frac{\Delta u}{\Delta x}\right) = 0\cdot\frac{du}{dx} = 0.$$

If $y = x^n(u = x)$, (V) takes the particular form,

$$\frac{dy}{dx} = nx^{n-1}. \qquad (V')$$

Although the above proof is valid only for positive integral values of n, formulas (V) and (V′) are true for all values of the exponent. This we shall assume without proof.

Exercises

Find the derivative of the following functions:

1. $y = 3x^3 - 5x^2 + 2x + 4.$

Solution. $\frac{dy}{dx} = 3\frac{d}{dx}(x^3) - 5\frac{d}{dx}(x^2) + 2\frac{d}{dx}(x).$ (See (II) and (III′).)

$= 9x^2 - 10x + 2.$ (See (V').)

Hence, $\frac{d}{dx}(3x^3 - 5x^2 + 2x + 4) = 9x^2 - 10x + 2.$

2. $y = (x^2 + 2)(3x^3 + 4x).$

Solution. $\frac{dy}{dx} = (x^2 + 2)\frac{d}{dx}(3x^3 + 4x) + (3x^3 + 4x)\frac{d}{dx}(x^2 + 2).$

(See (III).)

$= (x^2 + 2)(9x^2 + 4) + (3x^3 + 4x)\,2x.$

$= 15x^4 + 30x^2 + 8.$

Hence, $\frac{d}{dx}(x^2 + 2)(3x^3 + 4x) = 15x^4 + 30x^2 + 8.$

3. $y = \frac{x^2 + 3x}{x - 2}.$

Solution. $$\frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x^2+3x) - (x^2+3x)\frac{d}{dx}(x-2)}{(x-2)^2},$$

(See (IV).)

$$= \frac{(x-2)(2x+3) - (x^2+3x)}{(x-2)^2}$$

$$= \frac{x^2-4x-6}{(x-2)^2}.$$

Hence, $$\frac{d}{dx}\left(\frac{x^2+3x}{x-2}\right) = \frac{x^2-4x-6}{(x-2)^2}.$$

4. $y = (2x^3 + 3x + 2)^3$.

Solution. This function is of the form u^n, where $u = 2x^3 + 3x + 2$ and $n = 3$.

Hence, using (V) we obtain,

$$\frac{dy}{dx} = 3(2x^3 + 3x + 2)^2(6x^2 + 3).$$

5. $y = \sqrt[3]{3x^2 + 2x + 5}$.

Solution. $y = (3x^2 + 2x + 5)^{1/3}$.

$$\frac{dy}{dx} = \frac{1}{3}(3x^2 + 2x + 5)^{-2/3}(6x + 2). \quad \text{(See (V).)}$$

Hence, $$\frac{d}{dx}\sqrt[3]{3x^2 + 2x + 5} = \frac{6x+2}{3(3x^2+2x+5)^{2/3}}.$$

6. $y = 1 - 3x - 5x^2 - x^3$.

7. $y = x - 3x^3 - 2x^5$.

8. $y = \frac{1}{x} - \frac{1}{x^2} + \frac{3}{x^3}$. (Use (IV′).)

9. $y = \frac{3x-1}{2-2x}$.

10. $y = (4x^3 - 5x)(x^2 - 5x + 2)$.

11. $y = (x^2 + 3x)^3(x^3 + 5x + 2)$.

12. $y = \sqrt{x^3 - 9x^2 + 4}$.

13. $y = \dfrac{(1 + x)(1 - x^2)}{x^2}$.

Find the slope of each of the following curves at the point indicated:

14. $y = 3x^2 + 2x + 5$, where $x = -1$.

Solution. The slope at any point (x, y) is $\dfrac{dy}{dx} = 6x + 2$. (See **Ex. 1,** page 155.)

The slope at the point where $x = -1$ is therefore -4.

15. $y = \dfrac{x}{x + 3}$, where $x = 3$.

16. $y = (x + 2)(x^2 + 1)$, where $x = -1$.

Find the equation of the tangent to each of the following curves at the points indicated:

17. $y = 2x^2 + 3x + 1$, where $x = -2$.

Solution. At the point where $x = -2$, $y = 2(-2)^2 + 3(-2) + 1 = 3$. The slope of the tangent at the point $(-2, 3)$ is -5.

Hence, the equation of the tangent is

$$y - 3 = -5(x + 2). \quad \text{(See equation (7), page 53), or,}$$

$$5x + y + 7 = 0.$$

18. $y = x^3 - 3x^2 + 4x + 5$, where $x = 2$.

19. $y = 2x^3 - x^2 - 4$, where $x = 1$.

20. Find the tangent to the curve $y = 3x^2 - x$ which shall have 5 for its slope. Ans. $5x - y - 3 = 0$.

109. Increasing and decreasing functions. *A function, $y = f(x)$, is said to be increasing when it increases as x increases and is said to be decreasing when it decreases as x increases.* Assume that figure 54 is the graph of $y = f(x)$. Going from left to right, we notice that the curve is rising between the points P_1 and P_2, falling between the points P_2 and P_3, and rising to the right of P_3. In other words, the function $f(x)$ is increasing as x increases from x_1 to x_2, decreasing as x increases from x_2 to x_3, and increasing as x increases from x_3 to x_4, and so on. We notice also that the slope of the tangent is positive when the curve is rising and negative when the curve is falling. (See Figs. 18*a* and 18*b*.) That is, the derivative of $f(x)$

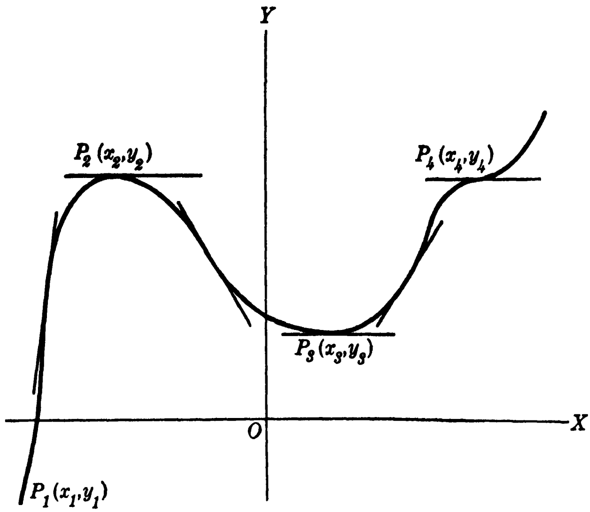

FIG. 54.

is positive when $f(x)$ is increasing and negative when $f(x)$ is decreasing.

Hence, we conclude,

If $\dfrac{dy}{dx} > 0, \quad y = f(x)$ increases.

If $\dfrac{dy}{dx} < 0, \quad y = f(x)$ decreases.

Example. Show that $y = x^2 + 4x + 3$ is decreasing when $x = -4$ and increasing when $x = 0$. Graph the function.

Solution.

$$y = x^2 + 4x + 3.$$

$$\frac{dy}{dx} = 2x + 4.$$

When $x = -4$, $\frac{dy}{dx} = -4.$

Hence, y is decreasing.

When $x = 0$, $\frac{dy}{dx} = 4.$

Hence, y is increasing.

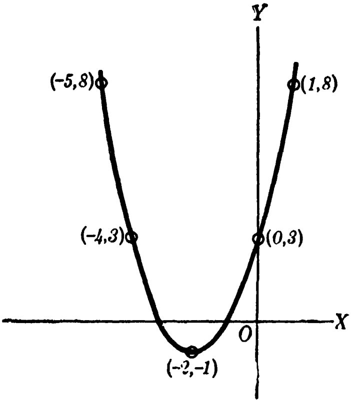

FIG. 55.

110. Maxima and minima. Maxima and minima values of quadratic functions were discussed in Art. 49. Maxima and minima values in general will now be discussed. *A maximum value of a function is that value where the function ceases to increase and begins to decrease. A minimum value of a function is that value where the function ceases to decrease and begins to increase. A maximum point is that point on the graph of a function where the graph ceases to rise and begins to fall. A minimum point is that point where the graph ceases to fall and begins to rise.*

Observing Fig. 54, we notice that P_2 is a *maximum point* and P_3 is a *minimum point.* It is evident that at such points the tangent is parallel to the X-axis; that is,

$$\frac{dy}{dx} = 0.$$

However, the vanishing of the derivative does not mean that the function necessarily has a maximum or a minimum. The tangent is parallel to the X-axis at the point P_4, but the function has neither a maximum nor a minimum there. It appears from the figure that the test is as follows:

At a point where $\frac{dy}{dx} = 0$, *if* $\frac{dy}{dx}$ *changes from positive to negative (as* x *increases),* y *is a maximum; if* $\frac{dy}{dx}$ *changes from negative to positive,* y *is a minimum; if* $\frac{dy}{dx}$ *does not change sign,* y *is neither a maximum nor a minimum.*

Example. Find the maximum and minimum values of the function, $y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 5$. Graph the function.

Solution.

$$y = \frac{x^3}{3} - \frac{x^2}{2} - 6x + 5. \tag{1}$$

$$\frac{dy}{dx} = x^2 - x - 6 = (x + 2)(x - 3). \tag{2}$$

When $\frac{dy}{dx} = 0, \quad x = -2, 3.$

When $x = -2, \quad y = 12\tfrac{1}{3}.$

When $x = 3, \quad y = -8\tfrac{1}{2}.$

When $x < -2$, we notice that $\frac{dy}{dx} > 0$ and when $x > -2$ we find that $\frac{dy}{dx} < 0$. Hence, the point $(-2, 12\tfrac{1}{3})$ is a maximum point on the graph of the function and $12\tfrac{1}{3}$ is a maximum value of the function.

When $x < 3$, $\frac{dy}{dx} < 0$; and when $x > 3$, $\frac{dy}{dx} > 0$. Hence, the point $(3, -8\frac{1}{2})$ is a minimum point on the graph and $-8\frac{1}{2}$ is a minimum value of the function. (See Fig. 56.)

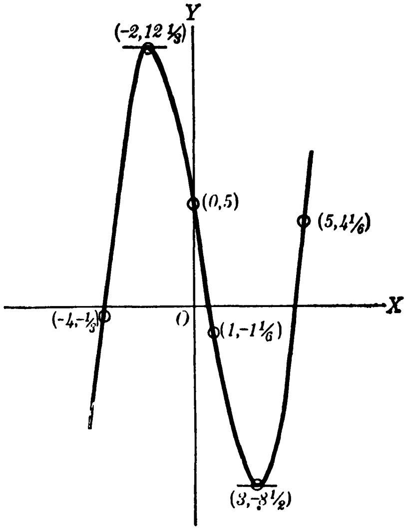

FIG. 56.

Exercises

In the following exercises determine the value of x for which $\frac{dy}{dx} = 0$. Determine the corresponding values of y and show whether these values are a maximum or a minimum.

1. $y = x^2 - 4x + 5$. Minimum at (2, 1).

2. $y = -x^2 + 6x + 7$. Maximum at (3, 16).

3. $y = x^3 + 3x^2 - 9x - 27$. Maximum at $(-3, 0)$, minimum at $(1, -32)$.

4. $y = 3x^3 - 9x^2 - 27x + 30$. $x = -1$, gives $y = 45$, maximum.
$x = 3$, gives $y = -51$, minimum.

5. $y = x^3 - 8$, $x = 0$, gives neither a maximum nor a minimum.

6. $y = x^3 - 3x^2 + 6x + 10$. Neither maximum nor minimum.

7. $y = \dfrac{1 - x + x^2}{1 + x - x^2}$. $x = \frac{1}{2}$, gives $y = \frac{3}{5}$, minimum.

8. $y = \dfrac{x^2 - 7x + 6}{x - 10}$. $x = 4$, gives maximim; $x = 16$, gives minimum.

111. Applications of the theory of maxima and minima. It was shown in Art. 110 that, at a point where the first derivative is 0, a function has either a maximum or a minimum value (provided the derivative changes sign at the point). This theory will now be applied to some practical problems.

Example. A box is to be made of a piece of card board 8 inches square by cutting equal squares out of the corners and turning up the sides. Find the volume of the largest box that can be made in this way.

Solution. Let x = the length of the side of each of the squares cut out. Then the volume of the box is

$$V = x(8 - 2x)^2. \tag{1}$$

$$\frac{dy}{dx} = (8 - 2x)(8 - 6x). \tag{2}$$

Making $\dfrac{dV}{dx} = 0$, we find,

$$x = 4, \tfrac{4}{3}.$$

When $x = 4$, $V = 0$. Hence, the value $x = 4$ can not be used.

When $x = \frac{4}{3}$, $V = \frac{1024}{27}$, and this is a maximum value.

Problems

1. Solve problems 5, 6, 7, 8, and 9, pages 70 and 71, making use of the derivative.

2. A box with a square base and open top is to hold 108 cubic feet. Find the dimensions that will make its construction most economical.

Solution. Let us assume one side of the base to be x and the altitude to be y. It is evident here that the thing desired is to minimize the surface. Now the surface consists of the four sides and the base. Hence, we may write:

$$S = 4xy + x^2. \tag{1}$$

Since the volume is to be 108 cubic feet, we may write

$$x^2y = 108, \text{ or } y = \frac{108}{x^2}. \tag{2}$$

Substituting the above value of y in (1) we obtain,

$$S = \frac{432}{x} + x^2. \tag{3}$$

$$\frac{dS}{dx} = \frac{-432}{x^2} + 2x. \tag{4}$$

When $$\frac{dS}{dx} = 0,$$

$$2x^3 = 432, \; x = 6. \tag{5}$$

Making $x = 6$ in (2), we see that $y = 3$. Hence, the dimensions of the box are $6 \times 6 \times 3$.

3. A silo is made in the form of a cylinder, with a hemispherical roof; there is a floor of the same thickness as the wall and roof. Find the most economical shape. Ans. Diameter = total height.

Silos are not built this way. Why not?

4. A watering trough is to hold 500 gallons. Find the dimensions that will make its construction most economical if its base is to be a rectangle with one side three times the other. (There are 231 cubic inches in one gallon.) Ans. Base 3.1 feet by 9.3 feet.

Altitude 2.32 feet.

CHAPTER XV

STATISTICS AND APPLICATIONS

112. Introduction. In Chapter V it was pointed out that many functional relations that can not be expressed by an algebraic equation may be exhibited by means of a graph. The graph usually gives a better view of a numerical situation than a table. By letting the eye follow the graph we get at once an approximate picture of fluctuations in the series of values. If we want to study such variations more closely or make comparisons between two or more sets of data, numerical methods are usually clearer and more convenient. *The branch of mathematics that deals with quantitative data affected to a marked extent by a multiplicity of causes is called statistics.*

113. Frequency tables. The simplest way of presenting a series of numerical values is simply to list the values in their natural order in a table. As for instance *Average Farm Prices December First* on pages 37 and 38. We may, however, group the values and get what we call a *frequency table.* The price of corn in the table referred to varies from 21.5¢ in 1896 to 136.5¢ in 1918. We divide the total range into *classes*, for instance 20¢ but less than 30¢, 30¢ but less than 40¢, and so on, and count the number of cases in each group. We find 5 cases between 20¢ and 29.9¢, 16 cases between 30¢ and 39.9¢, 15 cases between 40¢ and 49.9¢, 5 cases between 50¢ and 59.9¢, 10 cases between 60¢ and 69.9¢, 1 case between 70¢ and 79.9¢, 1 case between 80¢ and 89.9¢, 1 case between 90¢ and 99.9¢, 0 case between 100¢ and 109.9¢, 0 case between 110¢ and

119.9¢, 1 case between 120¢ and 129.9¢, and 2 cases between 130¢ and 139.9¢. These facts are recorded in a frequency table as follows:

Price	Number of Cases	Price	Number of Cases
20–29 9	5	80– 89 9	1
30–39.9	16	90– 99.9	1
40–49 9	15	100–109.9	0
50–59 9	5	110–119.9	0
60–69.9	10	120–129.9	1
70–79.9	1	130–139 9	2

The size of the class, or the *class interval*, is arbitrary, but should be governed by the total range and the number of cases.

Exercises

Construct similar frequency tables for prices on the other farm products listed, using the following class intervals:

1. Wheat, 10¢.
2. Oats, 5¢.
3. Barley, 5¢.
4. Rye, 10¢.
5. Potatoes, 10¢.
6. Hay, 50¢.

114. Measures of Central Tendency. If we study, for instance, two tables giving prices for a certain grade of hogs day by day for two years, we will find that there is much overlapping in prices. The question in which year were hog prices higher could not be answered directly from such tables. To make a comparison we must have a single price that is in some measure representative of the prices for the year, or what we call a *measure of central tendency*. We shall consider three such measures: *arithmetic mean*, *median*, and *mode*.

Arithmetic mean (or what is commonly called the average) *is simply the sum of the measures divided by their number.* Or, expressed in a formula,

$$M = \frac{\Sigma X}{N}. \tag{1}$$

Median is the middlemost measure, when the measures have been arranged in order of magnitude. For example, in the series

1, 3, 5, 9, 10, 12, 15, 19, 24, 30, 35

12 is the median, for there are five measures smaller than 12 and five measures larger than 12.

If the series has an even number of terms there is no middlemost measure, and we define the median as a measure halfway between the two middle measures. For example in the series

1, 3, 5, 9, 10, 12, 15, 19, 24, 30

11 is the median, because it is halfway between 10, the fifth measure, and 12, the sixth measure.

Mode is the measure that occurs most frequently in the series. Consider for example the following table of representative hog sales at Sioux City Stock Yard, January 4, 1926:

Number	Weight	Price	Number	Weight	Price
14	233	$10.90	4	237	$11.00
33	208	10.90	27	163	11.10
35	226	10.95	50	153	11.25
22	240	11.00	36	162	11.25
6	250	11.00	2	165	11.30
6	245	11.00	11	175	11.30

$11.25 is the mode, because the greatest number of hogs, or 86, were sold at this price. In market reports a modification of the mode is often used, namely the *bulk of sales*. For instance in the above report, bulk of sales is $10.95 to $11.25.

115. Determination of the arithmetic mean from a frequency table. We here make the assumption that the measures within the class are all concentrated at the *midpoint of the class interval.* For example, in the frequency table of prices of corn (Art. 113) there were 5 cases where the price was between 20.0¢ and 29.9¢. We here assume the price for all five cases within this class to be 25¢, the price for all 16 cases within the next class to be 35¢, and so on. We then obtain the following table (f stands for *frequency*, or number of cases in the class):

Class Interval	f	X	fX
20– 29.9	5	25	125
30– 39.9	16	35	560
40– 49.9	15	45	675
50– 59.9	5	55	275
60– 69.9	10	65	650
70– 79.9	1	75	75
80– 89 9	1	85	85
90– 99.9	1	95	95
100–109 9	0	105	0
110–119 9	0	115	0
120–129.9	1	125	125
130–139 9	2	135	270
		$\Sigma fX =$	2935

$$M = \frac{\Sigma fX}{N} \qquad M = \frac{2935}{57} = 51.5¢$$

This value of the arithmetic mean is only approximate and usually differs somewhat from the value obtained from the summation of the original values divided by their number. In our example we would have found 51.9¢ from the original data. The values computed by the two methods become more nearly equal as the number of cases increases and the size of the class interval decreases.

Example. Compute the average prices of wheat, oats, barley, rye, potatoes, and hay for the years 1870 to 1926 from the original data and from the frequency tables.

116. Determination of the median from a frequency table. If the values are arranged in a frequency table we make the assumption, when computing the median, that the values within an interval are uniformly distributed in the interval. For example take the frequency table showing the price of corn. There are 57 cases and according to the definition $\frac{57}{2}$ or 28.5 measures must be below and 28.5 measures above the median. If we start at the lower end we find that there are 21 cases below 40c and 15 cases in the class 40 − 49.9¢. The median must therefore be somewhere in this interval. Subtracting 21 from 28.5 we obtain 7.5, and the median must be the 7.5th measure in the class 40 − 49.9¢. As there are 15 measures in this class the 7.5th measure must be $\frac{7.5}{15} \cdot 10$ or 5. Add this value to 40 and we get 45¢ as the median. The same value would be obtained if we started at the higher end. In the groups above 50 there are 21 measures, and the median must be the (28.5 − 21) or 7.5th measure in the interval 40 − 49.9¢ *counting from the top.* $\frac{7.5}{15} \cdot 10 = 5$, which value should be subtracted from 50¢ which again gives us 45¢.

117. Variability. It is often desirable to have some measure of the variability of a series of values; for instance, prices of some farm product during a year. We have several such values of variability. Those considered here are the *range,* the *mean deviation*, the *quartile deviation*, and the *standard deviation.*

The range. Take the two series

(1) 5, 9, 10, 14, 16, 16, 18, 24, 31.

(2) 9, 12, 14, 15, 16, 17, 19, 20, 22.

Both have a mean of 16 and a median of 16; hence they are alike as far as central tendency is concerned. Yet, they are rather different. Series (1) includes measures from 5 to 31, while (2) varies only from 9 to 22. *The difference between the highest and the lowest value in a series of measures is called the range.* Series (1) has a range of 26, while series (2) has a range of only 13. The range is a measure of variability but gives a very incomplete picture of a series, being dependent only upon the highest and lowest measures.

The mean deviation. If we determine the amount that each of the terms in series (1) varies from the mean, we get the series

11, 7, 6, 2, 0, 1, 2, 8, 15.

the mean of which is 5.78. This value is called the *mean deviation.* For series (2) we get a mean deviation of 3.11. *The mean deviation is a measure of the tendency of the individual measures in a series to scatter.* The mean deviation may be calculated from any measures of central tendency, the *mean, median* or *mode.* It should, therefore, always be indicated from which measure of central tendency the mean deviation is calculated.

Exercise

Calculate the mean deviation from the mean for prices of corn, wheat, oats, barley, rye, potatoes, and hay for the years 1870–1926. Try to do this with prices arranged in frequency tables and compare the value so determined for one of the products with the value obtained from the use of the original tabulation.

The quartile deviation or semi-interquartile range. As we determined the median as a point on a scale of values below which half of the number of cases are found and above which the other half are found, so we may determine two other points such that one-fourth of the number of values are found below and three-fourths above one of these points, and three-fourths below and one-fourth above the other point. *Half of the difference between these two measures is called the semi-interquartile range and is a measure of the spread or variability of the series.*

If the series of values are given in order of their magnitude we may simply count off one-fourth of the number of cases from the top and one-fourth from the bottom of the series and take the mean of the values so obtained. If the series is given as a frequency table the work is done practically the same way as in calculating the median. In our previous example on prices of corn we have 57 cases or 14.25 in each quartile. The first quartile point is at $30 + \frac{14.25 - 5}{16} \times 10$ or 35.8¢ and the third quartile point (first from the top) at $70 - \frac{14.25 - 6}{10} \times 10$ or 61.8¢. The semi-interquartile range is therefore $\frac{61.8 - 35.8}{2}$ or 13¢.

Exercise

Calculate the semi-interquartile ranges for prices of wheat, oats, barley, rye, potatoes and hay for the years 1870–1916.

The standard deviation. The most generally used measure of variability is the *standard deviation,* obtained in the following way: *Calculate the deviations from the mean, square these deviations, add the squares, divide by the number of cases, and extract the square root of the quotient.* The standard deviation is usually designated by the Greek letter σ (sigma) and may be expressed by the following formula,

$$\sigma = \sqrt{\frac{d^2}{N}}, \tag{2}$$

where d represents deviations from the mean, and N the number of cases.

As the deviations are usually rather awkward numbers to handle the formula may be expressed in the original measures X. By substituting $d = X - M$; $M = \frac{\Sigma X}{N}$; $d = X - \frac{\Sigma X}{N}$ in (2) we have

$$\sigma = \sqrt{\frac{\Sigma\left(X - \frac{\Sigma X}{N}\right)^2}{N}},$$

$$\sigma = \sqrt{\frac{\Sigma\left(X^2 - \frac{2X\Sigma X}{N} + \left(\frac{\Sigma X}{N}\right)^2\right)}{N}},$$

$$\sigma = \sqrt{\frac{\Sigma X^2}{N} - \frac{2\Sigma(X\Sigma X)}{N} + \frac{\Sigma(\Sigma X)^2}{N^3}},$$

$$\sigma = \sqrt{\frac{\Sigma X^2}{N} - \left(\frac{\Sigma X}{N}\right)^2}, \tag{3}$$

since, $\Sigma(X\Sigma X) = (\Sigma X)^2$,

and $\frac{\Sigma(\Sigma X)^2}{N^3} = \frac{(\Sigma X)^2}{N^2}$.

Although formula (3) looks more formidable than (2), it is in reality much simpler. Expressed in words, the operations are as follows: Square the original measures, add the squares, and divide by the number of cases. This gives $\frac{\Sigma X^2}{N}$. Next, add the original measures, divide the sum by N, and square the quotient. This gives $\left(\frac{\Sigma X}{N}\right)^2$. Subtract $\left(\frac{\Sigma X}{N}\right)^2$ from $\frac{\Sigma X^2}{N}$ and extract the square root of the difference. This gives σ.

If the series of values is given in the form of a frequency table, we must multiply each value by its frequency, and our formulas become

$$\sigma = \sqrt{\frac{\Sigma f d^2}{N}}, \tag{4}$$

$$\sigma = \sqrt{\frac{\Sigma f X^2}{N} - \left(\frac{\Sigma f X}{N}\right)^2}. \tag{5}$$

The chief objection to the range as a measure of variability has already been illustrated. The addition of one or two extreme cases may increase the range to several times its former value without actually causing any great change in the tendency of the cases to group themselves about some central value. The quartile deviation almost entirely eliminates the effect of extreme cases, a condition which is not always wholly desirable. It is also unreliable in those instances in which the distribution of the items under discussion departs decidedly from symmetry. In such situations the mean deviation is much more useful. The standard deviation, making use, as it does, of the squares of the deviations of all items from the mean, is affected strongly by extreme cases but reduces the effect somewhat by taking the square root of the sum. The greatest advantage of this measure is the ease with which it lends itself to algebraic manipulation. In making use of any of these measures of variability to compare distributions, the size of the objects involved must be kept in mind. As an example, suppose we consider the physical measurements of a group of men. A range of two inches in their heights would be almost negligible, while a range of two inches in the lengths of their feet would indicate a wide variety of sizes.

If we divide the quartile measure by the sum of the two values which were used in its computation, and divide each of the other measures by the arithmetic mean of the distribution to which they are applied, our measures become coefficients and allow us to compare more conveniently distributions of items of widely different magnitudes.

Example. Calculate the standard deviation of prices of corn for 1870–1926.

Class Interval	f	X	fX	fX^2
20– 29.9	5	25	125	3,125
30– 39 9	16	35	560	19,600
40– 49 9	15	45	675	30,375
50– 59 9	5	55	275	15,125
60– 69.9	10	65	650	42,250
70– 79.9	1	75	75	5,625
80– 89 9	1	85	85	7,225
90– 99 9	1	95	95	9,025
100–109.9	0	105	0	0
110–119.9	0	115	0	0
120–129 9	1	125	125	15,625
130–139 9	2	135	270	36,450
	$\Sigma f = 57$		$\Sigma fX = 2935$	$184,425 = \Sigma fX^2$

$$\sigma = \sqrt{\frac{184,425}{57} - \left(\frac{2935}{57}\right)^2} = 25.2¢$$

Exercises

1. Calculate the standard deviation of prices of wheat, oats, barley, rye, potatoes, and hay during the years 1870–1926.

2. Show that formula (3) follows from (2).

118. Correlation. In the physical sciences the value of a variable is usually dependent upon, or, as we say, is a function of a single variable or at least very few other variables, and one or more constants. For example, the electric current that flows through a conductor depends upon the electromotive force and the resistance of the conductor. Expressed as a functional relationship we may write

$$C = f(E,R).$$

In the laboratory we are usually able to keep all of the independent variables except one constant and allow this one to vary at will, thus arriving at a mathematical formula for the relationship. We may for instance keep the electromotive force constant and vary the resistance; we than find that the current varies inversely as the resistance. Again we may keep the resistance constant and vary the electromotive force, thus finding that the current varies directly as the electromotive force. By properly selecting the units in which we measure we may reduce all the constants to the value 1 and establish the formula,

$$C = \frac{E}{R},$$

where C = the current in amperes, E = electromotive force in volts, and R = resistance in ohms.

In the biological and still more so in the social sciences the number of variables is usually large and it is difficult or in many cases impossible to keep certain variables constant while varying others in the course of an experiment. We often have to measure the various factors of a phenomenon as it occurs without our controlling influence and by means of statistical analysis of the observed values draw conclusions regarding their interdependence. We may be able to establish a *degree* of relationship even if we are unable to determine the *nature* of this relationship.

The degree of relationship between two series of values of two variables is usually measured by the *coefficient of correlation.* This coefficient may have values from +1 through 0 to −1. If there is a perfect agreement between the variation of the two variables so that both increase or decrease together the correlation is said to be perfect and positive. Such a correlation exists between the values of current and electromotive force if the resistance is kept constant. The coefficient of correlation would in this case be +1. If on the other hand an increase in one variable always is accompanied by a decrease in the other variable the correlation is perfect and negative, = −1. Such would be the relationship between current and resistance if voltage is kept constant. If there is no relationship between the variables but an increase in one is just as likely to be accompanied by a decrease as by an increase in the other variable, the coefficient of correlation is 0.

There may, however, be a *tendency* for one variable to increase or decrease as the other variable increases or decreases, although the correspondence is not perfect. In such a case we get a coefficient of correlation between 0 and +1. If, on the other hand one variable tends to increase as the other variable decreases although imperfectly, we get values of the coefficient of correlation between 0 and −1.

After this description of the meaning of the term coefficient of correlation we shall give the two most commonly used methods of computing said coefficient, omitting the rather complicated mathematical theory on which they are based.

119. The rank method of correlation. This coefficient is usually designated by ρ (rho, Greek letter) and differs slightly from the coefficient r determined by the product-moment formula as described below.

Let the following two series of values be the mean prices of wheat and corn for ten weeks:

.	Wheat	Corn
1st week	$1.25	$0 67
2nd week	1.28	0.65
3rd week	1.33	0.75
4th week	1.40	0.76
5th week	1.36	0.74
6th week	1.41	0.77
7th week	1.34	0.72
8th week	1.30	0 70
9th week	1.35	0.73
10th week	1.38	0.71

Rank the prices of each, assigning to the lowest price the rank 1 and to highest price the rank 10. We then get

	Wheat Rank	Corn Rank	Rank of Wheat Minus Rank of Corn = d	d^2
1st week	1	2	−1	1
2nd week	2	1	1	1
3rd week	4	8	−4	16
4th week	9	9	0	0
5th week	7	7	0	0
6th week	10	10	0	0
7th week	5	5	0	0
8th week	3	3	0	0
9th week	6	6	0	0
10th week	8	4	4	16
				$\Sigma d^2 = 34$

The coefficient of rank correlation, C, is then obtained by the formula,

$$C = 1 - \frac{6\Sigma d^2}{N(N^2 - 1)}, \qquad (6)$$

where N is the number of cases and d is the differences in rank.

$$C = 1 - \frac{6 \times 34}{10(10^2 - 1)}$$

$$= 1 - \tfrac{204}{990} = 1 - 0.206 = 0.794.$$

The coefficient of rank correlation takes into account the ranks of the variables only, but not their magnitude. If two or more terms are equal they are given the same rank. For instance, if the 12th and 13th terms are equal, they are both given the rank 12.5. If the 12th, 13th, and 14th are alike, all three are given the rank 13, etc.

Exercise

Determine by the rank method the correlations between prices of wheat on the one hand and prices of (*a*) corn, (*b*) oats, (*c*) barley, (*d*) rye, (*e*) potatoes, (*f*) hay, on the other hand, as given on pp. 37ff.

120. The product moment formula or the Pearson correlation coefficient. This formula is

$$r = \frac{\Sigma xy}{N \cdot \sigma_x \cdot \sigma_y}, \tag{7}$$

where x are the deviations of the terms in the X-series from their mean (with proper signs), y the variations of the terms in the Y-series from their mean, N the number of cases, σ_x the standard deviation of the X-series, and σ_y the standard deviation of the Y-series.

Recalling that $\sigma_x = \sqrt{\frac{\Sigma x^2}{N}}$ and $\sigma_y = \sqrt{\frac{\Sigma y^2}{N}}$ we may substitute these values in (7) and get

$$r = \frac{\Sigma xy}{N\sqrt{\frac{\Sigma x^2}{N}}\sqrt{\frac{\Sigma y^2}{N}}},$$

or
$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}. \qquad (8)$$

Example

	Price of Wheat in Cents X	Price of Corn in Cents Y	$X - M_x$ or x	$Y - M_y$ or y	xy	x^2	y^2
1st ...	125	67	−9	−5	+45	81	25
2nd.	128	65	−6	−7	+42	36	49
3rd .	133	75	−1	+3	− 3	1	9
4th	140	76	+6	+4	+24	36	16
5th	136	74	+2	+2	+ 4	4	4
6th	141	77	+7	+5	+35	49	25
7th .	134	72	0	0	0	0	0
8th	130	70	−4	−2	+ 8	16	4
9th	135	73	+1	+1	+ 1	1	1
10th	138	71	+4	−1	− 4	16	1

$\Sigma X = 1340$ $\Sigma Y = 720$ $\Sigma x^2 = 240$ $\Sigma y^2 = 134$ $\Sigma xy = +152$

$$M_x = \frac{\Sigma X}{N} = 134 \qquad M_y = \frac{\Sigma Y}{N} = 72$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \Sigma y^2}} = \frac{+152}{\sqrt{240 \times 134}} = 0.847.$$

If the means of the X- and Y-series come out with decimals, this method involves considerable numerical work. We may then employ to advantage a modification of the formula that uses the original X- and Y-values.

Recalling that

$$x = X - M_x = X - \frac{\Sigma X}{N},$$

$$y = Y - M_y = Y - \frac{\Sigma Y}{N},$$

and substituting these values in (8), we get,

$$r = \frac{\Sigma\left(X - \frac{\Sigma X}{N}\right)\left(Y - \frac{\Sigma Y}{N}\right)}{\sqrt{\Sigma\left(X - \frac{\Sigma X}{N}\right)^2 \Sigma\left(Y - \frac{\Sigma Y}{N}\right)^2}}$$

$$= \frac{\Sigma XY - \frac{\Sigma X \Sigma Y}{N}}{\sqrt{\left(\Sigma X^2 - \frac{(\Sigma X)^2}{N}\right)\left(\Sigma Y^2 - \frac{(\Sigma Y)^2}{N}\right)}} \qquad (9)$$

The work can be further reduced due to the fact that the subtraction of a constant term from either series does not affect the coefficient of correlation. We could, for instance, in our example subtract 124 from all the X-values and 64 from all the Y-values, thus materially reducing the size of the figures with which we have to operate.

Exercise

Determine by the product moment formula the correlations between prices of wheat, on the one hand, and prices of (*a*) corn, (*b*) oats, (*c*) barley, (*d*) rye, (*e*) potatoes, (*f*) hay, on the other hand, as given on pp. 37ff.

CHAPTER XVI

PROBABILITY

121. Meaning of probability. A box contains four white and five black balls. One ball is drawn at random and then replaced and this process is continued indefinitely. What proportion of the balls drawn will be black? Here there are nine balls to be drawn or we may say there are nine possibilities, and either of the nine balls is *equally likely* to be drawn or any one of the nine possibilities is *equally likely* to happen. Of the nine possibilities, any one of four would result in drawing a white ball and any one of five would result in drawing a black ball. We would say, then, that four possibilities of the nine are favorable to drawing a white ball and the other five possibilities are favorable to drawing a black ball. We put the above statement in another way by saying that in a single draw the probability of drawing a white ball is $\frac{4}{9}$ and the probability of drawing a black ball is $\frac{5}{9}$. This does not mean that out of only nine draws, exactly four would be white and five black. But it does mean that, if a single ball were drawn at random and were replaced and this process continued indefinitely, $\frac{4}{9}$ of the balls drawn would be white and $\frac{5}{9}$ would be black. Or the ratio of the number of white balls drawn to the number of black balls drawn would be as 4 to 5.

Reasoning similar to the above led LaPlace to formulate the following definition of probability: *If h is the number of possible ways that an event will happen and f is the number of possible ways that it will fail and all of the possibilities are equally likely,* *the probability that the event will happen is* $\frac{h}{h+f}$ *and the probability that it will fail is* $\frac{f}{h+f}$.

It is evident, then, that the sum of the probability that an event will happen and the probability that it will fail is 1, the symbol for certainty.

In analyzing a number of possibilities we must be sure that each of them is *equally likely* to happen before we attempt to apply the above definition of probability.

Example. What is the probability that a man, age 30 and in good health, will die before age 35? In this case we might reason thus: The event can happen in only one way and fail in only one way, and consequently, the probability that he will die before age 35 is $\frac{1}{2}$. But this reasoning is false for we are assuming that living five years and dying within five years are *equally likely* for a man now 30 years old. But this is not the actual experience. This example will be discussed in Art. 122.

122. Probability based upon observation or experience. There are many events in which it is impossible to enumerate all the *equally likely* ways in which the event can happen or fail. Yet by means of experience we may determine to a fair degree of accuracy the probability that a future event will happen at a certain time. If we have observed that an event has happened h times out of n possible ways, where n is a large number, we conclude that h/n is a fair estimate of the probability that the event will again happen and our confidence in this estimate increases as the number of possibilities, n, increases.

We are now ready to solve the problem which was stated in Art. 121. The American Experience Table of Mortality shows that out of 85,441 men living at age 30, the number of living at age 35 will be 81,822. Then the number dying before age

35 is 85,441 − 81,822 or 3619. Hence the probability that a man aged 30 will die before age 35 is $\frac{3619}{85,441} = .04235$. In this problem $n = 85,441$ and $h = 3619$.

We have previously stated that the value h/n is only an estimate, but it is accurate enough (when n is large enough) for many practical purposes.

123. Meaning of mortality table. If it were possible to trace a large number of persons, say 100,000, living at age 10 until the death of each occurred, and a record kept of the number living at each age x and the number dying between the ages x and $x + 1$, we would have a mortality table.

However, mortality tables are not constructed by observing a large number of individuals living at a certain age until the death of each, for it is evident that this method would not be practicable, but would be next to impossible, if not impossible. Mechanical methods have been devised for the construction of such tables, but the scope of this text does not permit of the discussion of these methods.

Table IV (back of book) is known as the American Experience Table of Mortality and is based upon the records of the Mutual Life Insurance Company of New York. It was first published in 1868 and is used by life insurance companies in America to determine the premium to charge for their policies. It will be used in this book as a basis for all computations dealing with mortality statistics. It consists of five columns as follows: The first gives the ages running from 10 to 95, the different ages being denoted by x; the second gives the number living at the beginning of each age x and is denoted by l_x; the third gives the number dying between ages x and $x + 1$ and is denoted by d_x; the fourth gives the probability of dying in the year from age x to age $x + 1$ and is denoted by q_x; and the fifth gives the probability of living a year from age x to age $x + 1$ and is denoted by p_x.

Exercises

1. What is the probability that a man aged 40 will live to be 65? What is the probability that a man of the same age will die before reaching age 65? What is the sum of the two probabilities? (See solution of problem in art. 122.)

2. Suppose 100,000 lives age 10 were insured for one year, by a company for \$1000 each. Using the American Experience Table as a basis and not figuring interest, what would be the cost to each individual? Ans. \$7.49.

3. What would be the cost of \$1000 insurance for one year on the life of an individual 30 years old? (Assume for convenience that 85,441 individuals are insured by the same company.) Ans. \$8.43.

124. Permutations. Number of permutations of things all different. Before discussing permutations we state the following principle, which is fundamental: *If one thing may be done in p ways and after it has been done in one of these ways, another thing may be done in q ways, then the two things together may be done in the order named in pq ways.*

It is evident that for each of the p ways of doing the first thing there are q ways of doing the second thing and the total number of ways of doing the two in succession is pq.

This principle may be extended to three or more things.

Example. A man may go from A to B over any one of 4 routes and from B to C over any one of 7 routes. In how many ways may he go from A to C through B? Ans. 28.

Each of the different ways that a number of things may be arranged is known as a permutation of those things. For example the different arrangements of the letters *abc* are—*abc*, *acb*, *bac*, *bca*, *cab*, *cba*. There are 3 different ways of selecting the first letter and after it has been selected in one of these ways there remain 2 ways of selecting the second letter. Then the first two letters may be selected in 3×2 or 6 ways. It is clear that we have no choice in the selection of the third letter and consequently the total number of permutations (or arrangements) of the three letters is 6.

Now suppose there are n things all different and we wish to find the number of permutations of these things taken r at a time, $n > r$.

Since only r of the n things are to be used at a time, there are only r places to be filled. The first place may be filled by any one of the n things and the second place by any one of the $n - 1$ remaining things. Then, the first and second places together may be filled in $n(n - 1)$ ways. The third place may be filled by any one of the $n - 2$ remaining things. Hence the first three places may be filled in $n(n - 1)(n - 2)$ ways. Reasoning in a similar way we see that after $r - 1$ places have been filled, there remain $n - (r - 1)$ things from which to fill the rth place. Applying the fundamental principle stated above we have

$$ {}_nP_r = n(n-1)(n-2) \dots (n-r+1) \qquad (1)$$

When $r = n$, (1) becomes,

$$ {}_nP_n = n(n-1)(n-2) \dots 3 \cdot 2 \cdot 1 = n! \qquad (2)$$

NOTE. The symbol ${}_nP_r$ is used to denote the number of permutations of n things taken r at a time. $n!$ is a symbol which stands for the product of all the integers from 1 up to and including n, and is read "factorial" n.

Exercises

1. A man has two suits of clothes, three shirts, four ties and two hats. In how many ways may he dress by changing suits, shirts, ties and hats?

2. How many arrangements of the letters in the word "Vermont" can be made, using in each arrangement

(*a*) 4 letters? (*b*) all the letters?

3. How many signals could be made from 4 different flags?

4. Four persons enter a street car in which there are 7 vacant seats. In how many ways may they be seated?

125. Combinations. Number of combinations of things all different. *By a combination we mean a group of things without any regard for order of arrangement of the individuals within the group.* For example *abc, acb, bac, bca, cab, cba* are the same *combination* of the letters *abc*, but each arrangement is a different *permutation.*

By the number of combinations of n things taken r at a time is meant the number of different groups that may be formed from n individuals when r individuals are placed in each group. For example *ab, ac, bc* are the different combinations of the letters *abc* when two letters are used at a time.

The symbol ${}_nC_r$ is used to stand for the number of combinations of n things taken r at a time. We will now derive an expression for ${}_nC_r$. For each one of the ${}_nC_r$ combinations there are $r!$ different permutations. And for all of the ${}_nC_r$ combinations there are ${}_nC_r r!$ permutations, which is the number of permutations of n things taken r at a time. Hence,

$$ {}_nC_r r! = {}_nP_r, $$

and
$$ {}_nC_r = \frac{{}_nP_r}{r!}. $$

Since,
$$ {}_nP_r = n(n-1)(n-2)\ \dots\ (n-r+1), $$

we have
$$ {}_nC_r = \frac{n(n-1)(n-2)\ \dots\ (n-r+1)}{r!}. \qquad (3) $$

Exercises

1. Find the number of combinations of 8 things taken 5 at a time.

Solution. Here $n = 8$ and $r = 5$.

Then, $${}_8C_5 = \frac{8\cdot7\cdot6\cdot5\cdot4}{5\cdot4\cdot3\cdot2} = 56.$$

2. How many committees of 5 men can be selected from a group of 12 men?

3. Out of 7 Englishmen and 6 Americans, how many committees of 3 Englishmen and 2 Americans can be chosen? Ans. 525.

4. How many different sums can be made up from a cent, a nickel, a dime, a quarter, and a dollar? Ans. 31.

5. An urn contains 4 white and 9 black balls. If 5 balls are drawn at random, what is the probability that (*a*) all are black, (*b*) 2 white and 3 black?

Solution. (*a*) The total number of ways that 5 balls may be drawn from 13 balls is ${}_{13}C_5$ or 1287 ways. And the number of ways that 5 black balls may be drawn is ${}_9C_5$ or 126 ways. Hence, the probability of drawing 5 black balls is $\frac{126}{1287}$ or $\frac{14}{143}$.

(*b*) 2 white balls may be drawn in ${}_4C_2$ ways or 6 ways. And for each one of these 6 ways of drawing 2 white balls, 3 black balls may be drawn in ${}_9C_3$ or 84 ways. Then 2 white balls and 3 black balls may be drawn together in $6 \cdot 84$ or 504 ways (see fundamental principle, art. 124). Hence, the probability of drawing 2 white and 3 black balls is $\frac{504}{1287}$ or $\frac{56}{143}$.

6. A bag contains 5 white, 6 black and 8 red balls. If 4 balls are drawn at random, what is the probability that (*a*) all are black, (*b*) 2 white and 2 red, (*c*) 2 black and 2 white, (*d*) 1 white, 1 black and 2 red?

126. Compound events. We may think of an event as composed of two or more simpler events. These component simpler events may be *independent, dependent* or *exclusive. Two or more events are said to be independent or dependent when the occurrence of any one of them at a given trial does not or does affect the occurrence of the others. Two or more events are said to be exclusive when the occurrence of any one of them on a particular occasion excludes the occurrence of another on that occasion.* We give now three theorems without proof.

Theorem I. *If $p_1, p_2 \ldots p_r$ are the separate probabilities of r independent events, the probability that all of these events will* *happen together at a given trial is the product of their separate probabilities*, that is,

$$p = p_1 \cdot p_2 \cdot p_3 \ \ldots \ p_r. \tag{4}$$

Theorem II. *Let p_1 be the probability of a first event; let p_2 be the probability of a second event after the first has happened; let p_3 be the probability of a third event after the first two have happened; and so on. Then the probability that all of these events will occur in order is*

$$p = p_1 \cdot p_2 \cdot p_3 \ \ldots \ p_r. \tag{5}$$

Theorem III. *If $p_1, p_2, \ldots p_r$ are the separate probabilities of r mutually exclusive events, the probability that one of these events will happen on a particular occasion when all of them are in question is*

$$P = p_1 + p_2 + p_3 \ \ldots + p_r. \tag{6}$$

Exercises

1. The probability that A will live 15 years is $\frac{1}{7}$, the probability that *B* will live 15 years is $\frac{1}{6}$, and the probability that *C* will live 15 years is $\frac{1}{8}$. What is the probability that all three will live 15 years?

Solution. We have here three independent events, where $p_1 = \frac{1}{7}$, $p_2 = \frac{1}{6}$, $p_3 = \frac{1}{8}$.

Hence, $P = \frac{1}{7} \cdot \frac{1}{6} \cdot \frac{1}{8} = \frac{1}{336}$.

2. Find the probability of drawing 2 white balls in succession from a bag containing 5 white and 6 black balls, if the first ball drawn is not replaced before the second drawing is made.

Solution. We have here two dependent events. The probability that the first draw will be white is $\frac{5}{5+6} = \frac{5}{11}$ the probability that the second draw will be white is $\frac{4}{4+6} = \frac{2}{5}$. Then $p_1 = \frac{5}{11}$ and $p_2 = \frac{2}{5}$. Hence,

$$P = \frac{5}{11} \cdot \frac{2}{5} = \frac{2}{11}. \qquad \text{((5) Art. 126)}$$

3. Five coins are tossed at once. What is the probability that all will be heads?

4. A bag contains 3 white, 4 black, and 6 red balls. One ball is drawn and not replaced, then a second ball is drawn and not replaced and then a third ball is drawn. What is the probability (*a*) that a ball of each color will be drawn, (*b*) that 2 blacks and 1 red will be drawn, (*c*) that all will be red?

5. Suppose that in example 4 the balls were replaced after each draw. Then answer (*a*), (*b*), and (*c*).

6. Three men of ages 25, 30, 32 respectively form a partnership. What is the probability (*a*) that all three will be living at the end of 8 years, (*b*) that the first two will be living, (*c*) that one only of the three will be living? Use the American Experience Table of Mortality.

7. A man and wife are 24 and 23 when they marry. What is the probability that they will both live to celebrate their Golden Wedding?

CHAPTER XVII

ANNUITIES AND INSURANCE

127. Meaning of life annuity. In Chapter X annuities certain (those that continue a certain time regardless of any future happening) were discussed. *By a life annuity we mean a succession of periodical payments which continue only during the life of the individual concerned.* It is clear then that the cost of such an annuity will depend upon the probability of living as well as upon the rate of interest. Before computing the cost of a life annuity we will discuss pure endowments.

128. Pure endowments. *A pure endowment is a sum of money payable to a person aged x, at a specified future date, provided the person survives until that date.* We will now find the cost of an endowment of 1 to be paid at the end of n years to a person whose present age is x. The symbol, $_nE_x$, will stand for the cost of such an endowment.

Suppose l_x individuals, all of age x, agree to contribute equally to a fund that will assure the payment of one dollar to each of the survivors at the end of n years. From the mortality table we see that out of the l_x individuals entering this agreement, l_{x+n} of them would be living at the end of n years. Consequently, it would require l_{x+n} dollars at that time. But the present value of this sum is

$$v^n \cdot l_{x+n} \qquad \text{(Equation (1), Art. 70)}$$

and since l_x persons are contributing equally to this fund, the share of each will be

$$v^n l_{x+n} \div l_x.$$

Hence,
$$ {}_nE_x = \frac{v^n l_{x+n}}{l_x}. \tag{1} $$

If the numerator and the denominator of (1) be multiplied by v_x, it becomes

$$ \frac{v^{x+n} l_{x+n}}{v^x l_x}, $$

and if we agree that the product $v^x l_x$ shall be denoted by D_x, then (1) becomes,

$$ {}_nE_x = \frac{D_{x+n}}{D_x}. \tag{2} $$

D_x is one of four symbols, called commutation symbols, that are used to facilitate insurance computations. See Table V in the back of this book. This table is based on the American Experience Table of Mortality and a $3\frac{1}{2}\%$ interest rate is used. There are other commutation tables based upon different tables of mortality and different rates of interest are used.

129. Present value (cost) of a life annuity. We now propose to find the present value of a life annuity of one dollar per annum payable to an individual, now aged x. The symbol, a_x, is used to denote such an annuity. We see that the present value of this annuity is merely the sum of pure endowments, payable at the end of one, two, three and so on years. Consequently,

$$ \begin{aligned} a_x &= {}_1E_x + {}_2E_x + {}_3E_x + \ldots \text{ to end of table.} \\ &= \frac{D_{x+1}}{D_x} + \frac{D_{x+2}}{D_x} + \frac{D_{x+3}}{Dx} + \ldots \text{ to end of table.} \\ &= \frac{D_{x+1} + D_{x+2} + D_{x+3} + \ldots \quad ((2), \text{ Art. } 128.)}{D_x} \quad (3) \end{aligned} $$

$$ a_x = \frac{N_{x+1}}{D_x}, \tag{4} $$

where

$$^*N_{x+1} = D_{x+1} + D_{x+2} + D_{x+3} + \ldots \text{ to end of table.} \quad (5)$$

130. Life annuity due. *When the first payment under an annuity is made immediately, we have what is called an annuity due.* The present value of an annuity due of 1 per annum to a person aged x is denoted by $\mathfrak{a}_x$. An annuity due differs from an ordinary annuity (Art. 129) only by an immediate payment. Consequently we have

$$\mathfrak{a}_x = 1 + a_x,$$

$$= 1 + \frac{N_{x+1}}{D_x} = \frac{D_x + N_{x+1}}{D_x}$$

$$= \frac{D_x + D_{x+1} + D_{x+2} + D_{x+3} + \ldots \text{ to end of table.}}{D_x} \quad (6)$$

$$= \frac{N_x}{D_x}, \quad (7)$$

where

$$N_x = D_x + D_{x+1} + D_{x+2} + \ldots \text{ to end of table.} \quad (8)$$

131. Temporary annuity. *When the payments under a life annuity stop after a certain time although the individual be still living, we have what is called a temporary annuity.* Such an annuity which ceases after n years is denoted by the symbol $a_{x\overline{n}|}$.

It is clear that the present value of a temporary annuity is equal to the sum of present values of pure endowments payable at end of 1, 2, 3, . . . , n years. Thus,

* See Table V.

$$a_{x\overline{n}|} = {}_1E_x + {}_2E_x + \ldots + {}_nE_x,$$

$$= \frac{D_{x+1} + D_{x+2} + \ldots + D_{x+n}}{D_x}$$

$$= \frac{D_{x+1} + D_{x+2} + \ldots \text{ to end of table}}{D_x}$$

$$- \frac{D_{x+n+1} + D_{x+n+2} + \ldots \text{ to end of table}}{D_x} \qquad (9)$$

$$a_{x\overline{n}|} = \frac{N_{x+1} - N_{x+n+1}}{D_x}. \qquad (10)$$

If the first of the n payments be made immediately and the last payment be made at the end of $n - 1$ years, we then have a temporary annuity due. Letting $\mathrm{a}_{x\overline{n}|}$ represent such an annuity, we get,

$$\mathrm{a}_{x\overline{n}|} = 1 + a_{x\overline{n-1}|},$$

$$= 1 + \frac{D_{x+1} + D_{x+2} + \ldots + D_{x+n-1}}{D_x}$$

$$= \frac{D_x + D_{x+1} + D_{x+2} + \ldots + D_{x+n-1}}{D_x}. \qquad (11)$$

$$= \frac{N_x - N_{x+n}}{D_x}. \qquad (12)$$

Exercises

1. Find the cost of a pure endowment of \$5000 due in 15 years and purchased at age 25, interest at $3\frac{1}{2}\%$.

Solution. Here $x = 25$, $n = 15$, and

$$ {}_{15}E_{25} = \frac{D_{40}}{D_{25}} = \frac{19727.4}{37673.6} = .523639.$$

Hence, $5000\ {}_{15}E_{25} = \$2618.20.$

2. What is the cost of a life annuity of \$500 per annum for a person aged 50, interest at $3\frac{1}{2}\%$?

Solution. From (4) Art. 129,

$$a_{50} = \frac{N_{51}}{D_{50}} = \frac{1691650}{12498.6}$$

$$= 13.534716.$$

The annuity of \$500 has a cost of

$$500a_{50} = 500(13.534716) = \$6767.36.$$

3. A man aged 60 has \$10,000 with which to buy a life annuity. What will be his annual income on a $3\frac{1}{2}\%$ basis?

Solution. Here we have the cost of an annuity and are required to find the annual rent. Hence, from (4) Art. 129, we have,

$$Ra_{60} = \$10,000,$$

$$R = \frac{\$10,000}{a_{60}}.$$

But,
$$a_{60} = \frac{N_{61}}{D_{60}} = \frac{73754.7}{7351.65} = 10.032401,$$

$$R = \frac{10,000}{10.032401} = \$996.77.$$

4. An heir, aged 14, is to receive \$30,000 when he becomes 21. What is the present value of his estate on a $3\frac{1}{2}\%$ basis?

5. What would be the present value of the estate in Ex. 4 on a 4% basis? Ans. \$21,597.30.

6. According to the terms of a will a person aged 30 is to receive a life income of \$6000, first payment at once An inheritance tax of 3% on the present value of the income must be paid immediately. Find the present value of the income and the amount of the tax. Ans. \$117,632.40, \$3,528.97.

7. A man carrying a \$10,000 life insurance policy arranges it so that the proceeds at his death shall be payable to his wife in annual installments for 20 years certain, first payment upon due proof of death. What would be the annual installment?

8. What would be the annual installment in Ex. 7, if payments were to be made throughout the life of his wife, assuming that she was 55 years of age at his death?

9. What would be the annual installment in Ex. 8, if the wife took a twenty-year temporary annuity?

132. Life insurance definitions. Life insurance is fundamentally sound only when a large group of individuals is considered. Each person contributes to a general fund from which the losses sustained by individuals of the group are paid. The organization that takes care of this fund and settles the claim for all losses is known as an *insurance company.* The deposit made to this fund by the individuals is called a *premium.* Since, the payment of this premium by the individuals insures a certain sum or *benefit* at his death, he is spoken of as the *insured* and the person to whom the benefit is paid at the death of the insured is called the *beneficiary.* The agreement made between the insured and the company is called a *policy* and the insured is sometimes spoken of as the *policy holder.* If all of those insured were of the same age all premiums would be the same, but since the policy holders are of different ages it is evident that the premiums vary. One of the main problems is to determine the premium to be paid for a certain benefit. It is clear that the premium will depend upon the probability of dying and also upon the rate of interest to be paid on funds left with the company. The premium based upon these two things only is known as a *net premium.* However, the insurance company has many expenses, in connection with the securing of policy holders, such as advertising, commissions, salaries, office supplies, et cetera, and consequently, must make a charge in addition to the net premium. The net premium plus this additional charge is called the *gross* or *office premium.* The premium may be *single,* or it may be paid annually, and this annual premium may sometimes be paid in semiannual, quarterly or even monthly installments. All premiums are paid in advance.

133. Ordinary life policy. *An ordinary life policy is one wherein the benefit is payable at death and at death only.* The net single premium on an ordinary life policy is the present value of this benefit. The symbol A_x will stand for the net single premium of a benefit of 1.

Let us assume that each of l_x persons all of age x, buys an ordinary life policy of 1. During the first year there will be d_x deaths, and consequently, at the end of the first year * the company will have to pay d_x in benefits. Hence, the present value of these benefits will be vd_x. There will be d_{x+1} deaths during the second year and the present value of these benefits will be v^2d_{x+1}, and so on. The sum of the present values of all future benefits will be given by the expression,

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \dots \text{ to end of table.}$$

Since l_x persons buy benefits of 1 each, we will obtain the present value of each person's benefit by dividing the above expression by l_x. Therefore,

$$A_x = \frac{vd_x + v^2d_{x+1} + v^3d_{x+3} + \dots \text{ to end of table.}}{l_x} \qquad (13)$$

If both numerator and denominator of (9) be multiplied by v^x, we get,

$$A_x = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + \dots \text{ to end of table,}}{v^x l_x}$$

$$= \frac{C_x + C_{x+1} + C_{x+2} + \dots \text{ to end of table.}}{D_x}$$

$$A_x = \frac{M_x}{D_x}, \qquad (14)$$

* In reality claims are paid upon due proof of the death of the insured, but we here assume that they are not paid until the end of the year.

where,

$$C_x{}^* = v^{x+1}d_x, \quad C_{x+1} = v^{x+2}d_{x+1}, \text{ and so on,}$$

and $M_x{}^* = C_x + C_{x+1} + C_{x+2} + \ldots$ to end of table.

Life insurance policies are seldom bought by a single premium. The common plan is to pay a fixed annual premium throughout the life of the policy. We denote the annual premium of an ordinary life policy of 1 by the symbol P_x. The payment of P_x, at the beginning of each year, for life forms a life annuity due and the present value of this annuity must be equivalent to the net single premium. Thus we have,

$$P_x \text{a}_x = A_x. \tag{15}$$

Solving for P_x, we get,

$$P_x = \frac{A_x}{\text{a}_x} = \frac{M_x}{N_x}, \tag{16}$$

since, $$A_x = \frac{M_x}{D_x} \quad \text{and} \quad \text{a}_x = \frac{N_x}{D_x}.$$

Exercises

1. What is the net single premium for an ordinary life policy for $10,000 on a person aged 25?

2. What is the annual premium on the policy of Ex. 1?

3. Compare annual premiums on ordinary life policies of $10,000 for ages 20 and 21 and for ages 50 and 51. Note the annual change in cost for the two periods of life.

134. Limited payment life policy. *The limited payment life policy is like the ordinary life policy* † *in that the benefit is payable at death and death only, but differs from it in that the*

* See Table V.

† The ordinary life policy and the limited payment life policy, are often spoken of as *whole life policies* in that the benefit of either is not payable until death.

equivalent of the net single premium is arranged to be paid in n annual payments. Here n is the number of annual payments that are to be made unless death should occur earlier. The standard forms of limited payment policies are usually for ten, fifteen, twenty or thirty payments but other forms may be written.

It is evident that the n annual premiums on the limited payment life policy form a temporary life annuity due. It is also evident that this annuity is equivalent to the net single premium A_x. Hence, if the net annual premium for a benefit of 1 be denoted by ${}_nP_x$, we may write,

$$ {}_nP_x \, a_{x\overline{n}|} = A_x. \quad ((11), \text{ Art. } 131.) \tag{17}$$

Solving for ${}_nP_x$ and substituting for $a_{x\overline{n}|}$ and A_x, we get,

$$ {}_nP_x = \frac{M_x}{N_x - N_x + n}. \tag{18}$$

Exercises

1. Find the net annual premium on a twenty-payment life policy for $2500 on a person aged 30.

Solution. Using (18), Art. 131, we have,

$$ {}_{20}P_{30} = \frac{M_{30}}{N_{30} - N_{50}} = \frac{10{,}259}{596{,}804 - 181{,}663}$$

$$ = \frac{10{,}259}{415{,}141} = 0.024712.$$

$$2500\,{}_{20}P_{30} = \$61.78.$$

2. Find the net annual premium for a fifteen-payment life policy of $10,000 issued at age 45.

3. Find the net annual premium on a twenty-payment life policy of $5000 for your age at nearest birthday.

4. Compare annual premiums on twenty-payment life policies of $20,000 for ages 25 and 26 and for ages 50 and 51. Note the annual change in cost for the two periods of life.

135. Term insurance. *Term insurance is temporary insurance as it provides for the payment of the benefit only in case death occurs within a certain period of n years.* After n years the policy becomes void. The stated period may be any number of years, but usually term policies are for five years, ten years, fifteen years and twenty years.

The symbol $A^1_{x\overline{n}|}$ is used to denote the net single premium on a n-year term policy of benefit 1, bought at age x.

If we assume that each of l_x persons all of age x, buys a term policy for n years, the present value of the payments made by the company will be given by

$$vd_x + v^2d_{x+1} + v^3d_{x+2} + \ldots v^nd_{x+n-1}. \tag{19}$$

Since each of l_x persons buys a benefit of 1, the present value of the benefit of each person will be gotten by dividing expression (19) by l_x, Hence,

$$A^1_{x\overline{n}|} = \frac{vd_x + v^2d_{x+1} + \ldots + v^nd_{x+n-1}.}{l_x} \tag{20}$$

If both the numerator and the denominator of (20) be multiplied by v^x, we get,

$$A^1_{x\overline{n}|} = \frac{v^{x+1}d_x + v^{x+2}d_{x+1} + \ldots v^{x+n}d_{x+n-1}}{v^xl_x}$$

$$= \frac{(v^{x+1}d_x + v^{x+2}d_{x+1} + \ldots \text{to end of table})}{v^xl_x}$$

$$- \frac{(v^{x+n+1}d_{x+n} + v^{x+n+2}d_{x+n+1} + \text{to end of table})}{v^xl_x}.$$

$$A^1_{x\overline{n}|} = \frac{M_x - M_{x+n}}{D_x}. \qquad ((14), \text{Art. } 133.) \tag{21}$$

When the term insurance is for one year only, the net premium is called the *natural premium*. It is given by making $n = 1$ in (21) Thus,

$$A^1_{x\overline{1|}} = \frac{M_x - M_{x+1}}{D_x} = \frac{C_x}{D_x}. \tag{22}$$

The net annual premium for a term policy of 1 for n years will be denoted by the symbol $P^1_{x\overline{n|}}$. It is evident that the annual premiums for a term policy constitute a temporary annuity due. This annuity is equivalent to the net single premium. Thus,

$$P^1_{x\overline{n|}}\,\mathrm{a}_{x\overline{n|}} = A^1_{x\overline{n|}}. \tag{23}$$

Solving for $P^1_{x\overline{n|}}$ and substituting for $\mathrm{a}_{x\overline{n|}}$ and $A^1_{x\overline{n|}}$, we get,

$$P^1_{x\overline{n|}} = \frac{M_x - M_{x+n}}{N_x - N_{x+n}}. \quad \text{((12), Art. 131 and (21), Art. 135.)} \tag{24}$$

Exercises

1. Find the net single premium for a term insurance of \$1000 for 15 years for a man aged 30.

Solution. From (21), Art. 135, we have,

$$A^1_{30\,\overline{15|}} = \frac{M_{30} - M_{45}}{D_{30}} = \frac{10259 - 7192.81}{30,440.8}$$

$$= \frac{3066.19}{20440.8} = 0.10072,$$

and $\quad 1000\,A^1_{30\,\overline{15|}} = \$100.72.$

2. Find the net single premium for a term insurance of \$25,000 for 5 years for a man aged 50.

3. What is the net annual premium for the insurance described in Ex. 2?

4. What are the natural premiums for ages 20, 30, 40, and 50 for an insurance of \$1000?

5. A person aged 35 buys a \$10,000 term policy which will terminate at age 65. Find the net annual premium.

136. Endowment insurance. *In an endowment policy the company agrees to pay a certain sum in event of the death of the insured within a specified period, known as the endowment period, and also agrees to pay this sum at the end of the endowment period, provided the insured be living to receive it.* From the above definition it is evident that an endowment insurance of 1 for n years may be considered as a term insurance of 1 for n years plus an n-year pure endowment of 1. (See Art. 128 and Art. 135.)

Thus, if we let the symbol $A_{x\overline{n}|}$ stand for the net single premium for an endowment of 1 for n years we have,

$$A_{x\overline{n}|} = A^1{}_{x\overline{n}|} + {}_nE_x$$

$$= \frac{M_x - M_{x+n}}{D_x} + \frac{D_{x+n}}{D_x}. \tag{25}$$

$$A_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{D_x}. \tag{26}$$

We shall now find the net annual premium for an endowment of 1 for n years, the premiums to be payable for k years. The symbol ${}_kP_{x\overline{n}|}$ will stand for the annual premium of such an endowment. It is clear that these premiums constitute a temporary annuity due that is equivalent to the net single premium. Hence,

$${}_kP_{x\overline{n}|}\,\mathrm{a}_{x\overline{k}|} = A_{x\overline{n}|}. \tag{27}$$

Solving for ${}_kP_{x\overline{n}|}$ and substituting for $\mathrm{a}_{x\overline{k}|}$ and $A_{x\overline{n}|}$, we get,

$${}_kP_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+k}}. \tag{28}$$

If the number of annual payments are to be equal to the number of years in the endowment period, then $k = n$, and (28) becomes,

$$P_{x\overline{n}|} = \frac{M_x - M_{x+n} + D_{x+n}}{N_x - N_{x+n}}. \tag{29}$$

Exercises

1. Find the net annual premium on a $10,000 20-payment, 30-year endowment policy taken at age 25.

Solution. From (28), we have,

$$\begin{aligned} {}_{20}P_{25\,\overline{30}|} &= \frac{M_{25} - M_{55} + D_{55}}{N_{25} - N_{45}} \\ &= \frac{11{,}631.1 - 5510.54 + 9733.40}{770{,}113 - 253{,}745} \\ &= \frac{15{,}853.96}{516{,}368} = 0.0307028. \end{aligned}$$

$$10{,}000\ {}_{20}P_{25\,\overline{30}|} = \$307.03.$$

2. Find the net single premium on a $1000 20-year endowment policy for a person aged 35.

3. Find the net annual premium for a $10,000 20-payment endowment policy maturing at age 60, taken at age 25.

4. Find the net annual premium on a $20,000 15-year endowment policy taken at age 55.

137. Meaning of reserves. By observing the table of mortality, we see that the probability of dying within any one year increases each year after the tenth year of age. Consequently, the natural premium will increase with each year's increase of age. The net annual premium will be much larger than the natural premium during the earlier years of the policy, but finally for the later years the natural premium will become larger than the net annual premium.

During the earlier years the difference between the net annual premium and the natural premium is set aside at interest annually. This fund grows from year to year and is held intact to meet the heavier mortality of the later years. This amount so held by the company is known as the reserve * or the value of its policies. This is unlike the reserve of a bank for it is not held to meet some unexpected emergency but it is a real liability of the company to be used to settle the claims of its policy-holders.

The above remarks may be illustrated as follows: Suppose a man aged 35 takes out a $1000 ordinary life policy. His net annual premium for that age on a $3\frac{1}{2}\%$ basis would be $19.91. The natural premium for that year would be $8.65, leaving a difference of $11.26 † to be placed in the reserve. However, at age 60 the natural premium would be $25.79, which is $5.88 larger than the net annual premium, this deficiency being cared for by the reserve.

Let us assume that each of 81,822 persons, all aged 25, buy an ordinary life policy of $1000. The total net annual premiums would amount to $1,629,076.02. This amount would accumulate to $1,686,093.68 by the end of the first year. According to the table of mortality the death losses to be paid at the end of the first year would amount to $732,000.00, leaving $954,093.68 in the reserve. This would leave a terminal reserve of $11.77 to each of the 81,090 survivors. The premiums received at the beginning of the second year amount to $1,614,501.90, which, when added to $954,093.68, makes a total of $2,568,595.58, and so on. The following table is self explanatory.

Table showing terminal reserves on an ordinary life policy for $1000 on the life of an individual aged 35 years.

* The reserve on any one policy at the end of any policy year is known as the terminal reserve for that year, or the policy value.

† This is the initial reserve for the first year.

Policy Year	Funds on Hand at Beginning of Year	Funds Accumulated at $3\frac{1}{2}\%$	Death Losses	Funds at End of Year	Amount of Credit of Each Survivor
1st.....	1,629,076.02	1,686,093.68	732,000	954,093.68	11.77
2nd....	2,568,595.58	2,658,496.43	737,000	1,921,496.43	23.91
3rd....	3,521,324.66	3,644,571.02	742,000	2,902,571.02	36.46
4th. ..	4,487,625.03	4,644,692.94	749,000	3,895,692.94	49.40
5th....	5,465,835.36	5,657,139.60	756,000	4,901,139.60	62.75

The above table illustrates what we mean by a reserve. Reserves, however, are not figured in this way. Formulas for finding the reserves on different kinds of policies and for any year may be derived but we shall not attempt this discussion here.

TABLE I.

COMMON LOGARITHMS OF NUMBERS.

N.	0	1	2	3	4	5	6	7	8	9
100	00 000	043	087	130	173	217	260	303	346	389
01	432	475	518	561	604	647	689	732	775	817
02	860	903	945	988	*030	*072	*115	*157	*199	*242
03	01 284	326	368	410	452	494	536	578	620	662
04	703	745	787	828	870	912	953	995	*036	*078
05	02 119	160	202	243	284	325	366	407	449	490
06	531	572	612	653	694	735	776	816	857	898
07	938	979	*019	*060	*100	*141	*181	*222	*262	*302
08	03 342	383	423	463	503	543	583	623	663	703
09	743	782	822	862	902	941	981	*021	*060	*100
110	04 139	179	218	258	297	336	376	415	454	493
11	532	571	610	650	689	727	766	805	844	883
12	922	961	999	*038	*077	*115	*154	*192	*231	*269
13	05 308	346	385	423	461	500	538	576	614	652
14	690	729	767	805	843	881	918	956	994	*032
15	06 070	108	145	183	221	258	296	333	371	408
16	446	483	521	558	595	633	670	707	744	781
17	819	856	893	930	967	*004	*041	*078	*115	*151
18	07 188	225	262	298	335	372	408	445	482	518
19	555	591	628	664	700	737	773	809	846	882
120	918	954	990	*027	*063	*099	*135	*171	*207	*243
21	08 279	314	350	386	422	458	493	529	565	600
22	636	672	707	743	778	814	849	884	920	955
23	991	*026	*061	*096	*132	*167	*202	*237	*272	*307
24	09 342	377	412	447	482	517	552	587	621	656
25	691	726	760	795	830	864	899	934	968	*003
26	10 037	072	106	140	175	209	243	278	312	346
27	380	415	449	483	517	551	585	619	653	687
28	721	755	789	823	857	890	924	958	992	*025
29	11 059	093	126	160	193	227	261	294	327	361
130	394	428	461	494	528	561	594	628	661	694
31	727	760	793	826	860	893	926	959	992	*024
32	12 057	090	123	156	189	222	254	287	320	352
33	385	418	450	483	516	548	581	613	646	678
34	710	743	775	808	840	872	905	937	969	*001
35	13 033	066	098	130	162	194	226	258	290	322
36	354	386	418	450	481	513	545	577	609	640
37	672	704	735	767	799	830	862	893	925	956
38	988	*019	*051	*082	*114	*145	*176	*208	*239	*270
39	14 301	333	364	395	426	457	489	520	551	582
140	613	644	675	706	737	768	799	829	860	891
41	922	953	983	*014	*045	*076	*106	*137	*168	*198
42	15 229	259	290	320	351	381	412	442	473	503
43	534	564	594	625	655	685	715	746	776	806
44	836	866	897	927	957	987	*017	*047	*077	*107
45	16 137	167	197	227	256	286	316	346	376	406
46	435	465	495	524	554	584	613	643	673	702
47	732	761	791	820	850	879	909	938	967	997
48	17 026	056	085	114	143	173	202	231	260	289
49	319	348	377	406	435	464	493	522	551	580
150	609	638	667	696	725	754	782	811	840	869
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	44	43	42
1	4.4	4.3	4.2
2	8.8	8.6	8.4
3	13.2	12.9	12.6
4	17.6	17.2	16.8
5	22.0	21.5	21.0
6	26.4	25.8	25.2
7	30.8	30.1	29.4
8	35.2	34.4	33.6
9	39.6	38.7	37.8
	41	**40**	**39**
1	4.1	4.0	3.9
2	8.2	8.0	7.8
3	12.3	12.0	11.7
4	16.4	16.0	15.6
5	20.5	20.0	19.5
6	24.6	24.0	23.4
7	28.7	28.0	27.3
8	32.8	32.0	31.2
9	36.9	36.0	35.1
	38	**37**	**36**
1	3.8	3.7	3.6
2	7.6	7.4	7.2
3	11.4	11.1	10.8
4	15.2	14.8	14.4
5	19.0	18.5	18.0
6	22.8	22.2	21.6
7	26.6	25.9	25.2
8	30.4	29.6	28.8
9	34.2	33.3	32.4
	35	**34**	**33**
1	3.5	3.4	3.3
2	7.0	6.8	6.6
3	10.5	10.2	9.9
4	14.0	13.6	13.2
5	17.5	17.0	16.5
6	21.0	20.4	19.8
7	24.5	23.8	23.1
8	28.0	27.2	26.4
9	31.5	30.6	29.7
	32	**31**	**30**
1	3.2	3.1	3.0
2	6.4	6.2	6.0
3	9.6	9.3	9.0
4	12.8	12.4	12.0
5	16.0	15.5	15.0
6	19.2	18.6	18.0
7	22.4	21.7	21.0
8	25.6	24.8	24.0
9	28.8	27.9	27.0

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
150	17 609	638	667	696	725	754	782	811	840	869
51	898	926	955	984	*013	*041	*070	*099	*127	*156
52	18 184	213	241	270	298	327	355	384	412	441
53	469	498	526	554	583	611	639	667	696	724
54	752	780	808	837	865	893	921	949	977	*005
55	19 033	061	089	117	145	173	201	229	257	285
56	312	340	368	396	424	451	479	507	535	562
57	590	618	645	673	700	728	756	783	811	838
58	866	893	921	948	976	*003	*030	*058	*085	*112
59	20 140	167	194	222	249	276	303	330	358	385
160	412	439	466	493	520	548	575	602	629	656
61	683	710	737	763	790	817	844	871	898	925
62	952	978	*005	*032	*059	*085	*112	*139	*165	*192
63	21 219	245	272	299	325	352	378	405	431	458
64	484	511	537	564	590	617	643	669	696	722
65	748	775	801	827	854	880	906	932	958	985
66	22 011	037	063	089	115	141	167	194	220	246
67	272	298	324	350	376	401	427	453	479	505
68	531	557	583	608	634	660	686	712	737	763
69	789	814	840	866	891	917	943	968	994	*019
170	23 045	070	096	121	147	172	198	223	249	274
71	300	325	350	376	401	426	452	477	502	528
72	553	578	603	629	654	679	704	729	754	779
73	805	830	855	880	905	930	955	980	*005	*030
74	24 055	080	105	130	155	180	204	229	254	279
75	304	329	353	378	403	428	452	477	502	527
76	551	576	601	625	650	674	699	724	748	773
77	797	822	846	871	895	920	944	969	993	*018
78	25 042	066	091	115	139	164	188	212	237	261
79	285	310	334	358	382	406	431	455	479	503
180	527	551	575	600	624	648	672	696	720	744
81	768	792	816	840	864	888	912	935	959	983
82	26 007	031	055	079	102	126	150	174	198	221
83	245	269	293	316	340	364	387	411	435	458
84	482	505	529	553	576	600	623	647	670	694
85	717	741	764	788	811	834	858	881	905	928
86	951	975	998	*021	*045	*068	*091	*114	*138	*161
87	27 184	207	231	254	277	300	323	346	370	393
88	416	439	462	485	508	531	554	577	600	623
89	646	669	692	715	738	761	784	807	830	852
190	875	898	921	944	967	989	*012	*035	*058	*081
91	28 103	126	149	171	194	217	240	262	285	307
92	330	353	375	398	421	443	466	488	511	533
93	556	578	601	623	646	668	691	713	735	758
94	780	803	825	847	870	892	914	937	959	981
95	29 003	026	048	070	092	115	137	159	181	203
96	226	248	270	292	314	336	358	380	403	425
97	447	469	491	513	535	557	579	601	623	645
98	667	688	710	732	754	776	798	820	842	863
99	885	907	929	951	973	994	*016	*038	*060	*081
200	30 103	125	146	168	190	211	233	255	276	298
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	29	28
1	2.9	2.8
2	5.8	5.6
3	8.7	8.4
4	11.6	11.2
5	14.5	14.0
6	17.4	16.8
7	20.3	19.6
8	23.2	22.4
9	26.1	25.2

	27	26
1	2.7	2.6
2	5.4	5.2
3	8.1	7.8
4	10.8	10.4
5	13.5	13.0
6	16.2	15.6
7	18.9	18.2
8	21.6	20.8
9	24.3	23.4

	25
1	2.5
2	5.0
3	7.5
4	10.0
5	12.5
6	15.0
7	17.5
8	20.0
9	22.5

	24	23
1	2.4	2.3
2	4.8	4.6
3	7.2	6.9
4	9.6	9.2
5	12.0	11.5
6	14.4	13.8
7	16.8	16.1
8	19.2	18.4
9	21.6	20.7

	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
200	30 103	125	146	168	190	211	233	255	276	298
01	320	341	363	384	406	428	449	471	492	514
02	535	557	578	600	621	643	664	685	707	728
03	750	771	792	814	835	856	878	899	920	942
04	963	984	*006	*027	*048	*069	*091	*112	*133	*154
05	31 175	197	218	239	260	281	302	323	345	366
06	387	408	429	450	471	492	513	534	555	576
07	597	618	639	660	681	702	723	744	765	785
08	806	827	848	869	890	911	931	952	973	994
09	32 015	035	056	077	098	118	139	160	181	201
210	222	243	263	284	305	325	346	366	387	408
11	428	449	469	490	510	531	552	572	593	613
12	634	654	675	695	715	736	756	777	797	818
13	838	858	879	899	919	940	960	980	*001	*021
14	33 041	062	082	102	122	143	163	183	203	224
15	244	264	284	304	325	345	365	385	405	425
16	445	465	486	506	526	546	566	586	606	626
17	646	666	686	706	726	746	766	786	806	826
18	846	866	885	905	925	945	965	985	*005	*025
19	34 044	064	084	104	124	143	163	183	203	223
220	242	262	282	301	321	341	361	380	400	420
21	439	459	479	498	518	537	557	577	596	616
22	635	655	674	694	713	733	753	772	792	811
23	830	850	869	889	908	928	947	967	986	*005
24	35 025	044	064	083	102	122	141	160	180	199
25	218	238	257	276	295	315	334	353	372	392
26	411	430	449	468	488	507	526	545	564	583
27	603	622	641	660	679	698	717	736	755	774
28	793	813	832	851	870	889	908	927	946	965
29	984	*003	*021	*040	*059	*078	*097	*116	*135	*154
230	36 173	192	211	229	248	267	286	305	324	342
31	361	380	399	418	436	455	474	493	511	530
32	549	568	586	605	624	642	661	680	698	717
33	736	754	773	791	810	829	847	866	884	903
34	922	940	959	977	996	*014	*033	*051	*070	*088
35	37 107	125	144	162	181	199	218	236	254	273
36	291	310	328	346	365	383	401	420	438	457
37	475	493	511	530	548	566	585	603	621	639
38	658	676	694	712	731	749	767	785	803	822
39	840	858	876	894	912	931	949	967	985	*003
240	38 021	039	057	075	093	112	130	148	166	184
41	202	220	238	256	274	292	310	328	346	364
42	382	399	417	435	453	471	489	507	525	543
43	561	578	596	614	632	650	668	686	703	721
44	739	757	775	792	810	828	846	863	881	899
45	917	934	952	970	987	*005	*023	*041	*058	*076
46	39 094	111	129	146	164	182	199	217	235	252
47	270	287	305	322	340	358	375	393	410	428
48	445	463	480	498	515	533	550	568	585	602
49	620	637	655	672	690	707	724	742	759	777
250	794	811	829	846	863	881	898	915	933	950
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	22	21	20	19	18	17
1	2.2	2.1	2.0	1.9	1.8	1.7
2	4.4	4.2	4.0	3.8	3.6	3.4
3	6.6	6.3	6.0	5.7	5.4	5.1
4	8.8	8.4	8.0	7.6	7.2	6.8
5	11.0	10.5	10.0	9.5	9.0	8.5
6	13.2	12.6	12.0	11.4	10.8	10.2
7	15.4	14.7	14.0	13.3	12.6	11.9
8	17.6	16.8	16.0	15.2	14.4	13.6
9	19.8	18.9	18.0	17.1	16.2	15.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
250	39 794	811	829	846	863	881	898	915	933	950
51	967	985	*002	*019	*037	*054	*071	*088	*106	*123
52	40 140	157	175	192	209	226	243	261	278	295
53	312	329	346	364	381	398	415	432	449	466
54	483	500	518	535	552	569	586	603	620	637
55	654	671	688	705	722	739	756	773	790	807
56	824	841	858	875	892	909	926	943	960	976
57	993	*010	*027	*044	*061	*078	*095	*111	*128	*145
58	41 162	179	196	212	229	246	263	280	296	313
59	330	347	363	380	397	414	430	447	464	481
260	497	514	531	547	564	581	597	614	631	647
61	664	681	697	714	731	747	764	780	797	814
62	830	847	863	880	896	913	929	946	963	979
63	996	*012	*029	*045	*062	*078	*095	*111	*127	*144
64	42 160	177	193	210	226	243	259	275	292	308
65	325	341	357	374	390	406	423	439	455	472
66	488	504	521	537	553	570	586	602	619	635
67	651	667	684	700	716	732	749	765	781	797
68	813	830	846	862	878	894	911	927	943	959
69	975	991	*008	*024	*040	*056	*072	*088	*104	*120
270	43 136	152	169	185	201	217	233	249	265	281
71	297	313	329	345	361	377	393	409	425	441
72	457	473	489	505	521	537	553	569	584	600
73	616	632	648	664	680	696	712	727	743	759
74	775	791	807	823	838	854	870	886	902	917
75	933	949	965	981	996	*012	*028	*044	*059	*075
76	44 091	107	122	138	154	170	185	201	217	232
77	248	264	279	295	311	326	342	358	373	389
78	404	420	436	451	467	483	498	514	529	545
79	560	576	592	607	623	638	654	669	685	700
280	716	731	747	762	778	793	809	824	840	855
81	871	886	902	917	932	948	963	979	994	*010
82	45 025	040	056	071	086	102	117	133	148	163
83	179	194	209	225	240	255	271	286	301	317
84	332	347	362	378	393	408	423	439	454	469
85	484	500	515	530	545	561	576	591	606	621
86	637	652	667	682	697	712	728	743	758	773
87	788	803	818	834	849	864	879	894	909	924
88	939	954	969	984	*000	*015	*030	*045	*060	*075
89	46 090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374
91	389	404	419	434	449	464	479	494	509	523
92	538	553	568	583	598	613	627	642	657	672
93	687	702	716	731	746	761	776	790	805	820
94	835	850	864	879	894	909	923	938	953	967
95	982	997	*012	*026	*041	*056	*070	*085	*100	*114
96	47 129	144	159	173	188	202	217	232	246	261
97	276	290	305	319	334	349	363	378	392	407
98	422	436	451	465	480	494	509	524	538	553
99	567	582	596	611	625	640	654	669	683	698
300	712	727	741	756	770	784	799	813	828	842
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	18	17	16	15	14
1	1.8	1.7	1.6	1.5	1.4
2	3.6	3.4	3.2	3.0	2.8
3	5.4	5.1	4.8	4.5	4.2
4	7.2	6.8	6.4	6.0	5.6
5	9.0	8.5	8.0	7.5	7.0
6	10.8	10.2	9.6	9.0	8.4
7	12.6	11.9	11.2	10.5	9.8
8	14.4	13.6	12.8	12.0	11.2
9	16.2	15.3	14.4	13.5	12.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
800	47 712	727	741	756	770	784	799	813	828	842
01	857	871	885	900	914	929	943	958	972	986
02	48 001	015	029	044	058	073	087	101	116	130
03	144	159	173	187	202	216	230	244	259	273
04	287	302	316	330	344	359	373	387	401	416
05	430	444	458	473	487	501	515	530	544	558
06	572	586	601	615	629	643	657	671	686	700
07	714	728	742	756	770	785	799	813	827	841
08	855	869	883	897	911	926	940	954	968	982
09	996	*010	*024	*038	*052	*066	*080	*094	*108	*122
810	49 136	150	164	178	192	206	220	234	248	262
11	276	290	304	318	332	346	360	374	388	402
12	415	429	443	457	471	485	499	513	527	541
13	554	568	582	596	610	624	638	651	665	679
14	693	707	721	734	748	762	776	790	803	817
15	831	845	859	872	886	900	914	927	941	955
16	969	982	996	*010	*024	*037	*051	*065	*079	*092
17	50 106	120	133	147	161	174	188	202	215	229
18	243	256	270	284	297	311	325	338	352	365
19	379	393	406	420	433	447	4[illegible]1	474	488	501
820	515	529	542	556	569	583	596	[illegible]10	623	637
21	651	664	678	691	705	718	732	745	759	772
22	786	799	813	826	840	853	866	880	893	907
23	920	934	947	961	974	987	*001	*014	*028	*041
24	51 055	068	081	095	108	121	135	148	162	175
25	188	202	215	228	242	255	268	282	295	308
26	322	335	348	362	375	388	402	415	428	441
27	455	468	481	495	508	521	534	548	561	574
28	587	601	614	627	640	654	667	680	693	706
29	720	733	746	759	772	786	799	812	825	838
830	851	865	878	891	904	917	930	943	957	970
31	983	996	*009	*022	*035	*048	*061	*075	*088	*101
32	52 114	127	140	153	166	179	192	205	218	231
33	244	257	270	284	297	310	323	336	349	362
34	375	388	401	414	427	440	453	466	479	492
35	504	517	530	543	556	569	582	595	608	621
36	634	647	660	673	686	699	711	724	737	750
37	763	776	789	802	815	827	840	853	866	879
38	892	905	917	930	943	956	969	982	994	*007
39	53 020	033	046	058	071	084	097	110	122	135
840	148	161	173	186	199	212	224	237	250	263
41	275	288	301	314	326	339	352	364	377	390
42	403	415	428	441	453	466	479	491	504	517
43	529	542	555	567	580	593	605	618	631	643
44	656	668	681	694	706	719	732	744	757	769
45	782	794	807	820	832	845	857	870	882	895
46	908	920	933	945	958	970	983	995	*008	*020
47	54 033	045	058	070	083	095	108	120	133	145
48	158	170	183	195	208	220	233	245	258	270
49	283	295	307	320	332	345	357	370	382	394
850	407	419	432	444	456	469	481	494	506	518
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**

Prop. Pts.

	15	14	13	12
1	1.5	1.4	1.3	1.2
2	3.0	2.8	2.6	2.4
3	4.5	4.2	3.9	3.6
4	6.0	5.6	5.2	4.8
5	7.5	7.0	6.5	6.0
6	9.0	8.4	7.8	7.2
7	10.5	9.8	9.1	8.4
8	12.0	11.2	10.4	9.6
9	13.5	12.6	11.7	10.8

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
850	54 407	419	432	444	456	469	481	494	506	518
51	531	543	555	568	580	593	605	617	630	642
52	654	667	679	691	704	716	728	741	753	765
53	777	790	802	814	827	839	851	864	876	888
54	900	913	925	937	949	962	974	986	998	*011
55	55 023	035	047	060	072	084	096	108	121	133
56	145	157	169	182	194	206	218	230	242	255
57	267	279	291	303	315	328	340	352	364	376
58	388	400	413	425	437	449	461	473	485	497
59	509	522	534	546	558	570	582	594	606	618
860	630	642	654	666	678	691	703	715	727	739
61	751	763	775	787	799	811	823	835	847	859
62	871	883	895	907	919	931	943	955	967	979
63	991	*003	*015	*027	*038	*050	*062	*074	*086	*098
64	56 110	122	134	146	158	170	182	194	205	217
65	229	241	253	265	277	289	301	312	324	336
66	348	360	372	384	396	407	419	431	443	455
67	467	478	490	502	514	526	538	549	561	573
68	585	597	608	620	632	644	656	667	679	691
69	703	714	726	738	750	761	773	785	797	808
870	820	832	844	855	867	879	891	902	914	926
71	937	949	961	972	984	996	*008	*019	*031	*043
72	57 054	066	078	089	101	113	124	136	148	159
73	171	183	194	206	217	229	241	252	264	276
74	287	299	310	322	334	345	357	368	380	392
75	403	415	426	438	449	461	473	484	496	507
76	519	530	542	553	565	576	588	600	611	623
77	634	646	657	669	680	692	703	715	726	738
78	749	761	772	784	795	807	818	830	841	852
79	864	875	887	898	910	921	933	944	955	967
880	978	990	*001	*013	*024	*035	*047	*058	*070	*081
81	58 092	104	115	127	138	149	161	172	184	195
82	206	218	229	240	252	263	274	286	297	309
83	320	331	343	354	365	377	388	399	410	422
84	433	444	456	467	478	490	501	512	524	535
85	546	557	569	580	591	602	614	625	636	647
86	659	670	681	692	704	715	726	737	749	760
87	771	782	794	805	816	827	838	850	861	872
88	883	894	906	917	928	939	950	961	973	984
89	995	*006	*017	*028	*040	*051	*062	*073	*084	*095
890	59 106	118	129	140	151	162	173	184	195	207
91	218	229	240	251	262	273	284	295	306	318
92	329	340	351	362	373	384	395	406	417	428
93	439	450	461	472	483	494	506	517	528	539
94	550	561	572	583	594	605	616	627	638	649
95	660	671	682	693	704	715	726	737	748	759
96	770	780	791	802	813	824	835	846	857	868
97	879	890	901	912	923	934	945	956	966	977
98	988	999	*010	*021	*032	*043	*054	*065	*076	*086
99	60 097	108	119	130	141	152	163	173	184	195
400	206	217	228	239	249	260	271	282	293	304
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	13
1	1.3
2	2.6
3	3.9
4	5.2
5	6.5
6	7.8
7	9.1
8	10.4
9	11.7

	12
1	1.2
2	2.4
3	3.6
4	4.8
5	6.0
6	7.2
7	8.4
8	9.6
9	10.8

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
400	60 206	217	228	239	249	260	271	282	293	304
01	314	325	336	347	358	369	379	390	401	412
02	423	433	444	455	466	477	487	498	509	520
03	531	541	552	563	574	584	595	606	617	627
04	638	649	660	670	681	692	703	713	724	735
05	746	756	767	778	788	799	810	821	831	842
06	853	863	874	885	895	906	917	927	938	949
07	959	970	981	991	*002	*013	*023	*034	*045	*055
08	61 066	077	087	098	109	119	130	140	151	162
09	172	183	194	204	215	225	236	247	257	268
410	278	289	300	310	321	331	342	352	363	374
11	384	395	405	416	426	437	448	458	469	479
12	490	500	511	521	532	542	553	563	574	584
13	595	606	616	627	637	648	658	669	679	690
14	700	711	721	731	742	752	763	773	784	794
15	805	815	826	836	847	857	868	878	888	899
16	909	920	930	941	951	962	972	982	993	*003
17	62 014	024	034	045	055	066	076	086	097	107
18	118	128	138	149	159	170	180	190	201	211
19	221	232	242	252	263	273	284	294	304	315
420	325	335	346	356	366	377	387	397	408	418
21	428	439	449	459	469	480	490	500	511	521
22	531	542	552	562	572	583	593	603	613	624
23	634	644	655	665	675	685	696	706	716	726
24	737	747	757	767	778	788	798	808	818	829
25	839	849	859	870	880	890	900	910	921	931
26	941	951	961	972	982	992	*002	*012	*022	*033
27	63 043	053	063	073	083	094	104	114	124	134
28	144	155	165	175	185	195	205	215	225	236
29	246	256	266	276	286	296	306	317	327	337
430	347	357	367	377	387	397	407	417	428	438
31	448	458	468	478	488	498	508	518	528	538
32	548	558	568	579	589	599	609	619	629	639
33	649	659	669	679	689	699	709	719	729	739
34	749	759	769	779	789	799	809	819	829	839
35	849	859	869	879	889	899	909	919	929	939
36	949	959	969	979	988	998	*008	*018	*028	*038
37	64 048	058	068	078	088	098	108	118	128	137
38	147	157	167	177	187	197	207	217	227	237
39	246	256	266	276	286	296	306	316	326	335
440	345	355	365	375	385	395	404	414	424	434
41	444	454	464	473	483	493	503	513	523	532
42	542	552	562	572	582	591	601	611	621	631
43	640	650	660	670	680	689	699	709	719	729
44	738	748	758	768	777	787	797	807	816	826
45	836	846	856	865	875	885	895	904	914	924
46	933	943	953	963	972	982	992	*002	*011	*021
47	65 031	040	050	060	070	079	089	099	108	118
48	128	137	147	157	167	176	186	196	205	215
49	225	234	244	254	263	273	283	292	302	312
450	321	331	341	350	360	369	379	389	398	408

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Prop. Pts.

	11
1	1.1
2	2.2
3	3.3
4	4.4
5	5.5
6	6.6
7	7.7
8	8.8
9	9.9

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
450	65 321	331	341	350	360	369	379	389	398	408
51	418	427	437	447	456	466	475	485	495	504
52	514	523	533	543	552	562	571	581	591	600
53	610	619	629	639	648	658	667	677	686	696
54	706	715	725	734	744	753	763	772	782	792
55	801	811	820	830	839	849	858	868	877	887
56	896	906	916	925	935	944	954	963	973	982
57	992	*001	*011	*020	*030	*039	*049	*058	*068	*077
58	66 087	096	106	115	124	134	143	153	162	172
59	181	191	200	210	219	229	238	247	257	266
460	276	285	295	304	314	323	332	342	351	361
61	370	380	389	398	408	417	427	436	445	455
62	464	474	483	492	502	511	521	530	539	549
63	558	567	577	586	596	605	614	624	633	642
64	652	661	671	680	689	699	708	717	727	736
65	745	755	764	773	783	792	801	811	820	829
66	839	848	857	867	876	885	894	904	913	922
67	932	941	950	960	969	978	987	997	*006	*015
68	67 025	034	043	052	062	071	080	089	099	108
69	117	127	136	145	154	164	173	182	191	201
470	210	219	228	237	247	256	265	274	284	293
71	302	311	321	330	339	348	357	367	376	385
72	394	403	413	422	431	440	449	459	468	477
73	486	495	504	514	523	532	541	550	560	569
74	578	587	596	605	614	624	633	642	651	660
75	669	679	688	697	706	715	724	733	742	752
76	761	770	779	788	797	806	815	825	834	843
77	852	861	870	879	888	897	906	916	925	934
78	943	952	961	970	979	988	997	*006	*015	*024
79	68 034	043	052	061	070	079	088	097	106	115
480	124	133	142	151	160	169	178	187	196	205
81	215	224	233	242	251	260	269	278	287	296
82	305	314	323	332	341	350	359	368	377	386
83	395	404	413	422	431	440	449	458	467	476
84	485	494	502	511	520	529	538	547	556	565
85	574	583	592	601	610	619	628	637	646	655
86	664	673	681	690	699	708	717	726	735	744
87	753	762	771	780	789	797	806	815	824	833
88	842	851	860	869	878	886	895	904	913	922
89	931	940	949	958	966	975	984	993	*002	*011
490	69 020	028	037	046	055	064	073	082	090	099
91	108	117	126	135	144	152	161	170	179	188
92	197	205	214	223	232	241	249	258	267	276
93	285	294	302	311	320	329	338	346	355	364
94	373	381	390	399	408	417	425	434	443	452
95	461	469	478	487	496	504	513	522	531	539
96	548	557	566	574	583	592	601	609	618	627
97	636	644	653	662	671	679	688	697	705	714
98	723	732	740	749	758	767	775	784	793	801
99	810	819	827	836	845	854	862	871	880	888
500	897	906	914	923	932	940	949	958	966	975

| N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

Prop. Pts.

	10
1	1.0
2	2.0
3	3.0
4	4.0
5	5.0
6	6.0
7	7.0
8	8.0
9	9.0

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

N.	0	1	2	3	4	5	6	7	8	9
500	69 897	906	914	923	932	940	949	958	966	975
01	984	992	*001	*010	*018	*027	*036	*044	*053	*062
02	70 070	079	088	096	105	114	122	131	140	148
03	157	165	174	183	191	200	209	217	226	234
04	243	252	260	269	278	286	295	303	312	321
05	329	338	346	355	364	372	381	389	398	406
06	415	424	432	441	449	458	467	475	484	492
07	501	509	518	526	535	544	552	561	569	578
08	586	595	603	612	621	629	638	646	655	663
09	672	680	689	697	706	714	723	731	740	749
510	757	766	774	783	791	800	808	817	825	834
11	842	851	859	868	876	885	893	902	910	919
12	927	935	944	952	961	969	978	986	995	*003
13	71 012	020	029	037	046	054	063	071	079	088
14	096	105	113	122	130	139	147	155	164	172
15	181	189	198	206	214	223	231	240	248	257
16	265	273	282	290	299	307	315	324	332	341
17	349	357	366	374	383	391	399	408	416	425
18	433	441	450	458	466	475	483	492	500	508
19	517	525	533	542	550	559	567	575	584	592
520	600	609	617	625	634	642	650	659	667	675
21	684	692	700	709	717	725	734	742	750	759
22	767	775	784	792	800	809	817	825	834	842
23	850	858	867	875	883	892	900	908	917	925
24	933	941	950	958	966	975	983	991	999	*008
25	72 016	024	032	041	049	057	066	074	082	090
26	099	107	115	123	132	140	148	156	165	173
27	181	189	198	206	214	222	230	239	247	255
28	263	272	280	288	296	304	313	321	329	337
29	346	354	362	370	378	387	395	403	411	419
530	428	436	444	452	460	469	477	48[illegible]	493	501
31	509	518	526	534	542	550	558	567	575	583
32	591	599	607	616	624	632	640	648	656	665
33	673	681	689	697	705	713	722	730	738	746
34	754	762	770	779	787	795	803	811	819	827
35	835	843	852	860	868	876	884	892	900	908
36	916	925	933	941	949	957	965	973	981	989
37	997	*006	*014	*022	*030	*038	*046	*054	*062	*070
38	73 078	086	094	102	111	119	127	135	143	151
39	159	167	175	183	191	199	207	215	223	231
540	239	247	255	263	272	280	288	296	304	312
41	320	328	336	344	352	360	368	376	384	392
42	400	408	416	424	432	440	448	456	464	472
43	480	488	496	504	512	520	528	536	544	552
44	560	568	576	584	592	600	608	616	624	632
45	640	648	656	664	672	679	687	695	703	711
46	719	727	735	743	751	759	767	775	783	791
47	799	807	815	823	830	838	846	854	862	870
48	878	886	894	902	910	918	926	933	941	949
49	957	965	973	981	989	997	*005	*013	*020	*028
550	74 036	044	052	060	068	076	084	092	099	107
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	9
1	0.9
2	1.8
3	2.7
4	3.6
5	4.5
6	5.4
7	6.3
8	7.2
9	8.1

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
550	74 036	044	052	060	068	076	084	092	099	107
51	115	123	131	139	147	155	162	170	178	186
52	194	202	210	218	225	233	241	249	257	265
53	273	280	288	296	304	312	320	327	335	343
54	351	359	367	374	382	390	398	406	414	421
55	429	437	445	453	461	468	476	484	492	500
56	507	515	523	531	539	547	554	562	570	578
57	586	593	601	609	617	624	632	640	648	656
58	663	671	679	687	695	702	710	718	726	733
59	741	749	757	764	772	780	788	796	803	811
560	819	827	834	842	850	858	865	873	881	889
61	896	904	912	920	927	935	943	950	958	966
62	974	981	989	997	*005	*012	*020	*028	*035	*043
63	75 051	059	066	074	082	089	097	105	113	120
64	128	136	143	151	159	166	174	182	189	197
65	205	213	220	228	236	243	251	259	266	274
66	282	289	297	305	312	320	328	335	343	351
67	358	366	374	381	389	397	404	412	420	427
68	435	442	450	458	465	473	481	488	496	504
69	511	519	526	534	542	549	557	565	572	580
570	587	595	603	610	618	626	633	641	648	656
71	664	671	679	686	694	702	709	717	724	732
72	740	747	755	762	770	778	785	793	800	808
73	815	823	831	838	846	853	861	868	876	884
74	891	899	906	914	921	929	937	944	952	959
75	967	974	982	989	997	*005	*012	*020	*027	*035
76	76 042	050	057	065	072	080	087	095	103	110
77	118	125	133	140	148	155	163	170	178	185
78	193	200	208	215	223	230	238	245	253	260
79	268	275	283	290	298	305	313	320	328	335
580	343	350	358	365	373	380	388	395	403	410
81	418	425	433	440	448	455	462	470	477	485
82	492	500	507	515	522	530	537	545	552	559
83	567	574	582	589	597	604	612	619	626	634
84	641	649	656	664	671	678	686	693	701	708
85	716	723	730	738	745	753	760	768	775	782
86	790	797	805	812	819	827	834	842	849	856
87	864	871	879	886	893	901	908	916	923	930
88	938	945	953	960	967	975	982	989	997	*004
89	77 012	019	026	034	041	048	056	063	070	078
590	085	093	100	107	115	122	129	137	144	151
91	159	166	173	181	188	195	203	210	217	225
92	232	240	247	254	262	269	276	283	291	298
93	305	313	320	327	335	342	349	357	364	371
94	379	386	393	401	408	415	422	430	437	444
95	452	459	466	474	481	488	495	503	510	517
96	525	532	539	546	554	561	568	576	583	590
97	597	605	612	619	627	634	641	648	656	663
98	670	677	685	692	699	706	714	721	728	735
99	743	750	757	764	772	779	786	793	801	808
600	815	822	830	837	844	851	859	866	873	880
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
600	77 815	822	830	837	844	851	859	866	873	880
01	887	895	902	909	916	924	931	938	945	952
02	960	967	974	981	988	996	*003	*010	*017	*025
03	78 032	039	046	053	061	068	075	082	089	097
04	104	111	118	125	132	140	147	154	161	168
05	176	183	190	197	204	211	219	226	233	240
06	247	254	262	269	276	283	290	297	305	312
07	319	326	333	340	347	355	362	369	376	383
08	390	398	405	412	419	426	433	440	447	455
09	462	469	476	483	490	497	504	512	519	526
610	533	540	547	554	561	569	576	583	590	597
11	604	611	618	625	63[illegible]	640	647	654	661	668
12	675	682	689	696	704	711	718	725	732	739
13	746	753	760	767	774	781	789	796	803	810
14	817	824	831	838	845	852	859	866	873	880
15	888	895	902	909	916	923	930	937	944	951
16	958	965	972	979	986	993	*000	*007	*014	*021
17	79 029	036	043	050	057	064	071	078	085	092
18	099	106	113	120	127	134	141	148	155	162
19	169	176	183	190	197	204	211	218	225	232
620	239	246	253	260	267	274	281	288	295	302
21	309	316	323	330	337	344	351	358	365	372
22	379	386	393	400	407	414	421	428	435	442
23	449	456	463	470	477	484	491	498	505	511
24	518	525	532	539	546	553	560	567	574	581
25	588	595	602	609	616	623	630	637	644	650
26	657	664	671	678	685	692	699	706	713	720
27	727	734	741	748	754	761	768	775	782	789
28	796	803	810	817	824	831	837	844	851	858
29	865	872	879	886	893	900	906	913	920	927
630	934	941	948	955	962	969	975	982	989	996
31	80 003	010	017	024	030	037	044	051	058	065
32	072	079	085	092	099	106	113	120	127	134
33	140	147	154	161	168	175	182	188	195	202
34	209	216	223	229	236	243	250	257	264	271
35	277	284	291	298	305	312	318	325	332	339
36	346	353	359	366	373	380	387	393	400	407
37	414	421	428	434	441	448	455	462	468	475
38	482	489	496	502	509	516	523	530	536	543
39	550	557	564	570	577	584	591	598	604	611
640	618	625	632	638	645	652	659	665	672	679
41	686	693	699	706	713	720	726	733	740	747
42	754	760	767	774	781	787	794	801	808	814
43	821	828	835	841	848	855	862	868	875	882
44	889	895	902	909	916	922	929	936	943	949
45	956	963	969	976	983	990	996	*003	*010	*017
46	81 023	030	037	043	050	057	064	070	077	084
47	090	097	104	111	117	124	131	137	144	151
48	158	164	171	178	184	191	198	204	211	218
49	224	231	238	245	251	258	265	271	278	285
650	291	298	305	311	318	325	331	338	345	351
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	8
1	0.8
2	1.6
3	2.4
4	3.2
5	4.0
6	4.8
7	5.6
8	6.4
9	7.2

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
650	81 291	298	305	311	318	325	331	338	345	351
51	358	365	371	378	385	391	398	405	411	418
52	425	431	438	445	451	458	465	471	478	485
53	491	498	505	511	518	525	531	538	544	551
54	558	564	571	578	584	591	598	604	611	617
55	624	631	637	644	651	657	664	671	677	684
56	690	697	704	710	717	723	730	737	743	750
57	757	763	770	776	783	790	796	803	809	816
58	823	829	836	842	849	856	862	869	875	882
59	889	895	902	908	915	921	928	935	941	948
660	954	961	968	974	981	987	994	*000	*007	*014
61	82 020	027	033	040	046	053	060	066	073	079
62	086	092	099	105	112	119	125	132	138	145
63	151	158	164	171	178	184	191	197	204	210
64	217	223	230	236	243	249	256	263	269	276
65	282	289	295	302	308	315	321	328	334	341
66	347	354	360	367	373	380	387	393	400	406
67	413	419	426	432	439	445	452	458	465	471
68	478	484	491	497	504	510	517	523	530	536
69	543	549	556	562	569	575	582	588	595	601
670	607	614	620	627	633	640	646	653	659	666
71	672	679	685	692	698	705	711	718	724	730
72	737	743	750	756	763	769	776	782	789	795
73	802	808	814	821	827	834	840	847	853	860
74	866	872	879	885	892	898	905	911	918	924
75	930	937	943	950	956	963	969	975	982	988
76	995	*001	*008	*014	*020	*027	*033	*040	*046	*052
77	83 059	065	072	078	085	091	097	104	110	117
78	123	129	136	142	149	155	161	168	174	181
79	187	193	200	206	213	219	225	232	238	245
680	251	257	264	270	276	283	289	296	302	308
81	315	321	327	334	340	347	353	359	366	372
82	378	385	391	398	404	410	417	423	429	436
83	442	448	455	461	467	474	480	487	493	499
84	506	512	518	525	531	537	544	550	556	563
85	569	575	582	588	594	601	607	613	620	626
86	632	639	645	651	658	664	670	677	683	689
87	696	702	708	715	721	727	734	740	746	753
88	759	765	771	778	784	790	797	803	809	816
89	822	828	835	841	847	853	860	866	872	879
690	885	891	897	904	910	916	923	929	935	942
91	948	954	960	967	973	979	985	992	998	*004
92	84 011	017	023	029	036	042	048	055	061	067
93	073	080	086	092	098	105	111	117	123	130
94	136	142	148	155	161	167	173	180	186	192
95	198	205	211	217	223	230	236	242	248	255
96	261	267	273	280	286	292	298	305	311	317
97	323	330	336	342	348	354	361	367	373	379
98	386	392	398	404	410	417	423	429	435	442
99	448	454	460	466	473	479	485	491	497	504
700	510	516	522	528	535	541	547	553	559	566
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
700	84 510	516	522	528	535	541	547	553	559	566
01	572	578	584	590	597	603	609	615	621	628
02	634	640	646	652	658	665	671	677	683	689
03	696	702	708	714	720	726	733	739	745	751
04	757	763	770	776	782	788	794	800	807	813
05	819	825	831	837	844	850	856	862	868	874
06	880	887	893	899	905	911	917	924	930	936
07	942	948	954	960	967	973	979	985	991	997
08	85 003	009	016	022	028	034	040	046	052	058
09	065	071	077	083	089	095	101	107	114	120
710	126	132	138	144	150	156	163	169	175	181
11	187	193	199	205	211	217	224	230	236	242
12	248	254	260	266	272	278	285	291	297	303
13	309	315	321	327	333	339	345	352	358	364
14	370	376	382	388	394	400	406	412	418	425
15	431	437	443	449	455	461	467	473	479	485
16	491	497	503	509	516	522	528	534	540	546
17	552	558	564	570	576	582	588	594	600	606
18	612	618	625	631	637	643	649	655	661	667
19	673	679	685	691	697	703	709	715	721	727
720	733	739	745	751	757	763	769	775	781	788
21	794	800	806	812	818	824	830	836	842	848
22	854	860	866	872	878	884	890	896	902	908
23	914	920	926	932	938	944	950	956	962	968
24	974	980	986	992	998	*004	*010	*016	*022	*028
25	86 034	040	046	052	058	064	070	076	082	088
26	094	100	106	112	118	124	130	136	141	147
27	153	159	165	171	177	183	189	195	201	207
28	213	219	225	231	237	243	249	255	261	267
29	273	279	285	291	297	303	308	314	320	326
730	332	338	344	350	356	362	368	374	380	386
31	392	398	404	410	415	421	427	433	439	445
32	451	457	463	469	475	481	487	493	499	504
33	510	516	522	528	534	540	546	552	558	564
34	570	576	581	587	593	599	605	611	617	623
35	629	635	641	646	652	658	664	670	676	682
36	688	694	700	705	711	717	723	729	735	741
37	747	753	759	764	770	776	782	788	794	800
38	806	812	817	823	829	835	841	847	853	859
39	864	870	876	882	888	894	900	906	911	917
740	923	929	935	941	947	953	958	964	970	976
41	982	988	994	999	*005	*011	*017	*023	*029	*035
42	87 040	046	052	058	064	070	075	081	087	093
43	099	105	111	116	122	128	134	140	146	151
44	157	163	169	175	181	186	192	198	204	210
45	216	221	227	233	239	245	251	256	262	268
46	274	280	286	291	297	303	309	315	320	326
47	332	338	344	349	355	361	367	373	379	384
48	390	396	402	408	413	419	425	431	437	442
49	448	454	460	466	471	477	483	489	495	500
750	506	512	518	523	529	535	541	547	552	558

N. | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | Prop. Pts.

Prop. Pts.

	7
1	0.7
2	1.4
3	2.1
4	2.8
5	3.5
6	4.2
7	4.9
8	5.6
9	6.3

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

N.	0	1	2	3	4	5	6	7	8	9
750	87 506	512	518	523	529	535	541	547	552	558
51	564	570	576	581	587	593	599	604	610	616
52	622	628	633	639	645	651	656	662	668	674
53	679	685	691	697	703	708	714	720	726	731
54	737	743	749	754	760	766	772	777	783	789
55	795	800	806	812	818	823	829	835	841	846
56	852	858	864	869	875	881	887	892	898	904
57	910	915	921	927	933	938	944	950	955	961
58	967	973	978	984	990	996	*001	*007	*013	*018
59	88 024	030	036	041	047	053	058	064	070	076
760	081	087	093	098	104	110	116	121	127	133
61	138	144	150	156	161	167	173	178	184	190
62	195	201	207	213	218	224	230	235	241	247
63	252	258	264	270	275	281	287	292	298	304
64	309	315	321	326	332	338	343	349	355	360
65	366	372	377	383	389	395	400	406	412	417
66	423	429	434	440	446	451	457	463	468	474
67	480	485	491	497	502	508	513	519	525	530
68	536	542	547	553	559	564	570	576	581	587
69	593	598	604	610	615	621	627	632	638	643
770	649	655	660	666	672	677	683	689	694	700
71	705	711	717	722	728	734	739	745	750	756
72	762	767	773	779	784	790	795	801	807	812
73	818	824	829	835	840	846	852	857	863	868
74	874	880	885	891	897	902	908	913	919	925
75	930	936	941	947	953	958	964	969	975	981
76	986	992	997	*003	*009	*014	*020	*025	*031	*037
77	89 042	048	053	059	064	070	076	081	087	092
78	098	104	109	115	120	126	131	137	143	148
79	154	159	165	170	176	182	187	193	198	204
780	209	215	221	226	232	237	243	248	254	260
81	265	271	276	282	287	293	298	304	310	315
82	321	326	332	337	343	348	354	360	365	371
83	376	382	387	393	398	404	409	415	421	426
84	432	437	443	448	454	459	465	470	476	481
85	487	492	498	504	509	515	520	526	531	537
86	542	548	553	559	564	570	575	581	586	592
87	597	603	609	614	620	625	631	636	642	647
88	653	658	664	669	675	680	686	691	697	702
89	708	713	719	724	730	735	741	746	752	757
790	763	768	774	779	785	790	796	801	807	812
91	818	823	829	834	840	845	851	856	862	867
92	873	878	883	889	894	900	905	911	916	922
93	927	933	938	944	949	955	960	966	971	977
94	982	988	993	998	*004	*009	*015	*020	*026	*031
95	90 037	042	048	053	059	064	069	075	080	086
96	091	097	102	108	113	119	124	129	135	140
97	146	151	157	162	168	173	179	184	189	195
98	200	206	211	217	222	227	233	238	244	249
99	255	260	266	271	276	282	287	293	298	304
800	309	314	320	325	331	336	342	347	352	358

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

N. 0 1 2 3 4 5 6 7 8 9 Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
800	90 309	314	320	325	331	336	342	347	352	358
01	363	369	374	380	385	390	396	401	407	412
02	417	423	428	434	439	445	450	455	461	466
03	472	477	482	488	493	499	504	509	515	520
04	526	531	536	542	547	553	558	563	569	574
05	580	585	590	596	601	607	612	617	623	628
06	634	639	644	650	655	660	666	671	677	682
07	687	693	698	703	709	714	720	725	730	736
08	741	747	752	757	763	768	773	779	784	789
09	795	800	806	811	816	822	827	832	838	843
810	849	854	859	865	870	875	881	886	891	897
11	902	907	913	918	924	929	934	940	945	950
12	956	961	966	972	977	982	988	993	998	*004
13	91 009	014	020	025	030	036	041	046	052	057
14	062	068	073	078	084	089	094	100	105	110
15	116	121	126	132	137	142	148	153	158	164
16	169	174	180	185	190	196	201	206	212	217
17	222	228	233	238	243	249	254	259	265	270
18	275	281	286	291	297	302	307	312	318	323
19	328	334	339	344	350	355	360	365	371	376
820	381	387	392	397	403	408	413	418	424	429
21	434	440	445	450	455	461	466	471	477	482
22	487	492	498	503	508	514	519	524	529	535
23	540	545	551	556	561	566	572	577	582	587
24	593	598	603	609	614	619	624	630	635	640
25	645	651	656	661	666	672	677	682	687	693
26	698	703	709	714	719	724	730	735	740	745
27	751	756	761	766	772	777	782	787	793	798
28	803	808	814	819	824	829	834	840	845	850
29	855	861	866	871	876	882	887	892	897	903
830	908	913	918	924	929	934	939	944	950	955
31	960	965	971	976	981	986	991	997	*002	*007
32	92 012	018	023	028	033	038	044	049	054	059
33	065	070	075	080	085	091	096	101	106	111
34	117	122	127	132	137	143	148	153	158	163
35	169	174	179	184	189	195	200	205	210	215
36	221	226	231	236	241	247	252	257	262	267
37	273	278	283	288	293	298	304	309	314	319
38	324	330	335	340	345	350	355	361	366	371
39	376	381	387	392	397	402	407	412	418	423
840	428	433	438	443	449	454	459	464	469	474
41	480	485	490	495	500	505	511	516	521	526
42	531	536	542	547	552	557	562	567	572	578
43	583	588	593	598	603	609	614	619	624	629
44	634	639	645	650	655	660	665	670	675	681
45	686	691	696	701	706	711	716	722	727	732
46	737	742	747	752	758	763	768	773	778	783
47	788	793	799	804	809	814	819	824	829	834
48	840	845	850	855	860	865	870	875	881	886
49	891	896	901	906	911	916	921	927	932	937
850	942	947	952	957	962	967	973	978	983	988
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	6
1	0.6
2	1.2
3	1.8
4	2.4
5	3.0
6	3.6
7	4.2
8	4.8
9	5.4

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9	Prop. Pts.
850	92 942	947	952	957	962	967	973	978	983	988	
51	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
52	93 044	049	054	059	064	069	075	080	085	090	
53	095	100	105	110	115	120	125	131	136	141	
54	146	151	156	161	166	171	176	181	186	192	
55	197	202	207	212	217	222	227	232	237	242	
56	247	252	258	263	268	273	278	283	288	293	6
57	298	303	308	313	318	323	328	334	339	344	1 0.6
58	349	354	359	364	369	374	379	384	389	394	2 1.2
59	399	404	409	414	420	425	430	435	440	445	3 1.8
860	450	455	460	465	470	475	480	485	490	495	4 2.4
61	500	505	510	515	520	526	531	536	541	546	5 3.0
62	551	556	561	566	571	576	581	586	591	596	6 3.6
63	601	606	611	616	621	626	631	636	641	646	7 4.2
64	651	656	661	666	671	676	682	687	692	697	8 4.8
65	702	707	712	717	722	727	732	737	742	747	9 5.4
66	752	757	762	767	772	777	782	787	792	797	
67	802	807	812	817	822	827	832	837	842	847	
68	852	857	862	867	872	877	882	887	892	897	
69	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
71	94 002	007	012	017	022	027	032	037	042	047	5
72	052	057	062	067	072	077	082	086	091	096	1 0.5
73	101	106	111	116	121	126	131	136	141	146	2 1.0
74	151	156	161	166	171	176	181	186	191	196	3 1.5
75	201	206	211	216	221	226	231	236	240	245	4 2.0
76	250	255	260	265	270	275	280	285	290	295	5 2.5
77	300	305	310	315	320	325	330	335	340	345	6 3.0
78	349	354	359	364	369	374	379	384	389	394	7 3.5
79	399	404	409	414	419	424	429	433	438	443	8 4.0
880	448	453	458	463	468	473	478	483	488	493	9 4.5
81	498	503	507	512	517	522	527	532	537	542	
82	547	552	557	562	567	571	576	581	586	591	
83	596	601	606	611	616	621	626	630	635	640	
84	645	650	655	660	665	670	675	680	685	689	
85	694	699	704	709	714	719	724	729	734	738	
86	743	748	753	758	763	768	773	778	783	787	4
87	792	797	802	807	812	817	822	827	832	836	1 0.4
88	841	846	851	856	861	866	871	876	880	885	2 0.8
89	890	895	900	905	910	915	919	924	929	934	3 1.2
890	939	944	949	954	959	963	968	973	978	983	4 1.6
91	988	993	998	*002	*007	*012	*017	*022	*027	*032	5 2.0
92	95 036	041	046	051	056	061	066	071	075	080	6 2.4
93	085	090	095	100	105	109	114	119	124	129	7 2.8
94	134	139	143	148	153	158	163	168	173	177	8 3.2
95	182	187	192	197	202	207	211	216	221	226	9 3.6
96	231	236	240	245	250	255	260	265	270	274	
97	279	284	289	294	299	303	308	313	318	323	
98	328	332	337	342	347	352	357	361	366	371	
99	376	381	386	390	395	400	405	410	415	419	
900	424	429	434	439	444	448	453	458	463	468	
N.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**Prop. Pts.**

N.	0	1	2	3	4	5	6	7	8	9
900	95 424	429	434	439	444	448	453	458	463	468
01	472	477	482	487	492	497	501	506	511	516
02	521	525	530	535	540	545̄	550	554	559	564
03	569	574	578	583	588	593	598	602	607	612
04	617	622	626	631	636	641	646	650	655	660
05	665̄	670	674	679	684	689	694	698	703	708
06	713	718	722	727	732	737	742	746	751	756
07	761	766	770	775	780	785̄	789	794	799	804
08	809	813	818	823	828	832	837	842	847	852
09	856	861	866	871	875	880	885	890	895̄	899
910	904	909	914	918	923	928	933	938	942	947
11	952	957	961	966	971	976	980	985	990	995̄
12	999	*004	*009	*014	*019	*023	*028	*033	*038	*042
13	96 047	052	057	061	066	071	076	080	085	090
14	095̄	099	104	109	114	118	123	128	133	137
15	142	147	152	156	161	166	171	175	180	185̄
16	190	194	199	204	209	213	218	223	227	232
17	237	242	246	251	256	261	265	270	275̄	280
18	284	289	294	298	303	308	313	317	322	327
19	332	336	341	346	350	355	360	365̄	369	374
920	379	384	388	393	398	402	407	412	417	421
21	426	431	435	440	445̄	450	454	459	464	468
22	473	478	483	487	492	497	501	506	511	515
23	520	525̄	530	534	539	544	548	553	558	562
24	567	572	577	581	586	591	595	600	605̄	609
25	614	619	624	628	633	638	642	647	652	656
26	661	666	670	675	680	685̄	689	694	699	703
27	708	713	717	722	727	731	736	741	745	750
28	755̄	759	764	769	774	778	783	788	792	797
29	802	806	811	816	820	825̄	830	834	839	844
930	848	853	858	862	867	872	876	881	886	890
31	895̄	900	904	909	914	918	923	928	932	937
32	942	946	951	956	960	965̄	970	974	979	984
33	988	993	997	*002	*007	*011	*016	*021	*025	*030
34	97 035̄	039	044	049	053	058	063	067	072	077
35	081	086	090	095	100	104	109	114	118	123
36	128	132	137	142	146	151	155	160	165̄	169
37	174	179	183	188	192	197	202	206	211	216
38	220	225̄	230	234	239	243	248	253	257	262
39	267	271	276	280	285	290	294	299	304	308
940	313	317	322	327	331	336	340	345	350	354
41	359	364	368	373	377	382	387	391	396	400
42	405	410	414	419	424	428	433	437	442	447
43	451	456	460	465̄	470	474	479	483	488	493
44	497	502	506	511	516	520	525̄	529	534	539
45	543	548	552	557	562	566	571	575	580	585̄
46	589	594	598	603	607	612	617	621	626	630
47	635̄	640	644	649	653	658	663	667	672	676
48	681	685	690	695̄	699	704	708	713	717	722
49	727	731	736	740	745̄	749	754	759	763	768
950	772	777	782	786	791	795	800	804	809	813
N.	0	1	2	3	4	5	6	7	8	9

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

Prop. Pts.

N.	0	1	2	3	4	5	6	7	8	9
950	97 772	777	782	786	791	795	800	804	809	813
51	818	823	827	832	836	841	845	850	855	859
52	864	868	873	877	882	886	891	896	900	905
53	909	914	918	923	928	932	937	941	946	950
54	955	959	964	968	973	978	982	987	991	996
55	98 000	005	009	014	019	023	028	032	037	041
56	046	050	055	059	064	068	073	078	082	087
57	091	096	100	105	109	114	118	123	127	132
58	137	141	146	150	155	159	164	168	173	177
59	182	186	191	195	200	204	209	214	218	223
960	227	232	236	241	245	250	254	259	263	268
61	272	277	281	286	290	295	299	304	308	313
62	318	322	327	331	336	340	345	349	354	358
63	363	367	372	376	381	385	390	394	399	403
64	408	412	417	421	426	430	435	439	444	448
65	453	457	462	466	471	475	480	484	489	493
66	498	502	507	511	516	520	525	529	534	538
67	543	547	552	556	561	565	570	574	579	583
68	588	592	597	601	605	610	614	619	623	628
69	632	637	641	646	650	655	659	664	668	673
970	677	682	686	691	695	700	704	709	713	717
71	722	726	731	735	740	744	749	753	758	762
72	767	771	776	780	784	789	793	798	802	807
73	811	816	820	825	829	834	838	843	847	851
74	856	860	865	869	874	878	883	887	892	896
75	900	905	909	914	918	923	927	932	936	941
76	945	949	954	958	963	967	972	976	981	985
77	989	994	998	*003	*007	*012	*016	*021	*025	*029
78	99 034	038	043	047	052	056	061	065	069	074
79	078	083	087	092	096	100	105	109	114	118
980	123	127	131	136	140	145	149	154	158	162
81	167	171	176	180	185	189	193	198	202	207
82	211	216	220	224	229	233	238	242	247	251
83	255	260	264	269	273	277	282	286	291	295
84	300	304	308	313	317	322	326	330	335	339
85	344	348	352	357	361	366	370	374	379	383
86	388	392	396	401	405	410	414	419	423	427
87	432	436	441	445	449	454	458	463	467	471
88	476	480	484	489	493	498	502	506	511	515
89	520	524	528	533	537	542	546	550	555	559
990	564	568	572	577	581	585	590	594	599	603
91	607	612	616	621	625	629	634	638	642	647
92	651	656	660	664	669	673	677	682	686	691
93	695	699	704	708	712	717	721	726	730	734
94	739	743	747	752	756	760	765	769	774	778
95	782	787	791	795	800	804	808	813	817	822
96	826	830	835	839	843	848	852	856	861	865
97	870	874	878	883	887	891	896	900	904	909
98	913	917	922	926	930	935	939	944	948	952
99	957	961	965	970	974	978	983	987	991	996
1000	00 000	004	009	013	017	022	026	030	035	039

Prop. Pts.

	5
1	0.5
2	1.0
3	1.5
4	2.0
5	2.5
6	3.0
7	3.5
8	4.0
9	4.5

	4
1	0.4
2	0.8
3	1.2
4	1.6
5	2.0
6	2.4
7	2.8
8	3.2
9	3.6

N. 0 1 2 3 4 5 6 7 8 9 Prop. Pts.

TABLE II.

LOGARITHMS

OF THE

SINE, COSINE, TANGENT, AND COTANGENT

FOR

EACH MINUTE OF THE QUADRANT.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	. . . .		. . . .		. . . .	0.00 000	60
1	6.46 373	30103	6.46 373	30103	3.53 627	0.00 000	59
2	6.76 476	17609	6.76 476	17609	3.23 524	0.00 000	58
3	6.94 085	12494	6.94 085	12494	3.05 915	0.00 000	57
4	7.06 579	9691	7.06 579	9691	2.93 421	0.00 000	56
5	7.16 270	7918	7.16 270	7918	2.83 730	0.00 000	55
6	7.24 188	6694	7.24 188	6694	2.75 812	0.00 000	54
7	7.30 882	5800	7.30 882	5800	2.69 118	0.00 000	53
8	7.36 682	5115	7.36 682	5115	2.63 318	0.00 000	52
9	7.41 797	4576	7.41 797	4576	2.58 203	0.00 000	51
10	7.46 373	4139	7.46 373	4139	2.53 627	0.00 000	50
11	7.50 512	3779	7.50 512	3779	2.49 488	0.00 000	49
12	7.54 291	3476	7.54 291	3476	2.45 709	0.00 000	48
13	7.57 767	3218	7.57 767	3219	2.42 233	0.00 000	47
14	7.60 985	2997	7.60 986	2996	2.39 014	0.00 000	46
15	7.63 982	2802	7.63 982	2803	2.36 018	0.00 000	45
16	7.66 784	2633	7.66 785	2633	2.33 215	0.00 000	44
17	7.69 417	2483	7.69 418	2482	2.30 582	9.99 999	43
18	7.71 900	2348	7.71 900	2348	2.28 100	9.99 999	42
19	7.74 248	2227	7.74 248	2228	2.25 752	9.99 999	41
20	7.76 475	2119	7.76 476	2119	2.23 524	9.99 999	40
21	7.78 594	2021	7.78 595	2020	2.21 405	9.99 999	39
22	7.80 615	1930	7.80 615	1931	2.19 385	9.99 999	38
23	7.82 545	1848	7.82 546	1848	2.17 454	9.99 999	37
24	7.84 393	1773	7.84 394	1773	2.15 606	9.99 999	36
25	7.86 166	1704	7.86 167	1704	2.13 833	9.99 999	35
26	7.87 870	1639	7.87 871	1639	2.12 129	9.99 999	34
27	7.89 509	1579	7.89 510	1579	2.10 490	9.99 999	33
28	7.91 088	1524	7.91 089	1524	2.08 911	9.99 999	32
29	7.92 612	1472	7.92 613	1473	2.07 387	9.99 998	31
30	7.94 084	1424	7.94 086	1424	2.05 914	9.99 998	30
31	7.95 508	1379	7.95 510	1379	2.04 490	9.99 998	29
32	7.96 887	1336	7.96 889	1336	2.03 111	9.99 998	28
33	7.98 223	1297	7.98 225	1297	2.01 775	9.99 998	27
34	7.99 520	1259	7.99 522	1259	2.00 478	9.99 998	26
35	8.00 779	1223	8.00 781	1223	1.99 219	9.99 998	25
36	8.02 002	1190	8.02 004	1190	1.97 996	9.99 998	24
37	8.03 192	1158	8.03 194	1159	1.96 806	9.99 997	23
38	8.04 350	1128	8.04 353	1128	1.95 647	9.99 997	22
39	8.05 478	1100	8.05 481	1100	1.94 519	9.99 997	21
40	8.06 578	1072	8.06 581	1072	1.93 419	9.99 997	20
41	8.07 650	1046	8.07 653	1047	1.92 347	9.99 997	19
42	8.08 696	1022	8.08 700	1022	1.91 300	9.99 997	18
43	8.09 718	999	8.09 722	998	1.90 278	9.99 997	17
44	8.10 717	976	8.10 720	976	1.89 280	9.99 996	16
45	8.11 693	954	8.11 696	955	1.88 304	9.99 996	15
46	8.12 647	934	8.12 651	934	1.87 349	9.99 996	14
47	8.13 581	914	8.13 585	915	1.86 415	9.99 996	13
48	8.14 495	896	8.14 500	895	1.85 500	9.99 996	12
49	8.15 391	877	8.15 395	878	1.84 605	9.99 996	11
50	8.16 268	860	8.16 273	860	1.83 727	9.99 995	10
51	8.17 128	843	8.17 133	843	1.82 867	9.99 995	9
52	8.17 971	827	8.17 976	828	1.82 024	9.99 995	8
53	8.18 798	812	8.18 804	812	1.81 196	9.99 995	7
54	8.19 610	797	8.19 616	797	1.80 384	9.99 995	6
55	8.20 407	782	8.20 413	782	1.79 587	9.99 994	5
56	8.21 189	769	8.21 195	769	1.78 805	9.99 994	4
57	8.21 958	755	8.21 964	756	1.78 036	9.99 994	3
58	8.22 713	743	8.22 720	742	1.77 280	9.99 994	2
59	8.23 456	730	8.23 462	730	1.76 538	9.99 994	1
60	8.24 186		8.24 192		1.75 808	9.99 993	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

d.	p p 1″
30103	501.72
17609	293.48
12494	208.23
9691	161.52
7918	131.97
6694	111.57
5800	96.67
5115	85.25
4576	76.27
4139	68.98
3779	62.98
3476	57.93
3219	53.65
3218	53.63
2997	49.95
2996	49.93
2803	46.72
2802	46.70
2633	43.88
2483	41.38
2482	41.37
2348	39.13
2228	37.13
2227	37.12
2119	35.32
2021	33.68
2020	33.67
1931	32.18
1930	32.17
1848	30.80
1773	29.55
1704	28.40
1639	27.32
1579	26.32
1524	25.40
1473	24.55
1472	24.53
1424	23.73
1379	22.98

d.	p. p. 1″	d.	p. p. 1″
1336	22.27	915	15.25
1297	21.62	914	15.23
1259	20.98	896	14.93
1223	20.38	895	14.92
1190	19.83	878	14.63
1159	19.32	877	14.62
1158	19.30	860	14.33
1128	18.80	843	14.05
1100	18.33	828	13.80
1072	17.87	827	13.78
1047	17.45	812	13.53
1046	17.43	797	13.28
1022	17.03	782	13.03
999	16.65	769	12.82
998	16.63	756	12.60
976	16.27	755	12.58
955	15.92	743	12.38
954	15.90	742	12.37
934	15.57	730	12.17

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.24 186		8.24 192		1.75 808	9.99 993	**60**
1	8.24 903	717	8.24 910	718	1.75 090	9.99 993	59
2	8.25 609	706	8.25 616	706	1.74 384	9.99 993	58
3	8.26 304	695	8.26 312	696	1.73 688	9.99 993	57
4	8.26 988	684	8.26 996	684	1.73 004	9.99 992	56
5	8.27 661	673	8.27 669	673	1.72 331	9.99 992	55
6	8.28 324	663	8.28 332	663	1.71 668	9.99 992	54
7	8.28 977	653	8.28 986	654	1.71 014	9.99 992	53
8	8.29 621	644	8.29 629	643	1.70 371	9.99 992	52
9	8.30 25$\bar{5}$	634	8.30 263	634	1.69 737	9.99 991	51
10	8.30 879	624	8.30 888	625	1.69 112	9.99 991	**50**
11	8.31 495	616	8.31 50$\bar{5}$	617	1.68 495	9.99 991	49
12	8.32 103	608	8.32 112	607	1.67 888	9.99 990	48
13	8.32 702	599	8.32 711	599	1.67 289	9.99 990	47
14	8.33 292	590	8.33 302	591	1.66 698	9.99 990	46
15	8.33 875	583	8.33 886	584	1.66 114	9.99 990	45
16	8.34 450	575	8.34 461	575	1.65 539	9.99 989	44
17	8.35 018	568	8.35 029	568	1.64 971	9.99 989	43
18	8.35 578	560	8.35 590	561	1.64 410	9.99 989	42
19	8.36 131	553	8.36 143	553	1.63 857	9.99 989	41
20	8.36 678	547	8.36 689	546	1.63 311	9.99 988	**40**
21	8.37 217	539	8.37 229	540	1.62 771	9.99 988	39
22	8.37 75$\bar{0}$	533	8.37 762	533	1.62 238	9.99 988	38
23	8.38 276	526	8.38 289	527	1.61 711	9.99 987	37
24	8.38 796	520	8.38 809	520	1.61 191	9.99 987	36
25	8.39 310	514	8.39 323	514	1.60 677	9.99 987	35
26	8.39 818	508	8.39 832	509	1.60 168	9.99 986	34
27	8.40 320	502	8.40 334	502	1.59 666	9.99 986	33
28	8.40 816	496	8.40 830	496	1.59 170	9.99 986	32
29	8.41 307	491	8.41 321	491	1.58 679	9.99 985	31
30	8.41 792	485	8.41 807	486	1.58 193	9.99 985	**30**
31	8.42 272	480	8.42 287	480	1.57 713	9.99 98$\bar{5}$	29
32	8.42 746	474	8.42 762	475	1.57 238	9.99 984	28
33	8.43 216	470	8.43 232	470	1.56 768	9.99 984	27
34	8.43 680	464	8.43 696	464	1.56 304	9.99 984	26
35	8.44 139	459	8.44 156	460	1.55 844	9.99 983	25
36	8.44 594	455	8.44 611	455	1.55 389	9.99 983	24
37	8.45 044	450	8.45 061	450	1.54 939	9.99 983	23
38	8.45 489	445	8.45 507	446	1.54 493	9.99 982	22
39	8.45 930	441	8.45 948	441	1.54 052	9.99 982	21
40	8.46 366	436	8.46 38$\bar{5}$	437	1.53 615	9.99 982	**20**
41	8.46 799	433	8.46 817	432	1.53 183	9.99 981	19
42	8.47 226	427	8.47 245	428	1.52 75$\bar{5}$	9.99 981	18
43	8.47 65$\bar{0}$	424	8.47 669	424	1.52 331	9.99 981	17
44	8.48 069	419	8.48 089	420	1.51 911	9.99 980	16
45	8.48 48$\bar{5}$	416	8.48 505	416	1.51 49$\bar{5}$	9.99 980	15
46	8.48 896	411	8.48 917	412	1.51 083	9.99 979	14
47	8.49 304	408	8.49 325	408	1.50 67$\bar{5}$	9.99 979	13
48	8.49 708	404	8.49 729	404	1.50 271	9.99 979	12
49	8.50 108	400	8.50 130	401	1.49 870	9.99 978	11
50	8.50 504	396	8.50 527	397	1.49 473	9.99 978	**10**
51	8.50 897	393	8.50 920	393	1.49 080	9.99 977	9
52	8.51 287	390	8.51 310	390	1.48 690	9.99 977	8
53	8.51 673	386	8.51 696	386	1.48 304	9.99 977	7
54	8.52 055	382	8.52 079	383	1.47 921	9.99 976	6
55	8.52 434	379	8.52 459	380	1.47 541	9.99 976	5
56	8.52 810	376	8.52 83$\bar{5}$	376	1.47 165	9.99 975	4
57	8.53 183	373	8.53 208	373	1.46 792	9.99 97$\bar{5}$	3
58	8.53 552	369	8.53 578	370	1.46 422	9.99 974	2
59	8.53 919	367	8.53 94$\bar{5}$	367	1.46 055	9.99 974	1
60	8.54 282	363	8.54 308	363	1.45 692	9.99 974	**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

d.	p. p. 1″	d.	p. p. 1″
718	11.97	485	8.08
717	11.95	480	8.00
706	11.77	475	7.92
696	11.60	474	7.90
695	11.58	470	7.83
684	11.40	464	7.73
673	11.22	460	7.67
663	11.05	459	7.65
654	10.90	455	7.58
653	10.88	450	7.50
644	10.73	446	7.43
643	10.72	445	7.42
634	10.57	441	7.35
625	10.42	437	7.28
624	10.40	436	7.27
617	10.28	433	7.22
616	10.27	432	7.20
608	10.13	428	7.13
607	10.12	427	7.12
599	9.98	424	7.07
591	9.85	420	7.00
590	9.83	419	6.98
584	9.73	416	6.93
583	9.72	412	6.87
575	9.58	411	6.85
568	9.47	408	6.80
561	9.35	404	6.73
560	9.33	401	6.68
553	9.22	400	6.67
547	9.12	397	6.62
546	9.10	396	6.60
540	9.00	393	6.55
539	8.98	390	6.50
533	8.88	386	6.43
527	8.78	383	6.38
526	8.77	382	6.37
520	8.67	380	6.33
514	8.57	379	6.32
509	8.48	376	6.27
508	8.47	373	6.22
502	8.37	370	6.17
496	8.27	369	6.15
491	8.18	367	6.12
486	8.10	363	6.05

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.54 282	360	8.54 308	361	1.45 692	9.99 974	60
1	8.54 642	357	8.54 669	358	1.45 331	9.99 973	59
2	8.54 999	355	8.55 027	355	1.44 973	9.99 973	58
3	8.55 354	351	8.55 382	352	1.44 618	9.99 972	57
4	8.55 705	349	8.55 734	349	1.44 266	9.99 972	56
5	8.56 054	346	8.56 083	346	1.43 917	9.99 971	55
6	8.56 400	343	8.56 429	344	1.43 571	9.99 971	54
7	8.56 743	341	8.56 773	341	1.43 227	9.99 970	53
8	8.57 084	337	8.57 114	338	1.42 886	9.99 970	52
9	8.57 421	336	8.57 452	336	1.42 548	9.99 969	51
10	8.57 757	332	8.57 788	333	1.42 212	9.99 969	50
11	8.58 089	330	8.58 121	330	1.41 879	9.99 968	49
12	8.58 419	328	8.58 451	328	1.41 549	9.99 968	48
13	8.58 747	325	8.58 779	326	1.41 221	9.99 967	47
14	8.59 072	323	8.59 105	323	1.40 895	9.99 967	46
15	8.59 395	320	8.59 428	321	1.40 572	9.99 967	45
16	8.59 715	318	8.59 749	319	1.40 251	9.99 966	44
17	8.60 033	316	8.60 068	316	1.39 932	9.99 966	43
18	8.60 349	313	8.60 384	314	1.39 616	9.99 965	42
19	8.60 662	311	8.60 698	311	1.39 302	9.99 964	41
20	8.60 973	309	8.61 009	310	1.38 991	9.99 964	40
21	8.61 282	307	8.61 319	307	1.38 681	9.99 963	39
22	8.61 589	305	8.61 626	305	1.38 374	9.99 963	38
23	8.61 894	302	8.61 931	303	1.38 069	9.99 962	37
24	8.62 196	301	8.62 234	301	1.37 766	9.99 962	36
25	8.62 497	298	8.62 535	299	1.37 465	9.99 961	35
26	8.62 795	296	8.62 834	297	1.37 166	9.99 961	34
27	8.63 091	294	8.63 131	295	1.36 869	9.99 960	33
28	8.63 385	293	8.63 426	292	1.36 574	9.99 960	32
29	8.63 678	290	8.63 718	291	1.36 282	9.99 959	31
30	8.63 968	288	8.64 009	289	1.35 991	9.99 959	30
31	8.64 256	287	8.64 298	287	1.35 702	9.99 958	29
32	8.64 543	284	8.64 585	285	1.35 415	9.99 958	28
33	8.64 827	283	8.64 870	284	1.35 130	9.99 957	27
34	8.65 110	281	8.65 154	281	1.34 846	9.99 956	26
35	8.65 391	279	8.65 435	280	1.34 565	9.99 956	25
36	8.65 670	277	8.65 715	278	1.34 285	9.99 955	24
37	8.65 947	276	8.65 993	276	1.34 007	9.99 955	23
38	8.66 223	274	8.66 269	274	1.33 731	9.99 954	22
39	8.66 497	272	8.66 543	273	1.33 457	9.99 954	21
40	8.66 769	270	8.66 816	271	1.33 184	9.99 953	20
41	8.67 039	269	8.67 087	269	1.32 913	9.99 952	19
42	8.67 308	267	8.67 356	268	1.32 644	9.99 952	18
43	8.67 575	266	8.67 624	266	1.32 376	9.99 951	17
44	8.67 841	263	8.67 890	264	1.32 110	9.99 951	16
45	8.68 104	263	8.68 154	263	1.31 846	9.99 950	15
46	8.68 367	260	8.68 417	261	1.31 583	9.99 949	14
47	8.68 627	259	8.68 678	260	1.31 322	9.99 949	13
48	8.68 886	258	8.68 938	258	1.31 062	9.99 948	12
49	8.69 144	256	8.69 196	257	1.30 804	9.99 948	11
50	8.69 400	254	8.69 453	255	1.30 547	9.99 947	10
51	8.69 654	253	8.69 708	254	1.30 292	9.99 946	9
52	8.69 907	252	8.69 962	252	1.30 038	9.99 946	8
53	8.70 159	250	8.70 214	251	1.29 786	9.99 945	7
54	8.70 409	249	8.70 465	249	1.29 535	9.99 944	6
55	8.70 658	247	8.70 714	248	1.29 286	9.99 944	5
56	8.70 905	246	8.70 962	246	1.29 038	9.99 943	4
57	8.71 151	244	8.71 208	245	1.28 792	9.99 942	3
58	8.71 395	243	8.71 453	244	1.28 547	9.99 942	2
59	8.71 638	242	8.71 697	243	1.28 303	9.99 941	1
60	8.71 880		8.71 940		1.28 060	9.99 940	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

d.	p. p. 1″	d.	p. p. 1″
361	6.02	291	4.85
360	6.00	290	4.83
358	5.97	289	4.82
357	5.95	288	4.80
355	5.92	287	4.78
352	5.87	285	4.75
351	5.85	284	4.73
349	5.82	283	4.72
346	5.77	281	4.68
344	5.73	280	4.67
343	5.72	279	4.65
341	5.68	278	4.63
338	5.63	277	4.62
337	5.62	276	4.60
336	5.60	274	4.57
333	5.55	273	4.55
332	5.53	272	4.53
330	5.50	271	4.52
328	5.47	270	4.50
326	5.43	269	4.48
325	5.42	268	4.47
323	5.38	267	4.45
321	5.35	266	4.43
320	5.33	264	4.40
319	5.32	263	4.38
318	5.30	261	4.35
316	5.27	260	4.33
314	5.23	259	4.32
313	5.22	258	4.30
311	5.18	257	4.28
310	5.17	256	4.27
309	5.15	255	4.25
307	5.12	254	4.23
305	5.08	253	4.22
303	5.05	252	4.20
302	5.03	251	4.18
301	5.02	250	4.17
299	4.98	249	4.15
298	4.97	248	4.13
297	4.95	247	4.12
296	4.93	246	4.10
295	4.92	245	4.08
294	4.90	244	4.07
293	4.88	243	4.05
292	4.87	242	4.03

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	8.71 880	240	8.71 940	241	1.28 060	9.99 940	60
1	8.72 120	239	8.72 181	239	1.27 819	9.99 940	59
2	8.72 359	238	8.72 420	239	1.27 580	9.99 939	58
3	8.72 597	237	8.72 659	237	1.27 341	9.99 938	57
4	8.72 834	235	8.72 896	236	1.27 104	9.99 938	56
5	8.73 069	234	8.73 132	234	1.26 868	9.99 937	55
6	8.73 303	232	8.73 366	234	1.26 634	9.99 936	54
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52
9	8.73 997	229	8.74 063	229	1.25 937	9.99 934	51
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	**50**
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	45
16	8.75 575	220	8.75 645	222	1.24 355	9.99 929	44
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43
18	8.76 015	219	8.76 087	219	1.23 913	9.99 928	42
19	8.76 234	217	8.76 306	219	1.23 694	9.99 927	41
20	8.76 451	216	8.76 525	217	1.23 475	9.99 926	**40**
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925	38
23	8.77 097	213	8.77 173	214	1.22 827	9.99 924	37
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	35
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34
27	8.77 943	209	8.78 022	210	1.21 978	9.99 921	33
28	8.78 152	208	8.78 232	209	1.21 768	9.99 920	32
29	8.78 360	208	8.78 441	208	1.21 559	9.99 920	31
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	**30**
31	8.78 774	205	8.78 855	206	1.21 145	9.99 918	29
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28
33	8.79 183	203	8.79 266	204	1.20 734	9.99 917	27
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	25
36	8.79 789	201	8.79 875	201	1.20 125	9.99 914	24
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	**20**
41	8.80 782	196	8.80 872	196	1.19 128	9.99 910	19
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18
43	8.81 173	194	8.81 264	195	1.18 736	9.99 909	17
44	8.81 367	193	8.81 459	194	1.18 541	9.99 908	16
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	15
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12
49	8.82 324	189	8.82 420	190	1.17 580	9.99 904	11
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	**10**
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8
53	8.83 075	186	8.83 175	186	1.16 825	9.99 900	7
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	5
56	8.83 630	183	8.83 732	184	1.16 268	9.99 898	4
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895	1
60	8.84 358		8.84 464		1.15 536	9.99 894	**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	238	234	229
6	23.8	23.4	22.9
7	27.8	27.3	26.7
8	31.7	31.2	30.5
9	35.7	35.1	34.4
10	39.7	39.0	38.2
20	79.3	78.0	76.3
30	119.0	117.0	114.5
40	158.7	156.0	152.7
50	198.3	195.0	190.8

	225	220	216
6	22.5	22.0	21.6
7	26.3	25.7	25.2
8	30.0	29.3	28.8
9	33.8	33.0	32.4
10	37.5	36.7	36.0
20	75.0	73.3	72.0
30	112.5	110.0	108.0
40	150.0	146.7	144.0
50	187.5	183.3	180.0

	212	208	204
6	21.2	20.8	20.4
7	24.7	24.3	23.8
8	28.3	27.7	27.2
9	31.8	31.2	30.6
10	35.3	34.7	34.0
20	70.7	69.3	68.0
30	106.0	104.0	102.0
40	141.3	138.7	136.0
50	176.7	173.3	170.0

	201	197	193
6	20.1	19.7	19.3
7	23.5	23.0	22.5
8	26.8	26.3	25.7
9	30.2	29.6	29.0
10	33.5	32.8	32.2
20	67.0	65.7	64.3
30	100.5	98.5	96.5
40	134.0	131.3	128.7
50	167.5	164.2	160.8

	189	185	181
6	18.9	18.5	18.1
7	22.1	21.6	21.1
8	25.2	24.7	24.1
9	28.4	27.8	27.2
10	31.5	30.8	30.2
20	63.0	61.7	60.3
30	94.5	92.5	90.5
40	126.0	123.3	120.7
50	157.5	154.2	150.8

	4	3	2	1
6	0.4	0.3	0.2	0.1
7	0.5	0.4	0.2	0.1
8	0.5	0.4	0.3	0.1
9	0.6	0.5	0.3	0.2
10	0.7	0.5	0.3	0.2
20	1.3	1.0	0.7	0.3
30	2.0	1.5	1.0	0.5
40	2.7	2.0	1.3	0.7
50	3.3	2.5	1.7	0.8

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.84 358	181	8.84 464	182	1.15 536	9.99 894	60
1	8.84 539	179	8.84 646	180	1.15 354	9.99 893	59
2	8.84 718	179	8.84 826	180	1.15 174	9.99 892	58
3	8.84 897	178	8.85 006	179	1.14 994	9.99 891	57
4	8.85 075	177	8.85 185̄	178	1.14 815	9.99 891	56
5	8.85 252	177	8.85 363	177	1.14 637	9.99 890	55
6	8.85 429	176	8.85 540	177	1.14 460	9.99 889	54
7	8.85 605̄	175	8.85 717	176	1.14 283	9.99 888	53
8	8.85 780	175	8.85 893	176	1.14 107	9.99 887	52
9	8.85 955̄	173	8.86 069	174	1.13 931	9.99 886	51
10	8.86 128	173	8.86 243	174	1.13 757	9.99 885	50
11	8.86 301	173	8.86 417	174	1.13 583	9.99 884	49
12	8.86 474	171	8.86 591	172	1.13 409	9.99 883	48
13	8.86 645	171	8.86 763	172	1.13 237	9.99 882	47
14	8.86 816	171	8.86 935	171	1.13 065̄	9.99 881	46
15	8.86 987	169	8.87 106	171	1.12 894	9.99 880	45
16	8.87 156	169	8.87 277	170	1.12 723	9.99 879	44
17	8.87 325	169	8.87 447	169	1.12 553	9.99 879	43
18	8.87 494	167	8.87 616	169	1.12 384	9.99 878	42
19	8.87 661	168	8.87 785̄	168	1.12 215	9.99 877	41
20	8.87 829	166	8.87 953	167	1.12 047	9.99 876	40
21	8.87 995̄	166	8.88 120	167	1.11 880	9.99 875̄	39
22	8.88 161	165	8.88 287	166	1.11 713	9.99 874	38
23	8.88 326	164	8.88 453	165	1.11 547	9.99 873	37
24	8.88 490	164	8.88 618	165	1.11 382	9.99 872	36
25	8.88 654	163	8.88 783	165	1.11 217	9.99 871	35
26	8.88 817	163	8.88 948	163	1.11 052	9.99 870	34
27	8.88 980	162	8.89 111	163	1.10 889	9.99 869	33
28	8.89 142	162	8.89 274	163	1.10 726	9.99 868	32
29	8.89 304	160	8.89 437	161	1.10 563	9.99 867	31
30	8.89 464	161	8.89 598	162	1.10 402	9.99 866	30
31	8.89 625̄	159	8.89 760	160	1.10 240	9.99 865̄	29
32	8.89 784	159	8.89 920	160	1.10 080	9.99 864	28
33	8.89 943	159	8.90 080	160	1.09 920	9.99 863	27
34	8.90 102	158	8.90 240	159	1.09 760	9.99 862	26
35	8.90 260	157	8.90 399	158	1.09 601	9.99 861	25
36	8.90 417	157	8.90 557	158	1.09 443	9.99 860	24
37	8.90 574	156	8.90 715̄	157	1.09 285	9.99 859	23
38	8.90 730	155	8.90 872	157	1.09 128	9.99 858	22
39	8.90 885	155	8.91 029	156	1.08 971	9.99 857	21
40	8.91 040	155	8.91 185̄	155	1.08 815	9.99 856	20
41	8.91 195̄	154	8.91 340	155	1.08 660	9.99 855̄	19
42	8.91 349	153	8.91 495	155	1.08 505̄	9.99 854	18
43	8.91 502	153	8.91 650	153	1.08 350	9.99 853	17
44	8.91 655	152	8.91 803	154	1.08 197	9.99 852	16
45	8.91 807	152	8.91 957	153	1.08 043	9.99 851	15
46	8.91 959	151	8.92 110	152	1.07 890	9.99 850	14
47	8.92 110	151	8.92 262	152	1.07 738	9.99 848	13
48	8.92 261	150	8.92 414	151	1.07 586	9.99 847	12
49	8.92 411	150	8.92 565̄	151	1.07 435	9.99 846	11
50	8.92 561	149	8.92 716	150	1.07 284	9.99 845	10
51	8.92 710	149	8.92 866	150	1.07 134	9.99 844	9
52	8.92 859	148	8.93 016	149	1.06 984	9.99 843	8
53	8.93 007	147	8.93 165̄	148	1.06 835	9.99 842	7
54	8.93 154	147	8.93 313	149	1.06 687	9.99 841	6
55	8.93 301	147	8.93 462	147	1.06 538	9.99 840	5
56	8.93 448	146	8.93 609	147	1.06 391	9.99 839	4
57	8.93 594	146	8.93 756	147	1.06 244	9.99 838	3
58	8.93 740	145	8.93 903	146	1.06 097	9.99 837	2
59	8.93 885̄	145	8.94 049	146	1.05 951	9.99 836	1
60	8.94 030		8.94 195		1.05 805̄	9.99 834	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	180	177	174
6	18.0	17.7	17.4
7	21.0	20.7	20.3
8	24.0	23.6	23.2
9	27.0	26.6	26.1
10	30.0	29.5	29.0
20	60.0	59.0	58.0
30	90.0	88.5	87.0
40	120.0	118.0	116.0
50	150.0	147.5	145.0

	171	169	167
6	17.1	16.9	16.7
7	20.0	19.7	19.5̄
8	22.8	22.5	22.3
9	25.7	25.4	25.1
10	28.5	28.2	27.8
20	57.0	56.3	55.7
30	85.5	84.5	83.5
40	114.0	112.7	111.3
50	142.5	140.8	139.2

	165	163	160
6	16.5	16.3	16.0
7	19.3	19.0	18.7
8	22.0	21.7	21.3
9	24.8	24.5̄	24.0
10	27.5	27.2	26.7
20	55.0	54.3	53.3
30	82.5	81.5	80.0
40	110.0	108.7	106.7
50	137.5	135.8	133.3

	157	155	153
6	15.7	15.5	15.3
7	18.3	18.1	17.9
8	20.9	20.7	20.4
9	23.6	23.3	23.0
10	26.2	25.8	25.5
20	52.3	51.7	51.0
30	78.5	77.5	76.5
40	104.7	103.3	102.0
50	130.8	129.2	127.5

	151	149	147
6	15.1	14.9	14.7
7	17.6	17.4	17.2
8	20.1	19.9	19.6
9	22.7	22.4	22.1
10	25.2	24.8	24.5
20	50.3	49.7	49.0
30	75.5	74.5	73.5
40	100.7	99.3	98.0
50	125.8	124.2	122.5

	146	2	1
6	14.6	0.2	0.1
7	17.0	0.2	0.1
8	19.5̄	0.3	0.1
9	21.9	0.3	0.2
10	24.3	0.3	0.2
20	48.7	0.7	0.3
30	73.0	1.0	0.5
40	97.3	1.3	0.7
50	121.7	1.7	0.8

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	8.94 030	144	8.94 195	145	1.05 805	9.99 834	60
1	8.94 174	143	8.94 340	145	1.05 660	9.99 833	59
2	8.94 317	144	8.94 485	145	1.05 515	9.99 832	58
3	8.94 461	142	8.94 630	143	1.05 370	9.99 831	57
4	8.94 603	143	8.94 773	144	1.05 227	9.99 830	56
5	8.94 746	141	8.94 917	143	1.05 083	9.99 829	55
6	8.94 887	142	8.95 060	142	1.04 940	9.99 828	54
7	8.95 029	141	8.95 202	142	1.04 798	9.99 827	53
8	8.95 170	140	8.95 344	142	1.04 656	9.99 825	52
9	8.95 310	140	8.95 486	141	1.04 514	9.99 824	51
10	8.95 450	139	8.95 627	140	1.04 373	9.99 823	50
11	8.95 589	139	8.95 767	141	1.04 233	9.99 822	49
12	8.95 728	139	8.95 908	139	1.04 092	9.99 821	48
13	8.95 867	138	8.96 047	140	1.03 953	9.99 820	47
14	8.96 005	138	8.96 187	138	1.03 813	9.99 819	46
15	8.96 143	137	8.96 325	139	1.03 675	9.99 817	45
16	8.96 280	137	8.96 464	138	1.03 536	9.99 816	44
17	8.96 417	136	8.96 602	137	1.03 398	9.99 815	43
18	8.96 553	136	8.96 739	138	1.03 261	9.99 814	42
19	8.96 689	136	8.96 877	136	1.03 123	9.99 813	41
20	8.96 825	135	8.97 013	137	1.02 987	9.99 812	40
21	8.96 960	135	8.97 150	135	1.02 850	9.99 810	39
22	8.97 095	134	8.97 285	136	1.02 715	9.99 809	38
23	8.97 229	134	8.97 421	135	1.02 579	9.99 808	37
24	8.97 363	133	8.97 556	135	1.02 444	9.99 807	36
25	8.97 496	133	8.97 691	134	1.02 309	9.99 806	35
26	8.97 629	133	8.97 825	134	1.02 175	9.99 804	34
27	8.97 762	132	8.97 959	133	1.02 041	9.99 803	33
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32
29	8.98 026	131	8.98 225	133	1.01 775	9.99 801	31
30	8.98 157	131	8.98 358	132	1.01 642	9.99 800	30
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29
32	8.98 419	130	8.98 622	131	1.01 378	9.99 797	28
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27
34	8.98 679	129	8.98 884	131	1.01 116	9.99 795	26
35	8.98 808	129	8.99 015	130	1.00 985	9.99 793	25
36	8.98 937	129	8.99 145	130	1.00 855	9.99 792	24
37	8.99 066	128	8.99 275	130	1.00 725	9.99 791	23
38	8.99 194	128	8.99 405	129	1.00 595	9.99 790	22
39	8.99 322	128	8.99 534	128	1.00 466	9.99 788	21
40	8.99 450	127	8.99 662	129	1.00 338	9.99 787	20
41	8.99 577	127	8.99 791	128	1.00 209	9.99 786	19
42	8.99 704	126	8.99 919	127	1.00 081	9.99 785	18
43	8.99 830	126	9.00 046	128	0.99 954	9.99 783	17
44	8.99 956	126	9.00 174	127	0.99 826	9.99 782	16
45	9.00 082	125	9.00 301	126	0.99 699	9.99 781	15
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14
47	9.00 332	124	9.00 553	126	0.99 447	9.99 778	13
48	9.00 456	125	9.00 679	126	0.99 321	9.99 777	12
49	9.00 581	123	9.00 805	125	0.99 195	9.99 776	11
50	9.00 704	124	9.00 930	125	0.99 070	9.99 775	10
51	9.00 828	123	9.01 055	124	0.98 945	9.99 773	9
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8
53	9.01 074	122	9.01 303	124	0.98 697	9.99 771	7
54	9.01 196	122	9.01 427	123	0.98 573	9.99 769	6
55	9.01 318	122	9.01 550	123	0.98 450	9.99 768	5
56	9.01 440	121	9.01 673	123	0.98 327	9.99 767	4
57	9.01 561	121	9.01 796	122	0.98 204	9.99 765	3
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2
59	9.01 803	120	9.02 040	122	0.97 960	9.99 763	1
60	9.01 923		9.02 162		0.97 838	9.99 761	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	145	143	141
6	14.5	14.3	14.1
7	16.9	16.7	16.5
8	19.3	19.1	18.8
9	21.8	21.5	21.2
10	24.2	23.8	23.5
20	48.3	47.7	47.0
30	72.5	71.5	70.5
40	96.7	95.3	94.0
50	120.8	119.2	117.5

	139	138	136
6	13.9	13.8	13.6
7	16.2	16.1	15.9
8	18.5	18.4	18.1
9	20.9	20.7	20.4
10	23.2	23.0	22.7
20	46.3	46.0	45.3
30	69.5	69.0	68.0
40	92.7	92.0	90.7
50	115.8	115.0	113.3

	135	133	131
6	13.5	13.3	13.1
7	15.8	15.5	15.3
8	18.0	17.7	17.5
9	20.3	20.0	19.7
10	22.5	22.2	21.8
20	45.0	44.3	43.7
30	67.5	66.5	65.5
40	90.0	88.7	87.3
50	112.5	110.8	109.2

	129	128	126
6	12.9	12.8	12.6
7	15.1	14.9	14.7
8	17.2	17.1	16.8
9	19.4	19.2	18.9
10	21.5	21.3	21.0
20	43.0	42.7	42.0
30	64.5	64.0	63.0
40	86.0	85.3	84.0
50	107.5	106.7	105.0

	125	123	122
6	12.5	12.3	12.2
7	14.6	14.4	14.2
8	16.7	16.4	16.3
9	18.8	18.5	18.3
10	20.8	20.5	20.3
20	41.7	41.0	40.7
30	62.5	61.5	61.0
40	83.3	82.0	81.3
50	104.2	102.5	101.7

	121	120	1
6	12.1	12.0	0.1
7	14.1	14.0	0.1
8	16.1	16.0	0.1
9	18.2	18.0	0.2
10	20.2	20.0	0.2
20	40.3	40.0	0.3
30	60.5	60.0	0.5
40	80.7	80.0	0.7
50	100.8	100.0	0.8

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.01 923	120	9.02 162	121	0.97 838	9.99 761	60
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	119	9.02 525	120	0.97 475̄	9.99 757	57
4	9.02 402	118	9.02 645	121	0.97 355̄	9.99 756	56
5	9.02 520	119	9.02 766	119	0.97 234	9.99 755̄	55
6	9.02 639	118	9.02 885	120	0.97 115̄	9.99 753	54
7	9.02 757	117	9.03 005̄	119	0.96 995	9.99 752	53
8	9.02 874	118	9.03 124	118	0.96 876	9.99 751	52
9	9.02 992	117	9.03 242	119	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	118	0.96 639	9.99 748	50
11	9.03 226	116	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	117	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	118	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	116	0.96 168	9.99 742	46
15	9.03 690	115	9.03 948	117	0.96 052	9.99 741	45
16	9.03 805̄	115	9.04 065	116	0.95 935̄	9.99 740	44
17	9.03 920	114	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	115	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	113	9.04 413	115	0.95 587	9.99 736	41
20	9.04 262	114	9.04 528	115	0.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	113	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	112	9.04 873	114	0.95 127	9.99 730	37
24	9.04 715	113	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	112	9.05 101	113	0.94 899	9.99 727	35
26	9.04 940	112	9.05 214	114	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	113	0.94 672	9.99 724	33
28	9.05 164	111	9.05 441	112	0.94 559	9.99 723	32
29	9.05 275̄	111	9.05 553	113	0.94 447	9.99 721	31
30	9.05 386	111	9.05 666	112	0.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	109	9.06 224	111	0.93 776	9.99 713	25
36	9.06 046	109	9.06 335̄	110	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	111	0.93 555̄	9.99 710	23
38	9.06 264	108	9.06 556	110	0.93 444	9.99 708	22
39	9.06 372	109	9.06 666	109	0.93 334	9.99 707	21
40	9.06 481	108	9.06 775	110	0.93 225̄	9.99 705	20
41	9.06 589	107	9.06 885̄	109	0.93 115	9.99 704	19
42	9.06 696	108	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	107	9.07 103	108	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	109	0.92 789	9.99 699	16
45	9.07 018	106	9.07 320	108	0.92 680	9.99 698	15
46	9.07 124	107	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	106	9.07 536	107	0.92 464	9.99 695̄	13
48	9.07 337	105	9.07 643	108	0.92 357	9.99 693	12
49	9.07 442	106	9.07 751	107	0.92 249	9.99 692	11
50	9.07 548	105	9.07 858	106	0.92 142	9.99 690	10
51	9.07 653	105	9.07 964	107	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	106	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	104	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	5
56	9.08 176	104	9.08 495̄	105	0.91 505	9.99 681	4
57	9.08 280	103	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295̄	9.99 678	2
59	9.08 486	103	9.08 810	104	0.91 190	9.99 677	1
60	9.08 589		9.08 914		0.91 086	9.99 675	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	121	120	119
6	12.1	12.0	11.9
7	14.1	14.0	13.9
8	16.1	16.0	15.9
9	18.2	18.0	17.9
10	20.2	20.0	19.8
20	40.3	40.0	39.7
30	60.5	60.0	59.5
40	80.7	80.0	79.3
50	100.8	100.0	99.2

	118	117	116
6	11.8	11.7	11.6
7	13.8	13.7	13.5̄
8	15.7	15.6	15.5̄
9	17.7	17.6	17.4
10	19.7	19.5	19.3
20	39.3	39.0	38.7
30	59.0	58.5	58.0
40	78.7	78.0	77.3
50	98.3	97.5	96.7

	115	114	113
6	11.5	11.4	11.3
7	13.4	13.3	13.2
8	15.3	15.2	15.1
9	17.3	17.1	17.0
10	19.2	19.0	18.8
20	38.3	38.0	37.7
30	57.5	57.0	56.5
40	76.7	76.0	75.3
50	95.8	95.0	94.2

	112	111	110
6	11.2	11.1	11.0
7	13.1	13.0	12.8
8	14.9	14.8	14.7
9	16.8	16.7	16.5
10	18.7	18.5	18.3
20	37.3	37.0	36.7
30	56.0	55.5	55.0
40	74.7	74.0	73.3
50	93.3	92.5	91.7

	109	108	107
6	10.9	10.8	10.7
7	12.7	12.6	12.5̄
8	14.5	14.4	14.3
9	16.4	16.2	16.1
10	18.2	18.0	17.8
20	36.3	36.0	35.7
30	54.5	54.0	53.5
40	72.7	72.0	71.3
50	90.8	90.0	89.2

	106	105	104
6	10.6	10.5	10.4
7	12.4	12.3	12.1
8	14.1	14.0	13.9
9	15.9	15.8	15.6
10	17.7	17.5	17.3
20	35.3	35.0	34.7
30	53.0	52.5	52.0
40	70.7	70.0	69.3
50	88.3	87.5	86.7

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	′
0	9.08 589	103	9.08 914	105	0.91 086	9.99 675	60
1	9.08 692	103	9.09 019	104	0.90 981	9.99 674	59
2	9.08 795	102	9.09 123	104	0.90 877	9.99 672	58
3	9.08 897	102	9.09 227	103	0.90 773	9.99 670	57
4	9.08 999	102	9.09 330	104	0.90 670	9.99 669	56
5	9.09 101	101	9.09 434	103	0.90 566	9.99 667	55
6	9.09 202	102	9.09 537	103	0.90 463	9.99 666	54
7	9.09 304	101	9.09 640	102	0.90 360	9.99 664	53
8	9.09 405	101	9.09 742	103	0.90 258	9.99 663	52
9	9.09 506	100	9.09 845	102	0.90 155	9.99 661	51
10	9.09 606	101	9.09 947	102	0.90 053	9.99 659	**50**
11	9.09 707	100	9.10 049	101	0.89 951	9.99 658	49
12	9.09 807	100	9.10 150	102	0.89 850	9.99 656	48
13	9.09 907	99	9.10 252	101	0.89 748	9.99 655	47
14	9.10 006	100	9.10 353	101	0.89 647	9.99 653	46
15	9.10 106	99	9.10 454	101	0.89 546	9.99 651	45
16	9.10 205	99	9.10 555	101	0.89 445	9.99 650	44
17	9.10 304	98	9.10 656	100	0.89 344	9.99 648	43
18	9.10 402	99	9.10 756	100	0.89 244	9.99 647	42
19	9.10 501	98	9.10 856	100	0.89 144	9.99 645	41
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	**40**
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39
22	9.10 795	98	9.11 155	99	0.88 845	9.99 640	38
23	9.10 893	97	9.11 254	99	0.88 746	9.99 638	37
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	35
26	9.11 184	97	9.11 551	98	0.88 449	9.99 633	34
27	9.11 281	96	9.11 649	98	0.88 351	9.99 632	33
28	9.11 377	97	9.11 747	98	0.88 253	9.99 630	32
29	9.11 474	96	9.11 845	98	0.88 155	9.99 629	31
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	**30**
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29
32	9.11 761	96	9.12 138	97	0.87 862	9.99 624	28
33	9.11 857	95	9.12 235	97	0.87 765	9.99 622	27
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	25
36	9.12 142	94	9.12 525	96	0.87 475	9.99 617	24
37	9.12 236	95	9.12 621	96	0.87 379	9.99 615	23
38	9.12 331	94	9.12 717	96	0.87 283	9.99 613	22
39	9.12 425	94	9.12 813	96	0.87 187	9.99 612	21
40	9.12 519	93	9.12 909	95	0.87 091	9.99 610	**20**
41	9.12 612	94	9.13 004	95	0.86 996	9.99 608	19
42	9.12 706	93	9.13 099	95	0.86 901	9.99 607	18
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605	17
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16
45	9.12 985	93	9.13 384	94	0.86 616	9.99 601	15
46	9.13 078	93	9.13 478	95	0.86 522	9.99 600	14
47	9.13 171	92	9.13 573	94	0.86 427	9.99 598	13
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12
49	9.13 355	92	9.13 761	93	0.86 239	9.99 595	11
50	9.13 447	92	9.13 854	94	0.86 146	9.99 593	**10**
51	9.13 539	91	9.13 948	93	0.86 052	9.99 591	9
52	9.13 630	92	9.14 041	93	0.85 959	9.99 589	8
53	9.13 722	91	9.14 134	93	0.85 866	9.99 588	7
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6
55	9.13 904	90	9.14 320	92	0.85 680	9.99 584	5
56	9.13 994	91	9.14 412	92	0.85 588	9.99 582	4
57	9.14 085	90	9.14 504	93	0.85 496	9.99 581	3
58	9.14 175	91	9.14 597	91	0.85 403	9.99 579	2
59	9.14 266	90	9.14 688	92	0.85 312	9.99 577	1
60	9.14 356		9.14 780		0.85 220	9.99 575	**0**
	L. Cos.	**d.**	**L. Cotg.**	**c. d.**	**L. Tang.**	**L. Sin.**	′

Prop. Pts.

	105	104	103
6	10.5	10.4	10.3
7	12.3	12.1	12.0
8	14.0	13.9	13.7
9	15.8	15.6	15.5
10	17.5	17.3	17.2
20	35.0	34.7	34.3
30	52.5	52.0	51.5
40	70.0	69.3	68.7
50	87.5	86.7	85.8

	102	101	100
6	10.2	10.1	10.0
7	11.9	11.8	11.7
8	13.6	13.5	13.3
9	15.3	15.2	15.0
10	17.0	16.8	16.7
20	34.0	33.7	33.3
30	51.0	50.5	50.0
40	68.0	67.3	66.7
50	85.0	84.2	83.3

	99	98	97
6	9.9	9.8	9.7
7	11.6	11.4	11.3
8	13.2	13.1	12.9
9	14.9	14.7	14.6
10	16.5	16.3	16.2
20	33.0	32.7	32.3
30	49.5	49.0	48.5
40	66.0	65.3	64.7
50	82.5	81.7	80.8

	96	95	94
6	9.6	9.5	9.4
7	11.2	11.1	11.0
8	12.8	12.7	12.5
9	14.4	14.3	14.1
10	16.0	15.8	15.7
20	32.0	31.7	31.3
30	48.0	47.5	47.0
40	64.0	63.3	62.7
50	80.0	79.2	78.3

	93	92	91
6	9.3	9.2	9.1
7	10.9	10.7	10.6
8	12.4	12.3	12.1
9	14.0	13.8	13.7
10	15.5	15.3	15.2
20	31.0	30.7	30.3
30	46.5	46.0	45.5
40	62.0	61.3	60.7
50	77.5	76.7	75.8

	90	2	1
6	9.0	0.2	0.1
7	10.5	0.2	0.1
8	12.0	0.3	0.1
9	13.5	0.3	0.2
10	15.0	0.3	0.2
20	30.0	0.7	0.3
30	45.0	1.0	0.5
40	60.0	1.3	0.7
50	75.0	1.7	0.8

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	60
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59
2	9.14 535	89	9.14 963	91	0.85 037	9.99 572	58
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57
4	9.14 714	89	9.15 145	91	0.84 855	9.99 568	56
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	55
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565	54
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	50
11	9.15 333	88	9.15 777	90	0.84 223	9.99 556	49
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46
15	9.15 683	87	9.16 135	89	0.83 865	9.99 548	45
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545	43
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	40
21	9.16 203	86	9.16 665	88	0.83 335	9.99 537	39
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	35
26	9.16 631	85	9.17 103	87	0.82 897	9.99 528	34
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	30
31	9.17 055	84	9.17 536	86	0.82 464	9.99 518	29
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515	27
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	25
36	9.17 474	84	9.17 965	86	0.82 035	9.99 509	24
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	20
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19
42	9.17 973	82	9.18 475	85	0.81 525	9.99 497	18
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	15
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	10
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8
53	9.18 871	81	9.19 395	83	0.80 605	9.99 476	7
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6
55	9.19 033	80	9.19 561	82	0.80 439	9.99 472	5
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4
57	9.19 193	80	9.19 725	82	0.80 275	9.99 468	3
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1
60	9.19 433		9.19 971		0.80 029	9.99 462	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	92	91	90
6	9.2	9.1	9.0
7	10.7	10.6	10.5
8	12.3	12.1	12.0
9	13.8	13.7	13.5
10	15.3	15.2	15.0
20	30.7	30.3	30.0
30	46.0	45.5	45.0
40	61.3	60.7	60.0
50	76.7	75.8	75.0

	89	88
6	8.9	8.8
7	10.4	10.3
8	11.9	11.7
9	13.4	13.2
10	14.8	14.7
20	29.7	29.3
30	44.5	44.0
40	59.3	58.7
50	74.2	73.3

	87	86	85
6	8.7	8.6	8.5
7	10.2	10.0	9.9
8	11.6	11.5	11.3
9	13.1	12.9	12.8
10	14.5	14.3	14.2
20	29.0	28.7	28.3
30	43.5	43.0	42.5
40	58.0	57.3	56.7
50	72.5	71.7	70.8

	84	83
6	8.4	8.3
7	9.8	9.7
8	11.2	11.1
9	12.6	12.5
10	14.0	13.8
20	28.0	27.7
30	42.0	41.5
40	56.0	55.3
50	70.0	69.2

	82	81	80
6	8.2	8.1	8.0
7	9.6	9.5	9.3
8	10.9	10.8	10.7
9	12.3	12.2	12.0
10	13.7	13.5	13.3
20	27.3	27.0	26.7
30	41.0	40.5	40.0
40	54.7	54.0	53.3
50	68.3	67.5	66.7

	2	1
6	0.2	0.1
7	0.2	0.1
8	0.3	0.1
9	0.3	0.2
10	0.3	0.2
20	0.7	0.3
30	1.0	0.5
40	1.3	0.7
50	1.7	0.8

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	60
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	55
6	9.19 909	79	9.20 459	81	0.79 541	9.99 450	54
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	50
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49
12	9.20 380	78	9.20 942	80	0.79 058	9.99 438	48
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	45
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	40
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415	37
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	35
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33
28	9.21 610	75	9.22 205	78	0.77 795	9.99 404	32
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	30
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26
35	9.22 137	74	9.22 747	77	0.77 253	9.99 390	25
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24
37	9.22 286	75	9.22 901	76	0.77 099	9.99 385	23
38	9.22 361	74	9.22 977	77	0.77 023	9.99 383	22
39	9.22 435	74	9.23 054	76	0.76 946	9.99 381	21
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	20
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19
42	9.22 657	74	9.23 283	76	0.76 717	9.99 375	18
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17
44	9.22 805	73	9.23 435	75	0.76 565	9.99 370	16
45	9.22 878	74	9.23 510	76	0.76 490	9.99 368	15
46	9.22 952	73	9.23 586	75	0.76 414	9.99 366	14
47	9.23 025	73	9.23 661	76	0.76 339	9.99 364	13
48	9.23 098	73	9.23 737	75	0.76 263	9.99 362	12
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	10
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8
53	9.23 462	73	9.24 112	74	0.75 888	9.99 351	7
54	9.23 535	72	9.24 186	75	0.75 814	9.99 348	6
55	9.23 607	72	9.24 261	74	0.75 739	9.99 346	5
56	9.23 679	73	9.24 335	75	0.75 665	9.99 344	4
57	9.23 752	71	9.24 410	74	0.75 590	9.99 342	3
58	9.23 823	72	9.24 484	74	0.75 516	9.99 340	2
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1
60	9.23 967		9.24 632		0.75 368	9.99 335	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	′

Prop. Pts.

	82	81	80
6	8.2	8.1	8.0
7	9.6	9.5	9.3
8	10.9	10.8	10.7
9	12.3	12.2	12.0
10	13.7	13.5	13.3
20	27.3	27.0	26.7
30	41.0	40.5	40.0
40	54.7	54.0	53.3
50	68.3	67.5	66.7

	79	78
6	7.9	7.8
7	9.2	9.1
8	10.5	10.4
9	11.9	11.7
10	13.2	13.0
20	26.3	26.0
30	39.5	39.0
40	52.7	52.0
50	65.8	65.0

	77	76
6	7.7	7.6
7	9.0	8.9
8	10.3	10.1
9	11.6	11.4
10	12.8	12.7
20	25.7	25.3
30	38.5	38.0
40	51.3	50.7
50	64.2	63.3

	75	74
6	7.5	7.4
7	8.8	8.6
8	10.0	9.9
9	11.3	11.1
10	12.5	12.3
20	25.0	24.7
30	37.5	37.0
40	50.0	49.3
50	62.5	61.7

	73	72	71
6	7.3	7.2	7.1
7	8.5	8.4	8.3
8	9.7	9.6	9.5
9	11.0	10.8	10.7
10	12.2	12.0	11.8
20	24.3	24.0	23.7
30	36.5	36.0	35.5
40	48.7	48.0	47.3
50	60.8	60.0	59.2

	3	2
6	0.3	0.2
7	0.4	0.2
8	0.4	0.3
9	0.5	0.3
10	0.5	0.3
20	1.0	0.7
30	1.5	1.0
40	2.0	1.3
50	2.5	1.7

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.23 967		9.24 632		0.75 368	9.99 335		60
1	9.24 039	72	9.24 706	74	0.75 294	9.99 333	2	59
2	9.24 110	71	9.24 779	73	0.75 221	9.99 331	2	58
3	9.24 181	71	9.24 853	74	0.75 147	9.99 328	3	57
4	9.24 253	72	9.24 926	73	0.75 074	9.99 326	2	56
5	9.24 324	71	9.25 000	74	0.75 000	9.99 324	2	55
6	9.24 395	71	9.25 073	73	0.74 927	9.99 322	2	54
7	9.24 466	71	9.25 146	73	0.74 854	9.99 319	3	53
8	9.24 536	70	9.25 219	73	0.74 781	9.99 317	2	52
9	9.24 607	71	9.25 292	73	0.74 708	9.99 315	2	51
10	9.24 677	70	9.25 365	73	0.74 635	9.99 313	2	50
11	9.24 748	71	9.25 437	72	0.74 563	9.99 310	3	49
12	9.24 818	70	9.25 510	73	0.74 490	9.99 308	2	48
13	9.24 888	70	9.25 582	72	0.74 418	9.99 306	2	47
14	9.24 958	70	9.25 655	73	0.74 345	9.99 304	2	46
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	3	45
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	2	44
17	9.25 168	70	9.25 871	72	0.74 129	9.99 297	2	43
18	9.25 237	69	9.25 943	72	0.74 057	9.99 294	3	42
19	9.25 307	70	9.26 015	72	0.73 985	9.99 292	2	41
20	9.25 376	69	9.26 086	71	0.73 914	9.99 290	2	40
21	9.25 445	69	9.26 158	72	0.73 842	9.99 288	2	39
22	9.25 514	69	9.26 229	71	0.73 771	9.99 285	3	38
23	9.25 583	69	9.26 301	72	0.73 699	9.99 283	2	37
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	2	36
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	3	35
26	9.25 790	69	9.26 514	71	0.73 486	9.99 276	2	34
27	9.25 858	68	9.26 585	71	0.73 415	9.99 274	2	33
28	9.25 927	69	9.26 655	70	0.73 345	9.99 271	3	32
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	2	31
30	9.26 063	68	9.26 797	71	0.73 203	9.99 267	2	30
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	3	29
32	9.26 199	68	9.26 937	70	0.73 063	9.99 262	2	28
33	9.26 267	68	9.27 008	71	0.72 992	9.99 260	2	27
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	3	26
35	9.26 403	68	9.27 148	70	0.72 852	9.99 255	2	25
36	9.26 470	67	9.27 218	70	0.72 782	9.99 252	3	24
37	9.26 538	68	9.27 288	70	0.72 712	9.99 250	2	23
38	9.26 605	67	9.27 357	69	0.72 643	9.99 248	2	22
39	9.26 672	67	9.27 427	70	0.72 573	9.99 245	3	21
40	9.26 739	67	9.27 496	69	0.72 504	9.99 243	2	20
41	9.26 806	67	9.27 566	70	0.72 434	9.99 241	2	19
42	9.26 873	67	9.27 635	69	0.72 365	9.99 238	3	18
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	2	17
44	9.27 007	67	9.27 773	69	0.72 227	9.99 233	3	16
45	9.27 073	66	9.27 842	69	0.72 158	9.99 231	2	15
46	9.27 140	67	9.27 911	69	0.72 089	9.99 229	2	14
47	9.27 206	66	9.27 980	69	0.72 020	9.99 226	3	13
48	9.27 273	67	9.28 049	69	0.71 951	9.99 224	2	12
49	9.27 339	66	9.28 117	68	0.71 883	9.99 221	3	11
50	9.27 405	66	9.28 186	69	0.71 814	9.99 219	2	10
51	9.27 471	66	9.28 254	68	0.71 746	9.99 217	2	9
52	9.27 537	66	9.28 323	69	0.71 677	9.99 214	3	8
53	9.27 602	65	9.28 391	68	0.71 609	9.99 212	2	7
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	3	6
55	9.27 734	66	9.28 527	68	0.71 473	9.99 207	2	5
56	9.27 799	65	9.28 595	68	0.71 405	9.99 204	3	4
57	9.27 864	65	9.28 662	67	0.71 338	9.99 202	2	3
58	9.27 930	66	9.28 730	68	0.71 270	9.99 200	2	2
59	9.27 995	65	9.28 798	68	0.71 202	9.99 197	3	1
60	9.28 060	65	9.28 865	67	0.71 135	9.99 195	2	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	74	73	72	71	70	69	68	67	66	65	3	2
6	7.4	7.3	7.2	7.1	7.0	6.9	6.8	6.7	6.6	6.5	0.3	0.2
7	8.6	8.5	8.4	8.3	8.2	8.1	7.9	7.8	7.7	7.6	0.4	0.2
8	9.9	9.7	9.6	9.5	9.3	9.2	9.1	8.9	8.8	8.7	0.4	0.3
9	11.1	11.0	10.8	10.7	10.5	10.4	10.2	10.1	9.9	9.8	0.5	0.3
10	12.3	12.2	12.0	11.8	11.7	11.5	11.3	11.2	11.0	10.8	0.5	0.3
20	24.7	24.3	24.0	23.7	23.3	23.0	22.7	22.3	22.0	21.7	1.0	0.7
30	37.0	36.5	36.0	35.5	35.0	34.5	34.0	33.5	33.0	32.5	1.5	1.0
40	49.3	48.7	48.0	47.3	46.7	46.0	45.3	44.7	44.0	43.3	2.0	1.3
50	61.7	60.8	60.0	59.2	58.3	57.5	56.7	55.8	55.0	54.2	2.5	1.7

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.28 060	65	9.28 865	68	0.71 135	9.99 195	3	60
1	9.28 125	65	9.28 933	67	0.71 067	9.99 192	2	59
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	3	58
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	2	57
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185	3	56
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	2	55
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	3	54
7	9.28 512	65	9.29 335	67	0.70 665	9.99 177	2	53
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175	3	52
9	9.28 641	64	9.29 468	67	0.70 532	9.99 172	2	51
10	9.28 705	64	9.29 535	66	0.70 465	9.99 170	3	50
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	2	49
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165	3	48
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	2	47
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	3	46
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	2	45
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155	3	44
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	2	43
18	9.29 214	63	9.30 064	66	0.69 936	9.99 150	3	42
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	2	41
20	9.29 340	63	9.30 195	66	0.69 805	9.99 145	3	40
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	2	39
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	3	38
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	2	37
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135	3	36
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	2	35
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	3	34
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	3	33
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	2	32
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	3	31
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	2	30
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	3	29
32	9.30 090	61	9.30 975	65	0.69 025	9.99 114	2	28
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	3	27
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	3	26
35	9.30 275	61	9.31 168	65	0.68 832	9.99 106	2	25
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	3	24
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	2	23
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	3	22
39	9.30 521	61	9.31 425	64	0.68 575	9.99 096	3	21
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	2	20
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	3	19
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	2	18
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	3	17
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	3	16
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	2	15
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	3	14
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	3	13
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	2	12
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	3	11
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	10
51	9.31 250	60	9.32 185	63	0.67 815	9.99 064	2	9
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	3	8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	3	7
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	2	6
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	3	5
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	3	4
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	2	3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	3	2
59	9.31 728	60	9.32 685	62	0.67 315	9.99 043	3	1
60	9.31 788		9.32 747		0.67 253	9.99 040		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	68	67		66	65		64	63		62	61		60	59		3	2
6	6.8	6.7	6	6.6	6.5	6	6.4	6.3	6	6.2	6.1	6	6.0	5.9	6	0.3	0.2
7	7.9	7.8	7	7.7	7.6	7	7.5	7.4	7	7.2	7.1	7	7.0	6.9	7	0.4	0.2
8	9.1	8.9	8	8.8	8.7	8	8.5	8.4	8	8.3	8.1	8	8.0	7.9	8	0.4	0.3
9	10.2	10.1	9	9.9	9.8	9	9.6	9.5	9	9.3	9.2	9	9.0	8.9	9	0.5	0.3
10	11.3	11.2	10	11.0	10.8	10	10.7	10.5	10	10.3	10.2	10	10.0	9.8	10	0.5	0.3
20	22.7	22.3	20	22.0	21.7	20	21.3	21.0	20	20.7	20.3	20	20.0	19.7	20	1.0	0.7
30	34.0	33.5	30	33.0	32.5	30	32.0	31.5	30	31.0	30.5	30	30.0	29.5	30	1.5	1.0
40	45.3	44.7	40	44.0	43.3	40	42.7	42.0	40	41.3	40.7	40	40.0	39.3	40	2.0	1.3
50	56.7	55.8	50	55.0	54.2	50	53.3	52.5	50	51.7	50.8	50	50.0	49.2	50	2.5	1.7

12°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	60
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	3	59
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	3	58
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	2	57
4	9.32 025	59	9.32 995	62	0.67 005	9.99 030	3	56
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	3	55
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	2	54
7	9.32 202	59	9.33 180	62	0.66 820	9.99 022	3	53
8	9.32 261	58	9.33 242	61	0.66 758	9.99 019	3	52
9	9.32 319	59	9.33 303	62	0.66 697	9.99 016	3	51
10	9.32 378	59	9.33 365	61	0.66 635	9.99 013	2	50
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	3	49
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	3	48
13	9.32 553	59	9.33 548	61	0.66 452	9.99 005	3	47
14	9.32 612	58	9.33 609	61	0.66 391	9.99 002	2	46
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	3	45
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	3	44
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	3	43
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	2	42
19	9.32 902	58	9.33 913	61	0.66 087	9.98 989	3	41
20	9.32 960	58	9.33 974	60	0.66 026	9.98 986	3	40
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	3	39
22	9.33 075	58	9.34 095	60	0.65 905	9.98 980	2	38
23	9.33 133	57	9.34 155	60	0.65 845	9.98 978	3	37
24	9.33 190	58	9.34 215	61	0.65 785	9.98 975	3	36
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	3	35
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	2	34
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	3	33
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	3	32
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	3	31
30	9.33 534	57	9.34 576	59	0.65 424	9.98 958	3	30
31	9.33 591	56	9.34 635	60	0.65 365	9.98 955	2	29
32	9.33 647	57	9.34 695	60	0.65 305	9.98 953	3	28
33	9.33 704	57	9.34 755	59	0.65 245	9.98 950	3	27
34	9.33 761	57	9.34 814	60	0.65 186	9.98 947	3	26
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	3	25
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	3	24
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	2	23
38	9.33 987	56	9.35 051	60	0.64 949	9.98 936	3	22
39	9.34 043	57	9.35 111	59	0.64 889	9.98 933	3	21
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	20
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	3	19
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	3	18
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	2	17
44	9.34 324	56	9.35 405	59	0.64 595	9.98 919	3	16
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	3	15
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	3	14
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	3	13
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	3	12
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	3	11
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	10
51	9.34 713	56	9.35 815	58	0.64 185	9.98 898	2	9
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	3	8
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	3	7
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	3	6
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	3	5
56	9.34 989	55	9.36 105	58	0.63 895	9.98 884	3	4
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3	3
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	3	2
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	3	1
60	9.35 209		9.36 336		0.63 664	9.98 872		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	63	62	61	60	59	58	57	56	55	3	2
6	6.3	6.2	6.1	6.0	5.9	5.8	5.7	5.6	5.5	0.3	0.2
7	7.4	7.2	7.1	7.0	6.9	6.8	6.7	6.5	6.4	0.4	0.2
8	8.4	8.3	8.1	8.0	7.9	7.7	7.6	7.5	7.3	0.4	0.3
9	9.5	9.3	9.2	9.0	8.9	8.7	8.6	8.4	8.3	0.5	0.3
10	10.5	10.3	10.2	10.0	9.8	9.7	9.5	9.3	9.2	0.5	0.3
20	21.0	20.7	20.3	20.0	19.7	19.3	19.0	18.7	18.3	1.0	0.7
30	31.5	31.0	30.5	30.0	29.5	29.0	28.5	28.0	27.5	1.5	1.0
40	42.0	41.3	40.7	40.0	39.3	38.7	38.0	37.3	36.7	2.0	1.3
50	52.5	51.7	50.8	50.0	49.2	48.3	47.5	46.7	45.8	2.5	1.7

77°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.35 209		9.36 336		0.63 664	9.98 872		60
1	9.35 263	54	9.36 394	58	0.63 606	9.98 869	3	59
2	9.35 318	55	9.36 452	58	0.63 548	9.98 867	2	58
3	9.35 373	55	9.36 509	57	0.63 491	9.98 864	3	57
4	9.35 427	54	9.36 566	57	0.63 434	9.98 861	3	56
5	9.35 481	54	9.36 624	58	0.63 376	9.98 858	3	55
6	9.35 536	55	9.36 681	57	0.63 319	9.98 855	3	54
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	3	53
8	9.35 644	54	9.36 795	57	0.63 205	9.98 849	3	52
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	3	51
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	50
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	3	49
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	3	48
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	3	47
14	9.35 968	54	9.37 137	57	0.62 863	9.98 831	3	46
15	9.36 022	54	9.37 193	56	0.62 807	9.98 828	3	45
16	9.36 075	53	9.37 250	57	0.62 750	9.98 825	3	44
17	9.36 129	54	9.37 306	56	0.62 694	9.98 822	3	43
18	9.36 182	53	9.37 363	57	0.62 637	9.98 819	3	42
19	9.36 236	54	9.37 419	56	0.62 581	9.98 816	3	41
20	9.36 289	53	9.37 476	57	0.62 524	9.98 813	3	40
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	3	39
22	9.36 395	53	9.37 588	56	0.62 412	9.98 807	3	38
23	9.36 449	54	9.37 644	56	0.62 356	9.98 804	3	37
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	3	36
25	9.36 555	53	9.37 756	56	0.62 244	9.98 798	3	35
26	9.36 608	53	9.37 812	56	0.62 188	9.98 795	3	34
27	9.36 660	52	9.37 868	56	0.62 132	9.98 792	3	33
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	3	32
29	9.36 766	53	9.37 980	56	0.62 020	9.98 786	3	31
30	9.36 819	53	9.38 035	55	0.61 965	9.98 783	3	30
31	9.36 871	52	9.38 091	56	0.61 909	9.98 780	3	29
32	9.36 924	53	9.38 147	56	0.61 853	9.98 777	3	28
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	3	27
34	9.37 028	52	9.38 257	55	0.61 743	9.98 771	3	26
35	9.37 081	53	9.38 313	56	0.61 687	9.98 768	3	25
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765	3	24
37	9.37 185	52	9.38 423	55	0.61 577	9.98 762	3	23
38	9.37 237	52	9.38 479	56	0.61 521	9.98 759	3	22
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	3	21
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	20
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750	3	19
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	4	18
43	9.37 497	52	9.38 754	55	0.61 246	9.98 743	3	17
44	9.37 549	52	9.38 808	54	0.61 192	9.98 740	3	16
45	9.37 600	51	9.38 863	55	0.61 137	9.98 737	3	15
46	9.37 652	52	9.38 918	55	0.61 082	9.98 734	3	14
47	9.37 703	51	9.38 972	54	0.61 028	9.98 731	3	13
48	9.37 755	52	9.39 027	55	0.60 973	9.98 728	3	12
49	9.37 806	51	9.39 082	55	0.60 918	9.98 725	3	11
50	9.37 858	52	9.39 136	54	0.60 864	9.98 722	3	10
51	9.37 909	51	9.39 190	54	0.60 810	9.98 719	3	9
52	9.37 960	51	9.39 245	55	0.60 755	9.98 715	4	8
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	3	7
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	3	6
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	3	5
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	3	4
57	9.38 215	51	9.39 515	54	0.60 485	9.98 700	3	3
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	3	2
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	3	1
60	9.38 368	51	9.39 677	54	0.60 323	9.98 690	4	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	58	57	56	55	54	53	52	51	4	3	2
6	5.8	5.7	5.6	5.5	5.4	5.3	5.2	5.1	0.4	0.3	0.2
7	6.8	6.7	6.5	6.4	6.3	6.2	6.1	6.0	0.5	0.4	0.2
8	7.7	7.6	7.5	7.3	7.2	7.1	6.9	6.8	0.5	0.4	0.3
9	8.7	8.6	8.4	8.3	8.1	8.0	7.8	7.7	0.6	0.5	0.3
10	9.7	9.5	9.3	9.2	9.0	8.8	8.7	8.5	0.7	0.5	0.3
20	19.3	19.0	18.7	18.3	18.0	17.7	17.3	17.0	1.3	1.0	0.7
30	29.0	28.5	28.0	27.5	27.0	26.5	26.0	25.5	2.0	1.5	1.0
40	38.7	38.0	37.3	36.7	36.0	35.3	34.7	34.0	2.7	2.0	1.3
50	48.3	47.5	46.7	45.8	45.0	44.2	43.3	42.5	3.3	2.5	1.7

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.38 368		9.39 677		0.60 323	9.98 690		60
1	9.38 418	50	9.39 731	54	0.60 269	9.98 687	3	59
2	9.38 469	51	9.39 785	54	0.60 215	9.98 684	3	58
3	9.38 519	50	9.39 838	53	0.60 162	9.98 681	3	57
4	9.38 570	51	9.39 892	54	0.60 108	9.98 678	3	56
5	9.38 620	50	9.39 945	53	0.60 055	9.98 675	3	55
6	9.38 670	50	9.39 999	54	0.60 001	9.98 671	4	54
7	9.38 721	51	9.40 052	53	0.59 948	9.98 668	3	53
8	9.38 771	50	9.40 106	54	0.59 894	9.98 665	3	52
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51
10	9.38 871	50	9.40 212	53	0.59 788	9.98 659	3	50
11	9.38 921	50	9.40 266	54	0.59 734	9.98 656	3	49
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	4	48
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47
14	9.39 071	50	9.40 425	53	0.59 575	9.98 646	3	46
15	9.39 121	50	9.40 478	53	0.59 522	9.98 643	3	45
16	9.39 170	49	9.40 531	53	0.59 469	9.98 640	3	44
17	9.39 220	50	9.40 584	53	0.59 416	9.98 636	4	43
18	9.39 270	50	9.40 636	52	0.59 364	9.98 633	3	42
19	9.39 319	49	9.40 689	53	0.59 311	9.98 630	3	41
20	9.39 369	50	9.40 742	53	0.59 258	9.98 627	3	40
21	9.39 418	49	9.40 795	53	0.59 205	9.98 623	4	39
22	9.39 467	49	9.40 847	52	0.59 153	9.98 620	3	38
23	9.39 517	50	9.40 900	53	0.59 100	9.98 617	3	37
24	9.39 566	49	9.40 952	52	0.59 048	9.98 614	3	36
25	9.39 615	49	9.41 005	53	0.58 995	9.98 610	4	35
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33
28	9.39 762	49	9.41 161	52	0.58 839	9.98 601	3	32
29	9.39 811	49	9.41 214	53	0.58 786	9.98 597	4	31
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	30
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29
32	9.39 958	49	9.41 370	52	0.58 630	9.98 588	3	28
33	9.40 006	48	9.41 422	52	0.58 578	9.98 584	4	27
34	9.40 055	49	9.41 474	52	0.58 526	9.98 581	3	26
35	9.40 103	48	9.41 526	52	0.58 474	9.98 578	3	25
36	9.40 152	49	9.41 578	52	0.58 422	9.98 574	4	24
37	9.40 200	48	9.41 629	51	0.58 371	9.98 571	3	23
38	9.40 249	49	9.41 681	52	0.58 319	9.98 568	3	22
39	9.40 297	48	9.41 733	52	0.58 267	9.98 565	3	21
40	9.40 346	49	9.41 784	51	0.58 216	9.98 561	4	20
41	9.40 394	48	9.41 836	52	0.58 164	9.98 558	3	19
42	9.40 442	48	9.41 887	51	0.58 113	9.98 555	3	18
43	9.40 490	48	9.41 939	52	0.58 061	9.98 551	4	17
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16
45	9.40 586	48	9.42 041	51	0.57 959	9.98 545	3	15
46	9.40 634	48	9.42 093	52	0.57 907	9.98 541	4	14
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13
48	9.40 730	48	9.42 195	51	0.57 805	9.98 535	3	12
49	9.40 778	48	9.42 246	51	0.57 754	9.98 531	4	11
50	9.40 825	47	9.42 297	51	0.57 703	9.98 528	3	10
51	9.40 873	48	9.42 348	51	0.57 652	9.98 525	3	9
52	9.40 921	48	9.42 399	51	0.57 601	9.98 521	4	8
53	9.40 968	47	9.42 450	51	0.57 550	9.98 518	3	7
54	9.41 016	48	9.42 501	51	0.57 499	9.98 515	3	6
55	9.41 063	47	9.42 552	51	0.57 448	9.98 511	4	5
56	9.41 111	48	9.42 603	51	0.57 397	9.98 508	3	4
57	9.41 158	47	9.42 653	50	0.57 347	9.98 505	3	3
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	4	2
59	9.41 252	47	9.42 755	51	0.57 245	9.98 498	3	1
60	9.41 300	48	9.42 805	50	0.57 195	9.98 494	4	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	54	53	52	51	50	49	48	47	4	3
6	5.4	5.3	5.2	5.1	5.0	4.9	4.8	4.7	0.4	0.3
7	6.3	6.2	6.1	6.0	5.8	5.7	5.6	5.5	0.5	0.4
8	7.2	7.1	6.9	6.8	6.7	6.5	6.4	6.3	0.5	0.4
9	8.1	8.0	7.8	7.7	7.5	7.4	7.2	7.1	0.6	0.5
10	9.0	8.8	8.7	8.5	8.3	8.2	8.0	7.8	0.7	0.5
20	18.0	17.7	17.3	17.0	16.7	16.3	16.0	15.7	1.3	1.0
30	27.0	26.5	26.0	25.5	25.0	24.5	24.0	23.5	2.0	1.5
40	36.0	35.3	34.7	34.0	33.3	32.7	32.0	31.3	2.7	2.0
50	45.0	44.2	43.3	42.5	41.7	40.8	40.0	39.2	3.3	2.5

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.41 300	47	9.42 805	51	0.57 195	9.98 494	3	60
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	4	56
5	9.41 535	47	9.43 057	51	0.56 943	9.98 477	3	55
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	4	53
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	3	52
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	4	51
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	3	50
11	9.41 815	46	9.43 358	50	0.56 642	9.98 457	4	49
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	3	48
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40
21	9.42 278	46	9.43 855	50	0.56 145	9.98 422	3	39
22	9.42 324	46	9.43 905	49	0.56 095	9.98 419	4	38
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	3	36
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	4	35
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	3	34
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	4	33
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	3	32
29	9.42 644	46	9.44 250	49	0.55 750	9.98 395	4	31
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	3	30
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	4	29
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	3	28
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	4	27
34	9.42 872	45	9.44 495	49	0.55 505	9.98 377	4	26
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	4	15
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	3	14
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	4	13
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	3	12
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	4	11
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	3	10
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	4	9
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8
53	9.43 724	45	9.45 415	48	0.54 585	9.98 309	3	7
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	4	3
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	3	2
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	4	1
60	9.44 034		9.45 750		0.54 250	9.98 284		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	51	50
6	5.1	5.0
7	6.0	5.8
8	6.8	6.7
9	7.7	7.5
10	8.5	8.3
20	17.0	16.7
30	25.5	25.0
40	34.0	33.3
50	42.5	41.7

	49	48
6	4.9	4.8
7	5.7	5.6
8	6.5	6.4
9	7.4	7.2
10	8.2	8.0
20	16.3	16.0
30	24.5	24.0
40	32.7	32.0
50	40.8	40.0

	47	46
6	4.7	4.6
7	5.5	5.4
8	6.3	6.1
9	7.1	6.9
10	7.8	7.7
20	15.7	15.3
30	23.5	23.0
40	31.3	30.7
50	39.2	38.3

	45	44
6	4.5	4.4
7	5.3	5.1
8	6.0	5.9
9	6.8	6.6
10	7.5	7.3
20	15.0	14.7
30	22.5	22.0
40	30.0	29.3
50	37.5	36.7

	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.44 034		9.45 750		0.54 250	9.98 284		60
1	9.44 078	44	9.45 797	47	0.54 203	9.98 281	3	59
2	9.44 122	44	9.45 845	48	0.54 155	9.98 277	4	58
3	9.44 166	44	9.45 892	47	0.54 108	9.98 273	4	57
4	9.44 210	44	9.45 940	48	0.54 060	9.98 270	3	56
5	9.44 253	43	9.45 987	47	0.54 013	9.98 266	4	55
6	9.44 297	44	9.46 035	48	0.53 965	9.98 262	4	54
7	9.44 341	44	9.46 082	47	0.53 918	9.98 259	3	53
8	9.44 385	44	9.46 130	48	0.53 870	9.98 255	4	52
9	9.44 428	43	9.46 177	47	0.53 823	9.98 251	4	51
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	3	50
11	9.44 516	44	9.46 271	47	0.53 729	9.98 244	4	49
12	9.44 559	43	9.46 319	48	0.53 681	9.98 240	4	48
13	9.44 602	43	9.46 366	47	0.53 634	9.98 237	3	47
14	9.44 646	44	9.46 413	47	0.53 587	9.98 233	4	46
15	9.44 689	43	9.46 460	47	0.53 540	9.98 229	4	45
16	9.44 733	44	9.46 507	47	0.53 493	9.98 226	3	44
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	4	42
19	9.44 862	43	9.46 648	47	0.53 352	9.98 215	3	41
20	9.44 905	43	9.46 694	46	0.53 306	9.98 211	4	40
21	9.44 948	43	9.46 741	47	0.53 259	9.98 207	4	39
22	9.44 992	44	9.46 788	47	0.53 212	9.98 204	3	38
23	9.45 035	43	9.46 835	47	0.53 165	9.98 200	4	37
24	9.45 077	42	9.46 881	46	0.53 119	9.98 196	4	36
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	4	35
26	9.45 163	43	9.46 975	47	0.53 025	9.98 189	3	34
27	9.45 206	43	9.47 021	46	0.52 979	9.98 185	4	33
28	9.45 249	43	9.47 068	47	0.52 932	9.98 181	4	32
29	9.45 292	43	9.47 114	46	0.52 886	9.98 177	4	31
30	9.45 334	42	9.47 160	46	0.52 840	9.98 174	3	30
31	9.45 377	43	9.47 207	47	0.52 793	9.98 170	4	29
32	9.45 419	42	9.47 253	46	0.52 747	9.98 166	4	28
33	9.45 462	43	9.47 299	46	0.52 701	9.98 162	4	27
34	9.45 504	42	9.47 346	47	0.52 654	9.98 159	3	26
35	9.45 547	43	9.47 392	46	0.52 608	9.98 155	4	25
36	9.45 589	42	9.47 438	46	0.52 562	9.98 151	4	24
37	9.45 632	43	9.47 484	46	0.52 516	9.98 147	4	23
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	3	22
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21
40	9.45 758	42	9.47 622	46	0.52 378	9.98 136	4	20
41	9.45 801	43	9.47 668	46	0.52 332	9.98 132	4	19
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	3	18
43	9.45 885	42	9.47 760	46	0.52 240	9.98 125	4	17
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16
45	9.45 969	42	9.47 852	46	0.52 148	9.98 117	4	15
46	9.46 011	42	9.47 897	45	0.52 103	9.98 113	4	14
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	3	13
48	9.46 095	42	9.47 989	46	0.52 011	9.98 106	4	12
49	9.46 136	41	9.48 035	46	0.51 965	9.98 102	4	11
50	9.46 178	42	9.48 080	45	0.51 920	9.98 098	4	10
51	9.46 220	42	9.48 126	46	0.51 874	9.98 094	4	9
52	9.46 262	42	9.48 171	45	0.51 829	9.98 090	4	8
53	9.46 303	41	9.48 217	46	0.51 783	9.98 087	3	7
54	9.46 345	42	9.48 262	45	0.51 738	9.98 083	4	6
55	9.46 386	41	9.48 307	45	0.51 693	9.98 079	4	5
56	9.46 428	42	9.48 353	46	0.51 647	9.98 075	4	4
57	9.46 469	41	9.48 398	45	0.51 602	9.98 071	4	3
58	9.46 511	42	9.48 443	45	0.51 557	9.98 067	4	2
59	9.46 552	41	9.48 489	46	0.51 511	9.98 063	4	1
60	9.46 594	42	9.48 534	45	0.51 466	9.98 060	3	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	48	47
6	4.8	4.7
7	5.6	5.5
8	6.4	6.3
9	7.2	7.1
10	8.0	7.8
20	16.0	15.7
30	24.0	23.5
40	32.0	31.3
50	40.0	39.2

	46	45
6	4.6	4.5
7	5.4	5.3
8	6.1	6.0
9	6.9	6.8
10	7.7	7.5
20	15.3	15.0
30	23.0	22.5
40	30.7	30.0
50	38.3	37.5

	44	43
6	4.4	4.3
7	5.1	5.0
8	5.9	5.7
9	6.6	6.5
10	7.3	7.2
20	14.7	14.3
30	22.0	21.5
40	29.3	28.7
50	36.7	35.8

	42	41
6	4.2	4.1
7	4.9	4.8
8	5.6	5.5
9	6.3	6.2
10	7.0	6.8
20	14.0	13.7
30	21.0	20.5
40	28.0	27.3
50	35.0	34.2

	4	3
6	0.4	0.3
7	0.5	0.4
8	0.5	0.4
9	0.6	0.5
10	0.7	0.5
20	1.3	1.0
30	2.0	1.5
40	2.7	2.0
50	3.3	2.5

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60
1	9.46 635	41	9.48 579	45	0.51 421	9.98 056	4	59
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56
5	9.46 800	41	9.48 759	45	0.51 241	9.98 040	4	55
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	3	53
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	4	52
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025	4	51
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49
12	9.47 086	41	9.49 073	45	0.50 927	9.98 013	4	48
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47
14	9.47 168	41	9.49 163	44	0.50 837	9.98 005	4	46
15	9.47 209	40	9.49 207	45	0.50 793	9.98 001	4	45
16	9.47 249	41	9.49 252	44	0.50 748	9.97 997	4	44
17	9.47 290	40	9.49 296	45	0.50 704	9.97 993	4	43
18	9.47 330	41	9.49 341	44	0.50 659	9.97 989	3	42
19	9.47 371	40	9.49 385	45	0.50 615	9.97 986	4	41
20	9.47 411	41	9.49 430	44	0.50 570	9.97 982	4	40
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36
25	9.47 613	41	9.49 652	44	0.50 348	9.97 962	4	35
26	9.47 654	40	9.49 696	44	0.50 304	9.97 958	4	34
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33
28	9.47 734	40	9.49 784	44	0.50 216	9.97 950	4	32
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	25
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24
37	9.48 094	39	9.50 180	43	0.49 820	9.97 914	4	23
38	9.48 133	40	9.50 223	44	0.49 777	9.97 910	4	22
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20
41	9.48 252	40	9.50 355	43	0.49 645	9.97 898	4	19
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17
44	9.48 371	40	9.50 485	44	0.49 515	9.97 886	4	16
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	15
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13
48	9.48 529	39	9.50 659	44	0.49 341	9.97 870	4	12
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	5	11
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	4	10
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	5
56	9.48 842	39	9.51 005	43	0.48 995	9.97 837	4	4
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3
58	9.48 920	39	9.51 092	43	0.48 908	9.97 829	4	2
59	9.48 959	39	9.51 135	43	0.48 865	9.97 825	4	1
60	9.48 998		9.51 178		0.48 822	9.97 821		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	45	44
6	4.5	4.4
7	5.3	5.1
8	6.0	5.9
9	6.8	6.6
10	7.5	7.3
20	15.0	14.7
30	22.5	22.0
40	30.0	29.3
50	37.5	36.7

	43
6	4.3
7	5.0
8	5.7
9	6.5
10	7.2
20	14.3
30	21.5
40	28.7
50	35.8

	42	41
6	4.2	4.1
7	4.9	4.8
8	5.6	5.5
9	6.3	6.2
10	7.0	6.8
20	14.0	13.7
30	21.0	20.5
40	28.0	27.3
50	35.0	34.2

	40	39
6	4.0	3.9
7	4.7	4.6
8	5.3	5.2
9	6.0	5.9
10	6.7	6.5
20	13.3	13.0
30	20.0	19.5
40	26.7	26.0
50	33.3	32.5

	5	4	3
6	0.5	0.4	0.3
7	0.6	0.5	0.4
8	0.7	0.5	0.4
9	0.8	0.6	0.5
10	0.8	0.7	0.5
20	1.7	1.3	1.0
30	2.5	2.0	1.5
40	3.3	2.7	2.0
50	4.2	3.3	2.5

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.48 998		9.51 178		0.48 822	9.97 821		60
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	4	59
2	9.49 076	39	9.51 264	43	0.48 736	9.97 812	5	58
3	9.49 115	39	9.51 306	42	0.48 694	9.97 808	4	57
4	9.49 153	38	9.51 349	43	0.48 651	9.97 804	4	56
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55
6	9.49 231	39	9.51 435	43	0.48 565	9.97 796	4	54
7	9.49 269	38	9.51 478	43	0.48 522	9.97 792	4	53
8	9.49 308	39	9.51 520	42	0.48 480	9.97 788	4	52
9	9.49 347	39	9.51 563	43	0.48 437	9.97 784	4	51
10	9.49 385	38	9.51 606	43	0.48 394	9.97 779	5	50
11	9.49 424	39	9.51 648	42	0.48 352	9.97 775	4	49
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48
13	9.49 500	38	9.51 734	43	0.48 266	9.97 767	4	47
14	9.49 539	39	9.51 776	42	0.48 224	9.97 763	4	46
15	9.49 577	38	9.51 819	43	0.48 181	9.97 759	4	45
16	9.49 615	38	9.51 861	42	0.48 139	9.97 754	5	44
17	9.49 654	39	9.51 903	42	0.48 097	9.97 750	4	43
18	9.49 692	38	9.51 946	43	0.48 054	9.97 746	4	42
19	9.49 730	38	9.51 988	42	0.48 012	9.97 742	4	41
20	9.49 768	38	9.52 031	43	0.47 969	9.97 738	4	40
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	4	39
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	5	38
23	9.49 882	38	9.52 157	42	0.47 843	9.97 725	4	37
24	9.49 920	38	9.52 200	43	0.47 800	9.97 721	4	36
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	4	34
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	5	33
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31
30	9.50 148	38	9.52 452	42	0.47 548	9.97 696	4	30
31	9.50 185	37	9.52 494	42	0.47 506	9.97 691	5	29
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28
33	9.50 261	38	9.52 578	42	0.47 422	9.97 683	4	27
34	9.50 298	37	9.52 620	42	0.47 380	9.97 679	4	26
35	9.50 336	38	9.52 661	41	0.47 339	9.97 674	5	25
36	9.50 374	38	9.52 703	42	0.47 297	9.97 670	4	24
37	9.50 411	37	9.52 745	42	0.47 255	9.97 666	4	23
38	9.50 449	38	9.52 787	42	0.47 213	9.97 662	4	22
39	9.50 486	37	9.52 829	42	0.47 171	9.97 657	5	21
40	9.50 523	37	9.52 870	41	0.47 130	9.97 653	4	20
41	9.50 561	38	9.52 912	42	0.47 088	9.97 649	4	19
42	9.50 598	37	9.52 953	41	0.47 047	9.97 645	4	18
43	9.50 635	37	9.52 995	42	0.47 005	9.97 640	5	17
44	9.50 673	38	9.53 037	42	0.46 963	9.97 636	4	16
45	9.50 710	37	9.53 078	41	0.46 922	9.97 632	4	15
46	9.50 747	37	9.53 120	42	0.46 880	9.97 628	4	14
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	5	13
48	9.50 821	37	9.53 202	41	0.46 798	9.97 619	4	12
49	9.50 858	37	9.53 244	42	0.46 756	9.97 615	4	11
50	9.50 896	38	9.53 285	41	0.46 715	9.97 610	5	10
51	9.50 933	37	9.53 327	42	0.46 673	9.97 606	4	9
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	4	8
53	9.51 007	37	9.53 409	41	0.46 591	9.97 597	5	7
54	9.51 043	36	9.53 450	41	0.46 550	9.97 593	4	6
55	9.51 080	37	9.53 492	42	0.46 508	9.97 589	4	5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	5	4
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3
58	9.51 191	37	9.53 615	41	0.46 385	9.97 576	4	2
59	9.51 227	36	9.53 656	41	0.46 344	9.97 571	5	1
60	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	43	42
6	4.3	4.2
7	5.0	4.9
8	5.7	5.6
9	6.5	6.3
10	7.2	7.0
20	14.3	14.0
30	21.5	21.0
40	28.7	28.0
50	35.8	35.0

	41
6	4.1
7	4.8
8	5.5
9	6.2
10	6.8
20	13.7
30	20.5
40	27.3
50	34.2

	39	38
6	3.9	3.8
7	4.6	4.4
8	5.2	5.1
9	5.9	5.7
10	6.5	6.3
20	13.0	12.7
30	19.5	19.0
40	26.0	25.3
50	32.5	31.7

	37	36
6	3.7	3.6
7	4.3	4.2
8	4.9	4.8
9	5.6	5.4
10	6.2	6.0
20	12.3	12.0
30	18.5	18.0
40	24.7	24.0
50	30.8	30.0

	5	4
6	0.5	0.4
7	0.6	0.5
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	60
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	4	58
3	9.51 374	37	9.53 820	41	0.46 180	9.97 554	4	57
4	9.51 411	36	9.53 861	41	0.46 139	9.97 550	5	56
5	9.51 447	37	9.53 902	41	0.46 098	9.97 545	4	55
6	9.51 484	36	9.53 943	41	0.46 057	9.97 541	5	54
7	9.51 520	37	9.53 984	41	0.46 016	9.97 536	4	53
8	9.51 557	36	9.54 025̄	40	0.45 975	9.97 532	4	52
9	9.51 593	36	9.54 065	41	0.45 935̄	9.97 528	5	51
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	‘	50
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49
12	9.51 702	36	9.54 187	41	0.45 813	9.97 515̄	5	48
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	4	47
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	5	46
15	9.51 811	36	9.54 309	41	0.45 691	9.97 501	4	45
16	9.51 847	36	9.54 350̄	40	0.45 650	9.97 497	5	44
17	9.51 883	36	9.54 390	41	0.45 610	9.97 492	4	43
18	9.51 919	36	9.54 431	40	0.45 569	9.97 488	4	42
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	5	41
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	4	40
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475̄	5	39
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	4	38
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	5	37
24	9.52 135̄	36	9.54 673	41	0.45 327	9.97 461	4	36
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	35
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	5	34
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	4	33
28	9.52 278	36	9.54 835̄	40	0.45 165	9.97 444	5	32
29	9.52 314	36	9.54 875̄	40	0.45 125	9.97 439	4	31
30	9.52 350	35	9.54 915̄	40	0.45 085	9.97 435̄	5	30
31	9.52 385	36	9.54 955	40	0.45 045̄	9.97 430	4	29
32	9.52 421	35	9.54 995	40	0.45 005̄	9.97 426	5	28
33	9.52 456	36	9.55 035	40	0.44 965̄	9.97 421	4	27
34	9.52 492	35	9.55 075	40	0.44 925̄	9.97 417	5	26
35	9.52 527	36	9.55 115	40	0.44 885̄	9.97 412	4	25
36	9.52 563	35	9.55 155	40	0.44 845̄	9.97 408	5	24
37	9.52 598	36	9.55 195	40	0.44 805̄	9.97 403	4	23
38	9.52 634	35	9.55 235	40	0.44 765̄	9.97 399	5	22
39	9.52 669	36	9.55 275	40	0.44 725̄	9.97 394	4	21
40	9.52 705̄	35	9.55 315̄	40	0.44 685	9.97 390	5	20
41	9.52 740	35	9.55 355̄	40	0.44 645	9.97 385	4	19
42	9.52 775	36	9.55 395̄	39	0.44 605	9.97 381	5	18
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	4	17
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	5	16
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	4	15
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	5	14
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	10
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	5
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	5	4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	4	3
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	5	2
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	4	1
60	9.53 405		9.56 107		0.43 893	9.97 299		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	41	40
6	4.1	4.0
7	4.8	4.7
8	5.5̄	5.3
9	6.2	6.0
10	6.8	6.7
20	13.7	13.3
30	20.5	20.0
40	27.3	26.7
50	34.2	33.3

	39
6	3.9
7	4.6
8	5.2
9	5.9
10	6.5
20	13.0
30	19.5
40	26.0
50	32.5

	37	36
6	3.7	3.6
7	4.3	4.2
8	4.9	4.8
9	5.6	5.4
10	6.2	6.0
20	12.3	12.0
30	18.5	18.0
40	24.7	24.0
50	30.8	30.0

	35	34
6	3.5	3.4
7	4.1	4.0
8	4.7	4.5
9	5.3	5.1
10	5.8	5.7
20	11.7	11.3
30	17.5	17.0
40	23.3	22.7
50	29.2	28.3

	5	4
6	0.5	0.4
7	0.6	0.5̄
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

Prop. Pts.

20°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.53 405		9.56 107		0.43 893	9.97 299		60
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59
2	9.53 475̄	35	9.56 185	39	0.43 815̄	9.97 289	5	58
3	9.53 509	34	9.56 224	39	0.43 776	9.97 285̄	4	57
4	9.53 544	35	9.56 264	40	0.43 736	9.97 280	5	56
5	9.53 578	34	9.56 303	39	0.43 697	9.97 276	4	55
6	9.53 613	35	9.56 342	39	0.43 658	9.97 271	5	54
7	9.53 647	34	9.56 381	39	0.43 619	9.97 266	5	53
8	9.53 682	35	9.56 420	39	0.43 580	9.97 262	4	52
9	9.53 716	34	9.56 459	39	0.43 541	9.97 257	5	51
10	9.53 751	35	9.56 498	39	0.43 502	9.97 252	5	50
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	4	49
12	9.53 819	34	9.56 576	39	0.43 424	9.97 243	5	48
13	9.53 854	35	9.56 615	39	0.43 385	9.97 238	5	47
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	4	46
15	9.53 922	34	9.56 693	39	0.43 307	9.97 229	5	45
16	9.53 957	35	9.56 732	39	0.43 268	9.97 224	5	44
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	4	43
18	9.54 025̄	34	9.56 810	39	0.43 190	9.97 215	5	42
19	9.54 059	34	9.56 849	39	0.43 151	9.97 210	5	41
20	9.54 093	34	9.56 887	38	0.43 113	9.97 206	4	40
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39
22	9.54 161	34	9.56 965̄	39	0.43 035	9.97 196	5	38
23	9.54 195	34	9.57 004	39	0.42 996	9.97 192	4	37
24	9.54 229	34	9.57 042	38	0.42 958	9.97 187	5	36
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	5	35
26	9.54 297	34	9.57 120	39	0.42 880	9.97 178	4	34
27	9.54 331	34	9.57 158	38	0.42 842	9.97 173	5	33
28	9.54 365̄	34	9.57 197	39	0.42 803	9.97 168	5	32
29	9.54 399	34	9.57 235	38	0.42 765̄	9.97 163	5	31
30	9.54 433	34	9.57 274	39	0.42 726	9.97 159	4	30
31	9.54 466	33	9.57 312	38	0.42 688	9.97 154	5	29
32	9.54 500	34	9.57 351	39	0.42 649	9.97 149	5	28
33	9.54 534	34	9.57 389	38	0.42 611	9.97 145̄	4	27
34	9.54 567	33	9.57 428	39	0.42 572	9.97 140	5	26
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25
36	9.54 635̄	34	9.57 504	38	0.42 496	9.97 130	5	24
37	9.54 668	33	9.57 543	39	0.42 457	9.97 126	4	23
38	9.54 702	34	9.57 581	38	0.42 419	9.97 121	5	22
39	9.54 735	33	9.57 619	38	0.42 381	9.97 116	5	21
40	9.54 769	34	9.57 658	39	0.42 342	9.97 111	5	20
41	9.54 802	33	9.57 696	38	0.42 304	9.97 107	4	19
42	9.54 836	34	9.57 734	38	0.42 266	9.97 102	5	18
43	9.54 869	33	9.57 772	38	0.42 228	9.97 097	5	17
44	9.54 903	34	9.57 810	38	0.42 190	9.97 092	5	16
45	9.54 936	33	9.57 849	39	0.42 151	9.97 087	5	15
46	9.54 969	33	9.57 887	38	0.42 113	9.97 083	4	14
47	9.55 003	34	9.57 925̄	38	0.42 075	9.97 078	5	13
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11
50	9.55 102	33	9.58 039	38	0.41 961	9.97 063	5	10
51	9.55 136	34	9.58 077	38	0.41 923	9.97 059	4	9
52	9.55 169	33	9.58 115̄	38	0.41 885	9.97 054	5	8
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7
54	9.55 235̄	33	9.58 191	38	0.41 809	9.97 044	5	6
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	5	5
56	9.55 301	33	9.58 267	38	0.41 733	9.97 035̄	4	4
57	9.55 334	33	9.58 304	37	0.41 696	9.97 030	5	3
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025̄	5	2
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1
60	9.55 433	33	9.58 418	38	0.41 582	9.97 015	5	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	40	39
6	4.0	3.9
7	4.7	4.6
8	5.3	5.2
9	6.0	5.9
10	6.7	6.5
20	13.3	13.0
30	20.0	19.5
40	26.7	26.0
50	33.3	32.5

	38	37
6	3.8	3.7
7	4.4	4.3
8	5.1	4.9
9	5.7	5.6
10	6.3	6.2
20	12.7	12.3
30	19.0	18.5
40	25.3	24.7
50	31.7	30.8

	35
6	3.5
7	4.1
8	4.7
9	5.3
10	5.8
20	11.7
30	17.5
40	23.3
50	29.2

	34	33
6	3.4	3.3
7	4.0	3.9
8	4.5	4.4
9	5.1	5̄.0
10	5.7	5.5
20	11.3	11.0
30	17.0	16.5
40	22.7	22.0
50	28.3	27.5

	5	4
6	0.5	0.4
7	0.6	0.5̄
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	60
1	9.55 466	33	9.58 455	38	0.41 545	9.97 010	5	59
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	4	58
3	9.55 532	32	9.58 531	38	0.41 469	9.97 001	5	57
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	55
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54
7	9.55 663	32	9.58 681	38	0.41 319	9.96 981	5	53
8	9.55 695	33	9.58 719	38	0.41 281	9.96 976	5	52
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	4	50
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	5	49
12	9.55 826	32	9.58 869	38	0.41 131	9.96 957	5	48
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	45
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	40
21	9.56 118	32	9.59 205	38	0.40 795	9.96 912	5	39
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	4	38
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	5	37
24	9.56 215	32	9.59 317	37	0.40 683	9.96 898	5	36
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	35
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	30
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26
35	9.56 568	31	9.59 725	37	0.40 275	9.96 843	5	25
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23
38	9.56 663	32	9.59 835	37	0.40 165	9.96 828	5	22
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	20
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19
42	9.56 790	32	9.59 983	36	0.40 017	9.96 808	5	18
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17
44	9.56 854	32	9.60 056	37	0.39 944	9.96 798	5	16
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	15
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14
47	9.56 949	31	9.60 166	37	0.39 834	9.96 783	5	13
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	6	12
49	9.57 012	32	9.60 240	36	0.39 760	9.96 772	5	11
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	10
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8
53	9.57 138	31	9.60 386	36	0.39 614	9.96 752	5	7
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	5
56	9.57 232	32	9.60 495	37	0.39 505	9.96 737	5	4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3
58	9.57 295	31	9.60 568	37	0.39 432	9.96 727	5	2
59	9.57 326	32	9.60 605	36	0.39 395	9.96 722	5	1
60	9.57 358		9.60 641		0.39 359	9.96 717		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	38	37
6	3.8	3.7
7	4.4	4.3
8	5.1	4.9
9	5.7	5.6
10	6.3	6.2
20	12.7	12.3
30	19.0	18.5
40	25.3	24.7
50	31.7	30.8

	36	33
6	3.6	3.3
7	4.2	3.9
8	4.8	4.4
9	5.4	5.0
10	6.0	5.5
20	12.0	11.0
30	18.0	16.5
40	24.0	22.0
50	30.0	27.5

	32
6	3.2
7	3.7
8	4.3
9	4.8
10	5.3
20	10.7
30	16.0
40	21.3
50	26.7

	31	6
6	3.1	0.6
7	3.6	0.7
8	4.1	0.8
9	4.7	0.9
10	5.2	1.0
20	10.3	2.0
30	15.5	3.0
40	20.7	4.0
50	25.8	5.0

	5	4
6	0.5	0.4
7	0.6	0.5
8	0.7	0.5
9	0.8	0.6
10	0.8	0.7
20	1.7	1.3
30	2.5	2.0
40	3.3	2.7
50	4.2	3.3

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	60
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	6	52
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	5	51
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	50
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	6	45
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	5	44
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	6	40
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	5	39
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	6	35
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	5	34
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33
28	9.58 223	30	9.61 651	36	0.38 349	9.96 572	5	32
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	6	30
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	5	29
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	6	26
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	5	25
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	6	22
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	5	21
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	20
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	6	19
42	9.58 648	30	9.62 150	35	0.37 850	9.96 498	5	18
43	9.58 678	31	9.62 185	36	0.37 815	9.96 493	5	17
44	9.58 709	30	9.62 221	35	0.37 779	9.96 488	5	16
45	9.58 739	30	9.62 256	36	0.37 744	9.96 483	6	15
46	9.58 769	30	9.62 292	35	0.37 708	9.96 477	5	14
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13
48	9.58 829	30	9.62 362	36	0.37 638	9.96 467	6	12
49	9.58 859	30	9.62 398	35	0.37 602	9.96 461	5	11
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	10
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	6	9
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	5	8
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	6	6
55	9.59 039	30	9.62 609	36	0.37 391	9.96 429	5	5
56	9.59 069	29	9.62 645	35	0.37 355	9.96 424	5	4
57	9.59 098	30	9.62 680	35	0.37 320	9.96 419	6	3
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	5	2
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1
60	9.59 188		9.62 785		0.37 215	9.96 403		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	37	36
6	3.7	3.6
7	4.3	4.2
8	4.9	4.8
9	5.6	5.4
10	6.2	6.0
20	12.3	12.0
30	18.5	18.0
40	24.7	24.0
50	30.8	30.0

	35
6	3.5
7	4.1
8	4.7
9	5.3
10	5.8
20	11.7
30	17.5
40	23.3
50	29.2

	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.7
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	30	29
6	3.0	2.9
7	3.5	3.4
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.59 188	30	9.62 785	35	0.37 215̄	9.96 403	6	60
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59
2	9.59 247	30	9.62 855	35	0.37 145̄	9.96 392	5	58
3	9.59 277	30	9.62 890	36	0.37 110	9.96 387	6	57
4	9.59 307	29	9.62 926	35	0.37 074	9.96 381	5	56
5	9.59 336	30	9.62 961	35	0.37 039	9.96 376	6	55
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	5	54
7	9.59 396	29	9.63 031	35	0.36 969	9.96 365̄	5	53
8	9.59 425	30	9.63 066	35	0.36 934	9.96 360	6	52
9	9.59 455̄	29	9.63 101	34	0.36 899	9.96 354	5	51
10	9.59 484	30	9.63 135	35	0.36 865̄	9.96 349	6	50
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	5	49
12	9.59 543	30	9.63 205	35	0.36 795̄	9.96 338	5	48
13	9.59 573	29	9.63 240	35	0.36 760	9.96 333	6	47
14	9.59 602	30	9.63 275	35	0.36 725̄	9.96 327	5	46
15	9.59 632	29	9.63 310	35	0.36 690	9.96 322	6	45
16	9.59 661	29	9.63 345̄	34	0.36 655	9.96 316	5	44
17	9.59 690	30	9.63 379	35	0.36 621	9.96 311	6	43
18	9.59 720	29	9.63 414	35	0.36 586	9.96 305	5	42
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	40
21	9.59 808	29	9.63 519	34	0.36 481	9.96 289	5	39
22	9.59 837	29	9.63 553	35	0.36 447	9.96 284	6	38
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	5	37
24	9.59 895	29	9.63 623	34	0.36 377	9.96 273	6	36
25	9.59 924	30	9.63 657	35	0.36 343	9.96 267	5	35
26	9.59 954	29	9.63 692	34	0.36 308	9.96 262	6	34
27	9.59 983	29	9.63 726	35	0.36 274	9.96 256	5	33
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	6	32
29	9.60 041	29	9.63 796	34	0.36 204	9.96 245	5	31
30	9.60 070	29	9.63 830	35	0.36 170	9.96 240	6	30
31	9.60 099	29	9.63 865̄	34	0.36 135	9.96 234	5	29
32	9.60 128	29	9.63 899	35	0.36 101	9.96 229	6	28
33	9.60 157	29	9.63 934	34	0.36 066	9.96 223	5	27
34	9.60 186	29	9.63 968	35	0.36 032	9.96 218	6	26
35	9.60 215̄	29	9.64 003	34	0.35 997	9.96 212	5	25
36	9.60 244	29	9.64 037	35	0.35 963	9.96 207	6	24
37	9.60 273	29	9.64 072	34	0.35 928	9.96 201	5	23
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	6	22
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	5	21
40	9.60 359	29	9.64 175̄	34	0.35 825	9.96 185̄	6	20
41	9.60 388	29	9.64 209	34	0.35 791	9.96 179	5	19
42	9.60 417	29	9.64 243	35	0.35 757	9.96 174	6	18
43	9.60 446	28	9.64 278	34	0.35 722	9.96 168	6	17
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14
47	9.60 561	28	9.64 415̄	34	0.35 585	9.96 146	6	13
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12
49	9.60 618	28	9.64 483	34	0.35 517	9.96 135̄	6	11
50	9.60 646	29	9.64 517	35	0.35 483	9.96 129	6	10
51	9.60 675	29	9.64 552	34	0.35 448	9.96 123	5	9
52	9.60 704	28	9.64 586	34	0.35 414	9.96 118	6	8
53	9.60 732	29	9.64 620	34	0.35 380	9.96 112	5	7
54	9.60 761	28	9.64 654	34	0.35 346	9.96 107	6	6
55	9.60 789	29	9.64 688	34	0.35 312	9.96 101	6	5
56	9.60 818	28	9.64 722	34	0.35 278	9.96 095	5	4
57	9.60 846	29	9.64 756	34	0.35 244	9.96 090	6	3
58	9.60 875̄	28	9.64 790	34	0.35 210	9.96 084	5	2
59	9.60 903	28	9.64 824	34	0.35 176	9.96 079	6	1
60	9.60 931		9.64 858		0.35 142	9.96 073		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	36	35
6	3.6	3.5
7	4.2	4.1
8	4.8	4.7
9	5.4	5.3
10	6.0	5.8
20	12.0	11.7
30	18.0	17.5
40	24.0	23.3
50	30.0	29.2

	34
6	3.4
7	4.0
8	4.5
9	5.1
10	5.7
20	11.3
30	17.0
40	22.7
50	28.3

	30	29
6	3.0	2.9
7	3.5	3.4
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

	28
6	2.8
7	3.3
8	3.7
9	4.2
10	4.7
20	9.3
30	14.0
40	18.7
50	23.3

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.60 931		9.64 858		0.35 142	9.96 073		60
1	9.60 960	29	9.64 892	34	0.35 108	9.96 067	6	59
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	5	58
3	9.61 016	28	9.64 960	34	0.35 040	9.96 056	6	57
4	9.61 045	29	9.64 994	34	0.35 006	9.96 050	6	56
5	9.61 073	28	9.65 028	34	0.34 972	9.96 045	5	55
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	6	54
7	9.61 129	28	9.65 096	34	0.34 904	9.96 034	5	53
8	9.61 158	29	9.65 130	34	0.34 870	9.96 028	6	52
9	9.61 186	28	9.65 164	34	0.34 836	9.96 022	6	51
10	9.61 214	28	9.65 197	33	0.34 803	9.96 017	5	50
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	6	49
12	9.61 270	28	9.65 265	34	0.34 735	9.96 005	6	48
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	5	47
14	9.61 326	28	9.65 333	34	0.34 667	9.95 994	6	46
15	9.61 354	28	9.65 366	33	0.34 634	9.95 988	6	45
16	9.61 382	28	9.65 400	34	0.34 600	9.95 982	6	44
17	9.61 411	29	9.65 434	34	0.34 566	9.95 977	5	43
18	9.61 438	27	9.65 467	33	0.34 533	9.95 971	6	42
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	6	41
20	9.61 494	28	9.65 535	34	0.34 465	9.95 960	5	40
21	9.61 522	28	9.65 568	33	0.34 432	9.95 954	6	39
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	38
23	9.61 578	28	9.65 636	34	0.34 364	9.95 942	6	37
24	9.61 606	28	9.65 669	33	0.34 331	9.95 937	5	36
25	9.61 634	28	9.65 703	34	0.34 297	9.95 931	6	35
26	9.61 662	28	9.65 736	33	0.34 264	9.95 925	6	34
27	9.61 689	27	9.65 770	34	0.34 230	9.95 920	5	33
28	9.61 717	28	9.65 803	33	0.34 197	9.95 914	6	32
29	9.61 745	28	9.65 837	34	0.34 163	9.95 908	6	31
30	9.61 773	28	9.65 870	33	0.34 130	9.95 902	6	30
31	9.61 800	27	9.65 904	34	0.34 096	9.95 897	5	29
32	9.61 828	28	9.65 937	33	0.34 063	9.95 891	6	28
33	9.61 856	28	9.65 971	34	0.34 029	9.95 885	6	27
34	9.61 883	27	9.66 004	33	0.33 996	9.95 879	6	26
35	9.61 911	28	9.66 038	34	0.33 962	9.95 873	6	25
36	9.61 939	28	9.66 071	33	0.33 929	9.95 868	5	24
37	9.61 966	27	9.66 104	33	0.33 896	9.95 862	6	23
38	9.61 994	28	9.66 138	34	0.33 862	9.95 856	6	22
39	9.62 021	27	9.66 171	33	0.33 829	9.95 850	6	21
40	9.62 049	28	9.66 204	33	0.33 796	9.95 844	6	20
41	9.62 076	27	9.66 238	34	0.33 762	9.95 839	5	19
42	9.62 104	28	9.66 271	33	0.33 729	9.95 833	6	18
43	9.62 131	27	9.66 304	33	0.33 696	9.95 827	6	17
44	9.62 159	28	9.66 337	33	0.33 663	9.95 821	6	16
45	9.62 186	27	9.66 371	34	0.33 629	9.95 815	6	15
46	9.62 214	28	9.66 404	33	0.33 596	9.95 810	5	14
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	6	13
48	9.62 268	27	9.66 470	33	0.33 530	9.95 798	6	12
49	9.62 296	28	9.66 503	33	0.33 497	9.95 792	6	11
50	9.62 323	27	9.66 537	34	0.33 463	9.95 786	6	10
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	6	9
52	9.62 377	27	9.66 603	33	0.33 397	9.95 775	5	8
53	9.62 405	28	9.66 636	33	0.33 364	9.95 769	6	7
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	6
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	6	5
56	9.62 486	27	9.66 735	33	0.33 265	9.95 751	6	4
57	9.62 513	27	9.66 768	33	0.33 232	9.95 745	6	3
58	9.62 541	28	9.66 801	33	0.33 199	9.95 739	6	2
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	6	1
60	9.62 595	27	9.66 867	33	0.33 133	9.95 728	5	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	34	33
6	3.4	3.3
7	4.0	3.9
8	4.5	4.4
9	5.1	5.0
10	5.7	5.5
20	11.3	11.0
30	17.0	16.5
40	22.7	22.0
50	28.3	27.5

	29
6	2.9
7	3.4
8	3.9
9	4.4
10	4.8
20	9.7
30	14.5
40	19.3
50	24.2

	28
6	2.8
7	3.3
8	3.7
9	4.2
10	4.7
20	9.3
30	14.0
40	18.7
50	23.3

	27
6	2.7
7	3.2
8	3.6
9	4.1
10	4.5
20	9.0
30	13.5
40	18.0
50	22.5

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.62 595		9.66 867		0.33 133	9.95 728		60
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	33	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	32	0.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	0.32 804	9.95 668	6	50
11	9.62 892	27	9.67 229	33	0.32 771	9.95 663	5	49
12	9.62 918	26	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	33	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 32[illegible]	32	0.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	45
16	9.63 026	27	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	26	9.67 426	33	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	32	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 [illegible]15	6	41
20	9.63 133	27	9.67 524	33	0.32 476	9.95 609	6	40
21	9.63 159	26	9.67 556	32	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	27	9.67 622	33	0.32 378	9.95 591	6	37
24	9.63 239	26	9.67 654	32	0.32 346	9.95 585	6	36
25	9.63 266	27	9.67 687	33	0.32 3[illegible]3	9.95 579	6	35
26	9.63 292	26	9.67 719	32	0.32 281	9.95 573	6	34
27	9.63 319	27	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	26	9.67 785	33	0.32 215	9.95 561	6	32
29	9.63 372	27	9.67 817	32	0.32 183	9.95 555	6	31
30	9.63 398	26	9.67 850	33	0.32 150	9.95 549	6	30
31	9.63 425	27	9.67 882	32	0.32 118	9.95 543	6	29
32	9.63 451	26	9.67 915	33	0.32 085	9.95 537	6	28
33	9.63 478	27	9.67 947	32	0.32 053	9.95 531	6	27
34	9.63 504	26	9.67 980	33	0.32 020	9.95 525	6	26
35	9.63 531	27	9.68 012	32	0.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	32	0.31 956	9.95 513	6	24
37	9.63 583	26	9.68 077	33	0.31 923	9.95 507	6	23
38	9.63 610	27	9.68 109	32	0.31 891	9.95 500	7	22
39	9.63 636	26	9.68 142	33	0.31 858	9.95 494	6	21
40	9.63 662	26	9.68 174	32	0.31 826	9.95 488	6	20
41	9.63 689	27	9.68 206	32	0.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	33	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	26	9.68 30[illegible]	32	0.31 697	9.95 464	6	16
45	9.63 794	27	9.68 336	33	0.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	32	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	33	0.31 535	9.95 434	6	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	7	10
51	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
52	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
53	9.64 002	26	9.68 593	32	0.31 407	9.95 409	6	7
54	9.64 028	26	9.68 626	33	0.31 374	9.95 403	6	6
55	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	5
56	9.64 080	26	9.68 690	32	0.31 310	9.95 391	6	4
57	9.64 106	26	9.68 722	32	0.31 278	9.95 384	7	3
58	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
59	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
60	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	33	32
6	3.3	3.2
7	3.9	3.7
8	4.4	4.3
9	5.0	4.8
10	5.5	5.3
20	11.0	10.7
30	16.5	16.0
40	22.0	21.3
50	27.5	26.7

	27
6	2.7
7	3.2
8	3.6
9	4.1
10	4.5
20	9.0
30	13.5
40	18.0
50	22.5

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	7
6	0.7
7	0.8
8	0.9
9	1.1
10	1.2
20	2.3
30	3.5
40	4.7
50	5.8

	6	5
6	0.6	0.5
7	0.7	0.6
8	0.8	0.7
9	0.9	0.8
10	1.0	0.8
20	2.0	1.7
30	3.0	2.5
40	4.0	3.3
50	5.0	4.2

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.		Prop. Pts.
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	**60**	
1	9.64 210	26	9.68 850	32	0.31 150	9.95 360	6	59	
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58	
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	7	57	
4	9.64 288	25	9.68 946	32	0.31 054	9.95 341	6	56	
5	9.64 313	26	9.68 978	32	0.31 022	9.95 335	6	55	
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54	
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53	
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	7	52	
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	6	51	
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	**50**	
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49	
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48	
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	7	47	
14	9.64 545	26	9.69 266	32	0.30 734	9.95 279	6	46	
15	9.64 571	25	9.69 298	31	0.30 702	9.95 273	6	45	
16	9.64 596	26	9.69 329	32	0.30 671	9.95 267	6	44	
17	9.64 622	25	9.69 361	32	0.30 639	9.95 261	7	43	
18	9.64 647	26	9.69 393	32	0.30 607	9.95 254	6	42	
19	9.64 673	25	9.69 425	32	0.30 575	9.95 248	6	41	
20	9.64 698	26	9.69 457	31	0.30 543	9.95 242	6	**40**	
21	9.64 724	25	9.69 488	32	0.30 512	9.95 236	7	39	
22	9.64 749	26	9.69 520	32	0.30 480	9.95 229	6	38	
23	9.64 775	25	9.69 552	32	0.30 448	9.95 223	6	37	
24	9.64 800	26	9.69 584	31	0.30 416	9.95 217	6	36	
25	9.64 826	25	9.69 615	32	0.30 385	9.95 211	7	35	
26	9.64 851	26	9.69 647	32	0.30 353	9.95 204	6	34	
27	9.64 877	25	9.69 679	31	0.30 321	9.95 198	6	33	
28	9.64 902	25	9.69 710	32	0.30 290	9.95 192	7	32	
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	6	31	
30	9.64 953	25	9.69 774	31	0.30 226	9.95 179	6	**30**	
31	9.64 978	25	9.69 805	32	0.30 195	9.95 173	6	29	
32	9.65 003	26	9.69 837	31	0.30 163	9.95 167	7	28	
33	9.65 029	25	9.69 868	32	0.30 132	9.95 160	6	27	
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26	
35	9.65 079	25	9.69 932	31	0.30 068	9.95 148	7	25	
36	9.65 104	26	9.69 963	32	0.30 037	9.95 141	6	24	
37	9.65 130	25	9.69 995	31	0.30 005	9.95 135	6	23	
38	9.65 155	25	9.70 026	32	0.29 974	9.95 129	7	22	
39	9.65 180	25	9.70 058	31	0.29 942	9.95 122	6	21	
40	9.65 205	25	9.70 089	32	0.29 911	9.95 116	6	**20**	
41	9.65 230	25	9.70 121	31	0.29 879	9.95 110	7	19	
42	9.65 255	26	9.70 152	32	0.29 848	9.95 103	6	18	
43	9.65 281	25	9.70 184	31	0.29 816	9.95 097	7	17	
44	9.65 306	25	9.70 215	32	0.29 785	9.95 090	6	16	
45	9.65 331	25	9.70 247	31	0.29 753	9.95 084	6	15	
46	9.65 356	25	9.70 278	31	0.29 722	9.95 078	7	14	
47	9.65 381	25	9.70 309	32	0.29 691	9.95 071	6	13	
48	9.65 406	25	9.70 341	31	0.29 659	9.95 065	6	12	
49	9.65 431	25	9.70 372	32	0.29 628	9.95 059	7	11	
50	9.65 456	25	9.70 404	31	0.29 596	9.95 052	6	**10**	
51	9.65 481	25	9.70 435	31	0.29 565	9.95 046	7	9	
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8	
53	9.65 531	25	9.70 498	31	0.29 502	9.95 033	6	7	
54	9.65 556	24	9.70 529	31	0.29 471	9.95 027	7	6	
55	9.65 580	25	9.70 560	32	0.29 440	9.95 020	6	5	
56	9.65 605	25	9.70 592	31	0.29 408	9.95 014	7	4	
57	9.65 630	25	9.70 623	31	0.29 377	9.95 007	6	3	
58	9.65 655	25	9.70 654	31	0.29 346	9.95 001	6	2	
59	9.65 680	25	9.70 685	32	0.29 315	9.94 995	7	1	
60	9.65 705		9.70 717		0.29 283	9.94 988		**0**	
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′	Prop. Pts.

Prop. Pts.

	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.7
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	25
6	2.5
7	2.9
8	3.3
9	3.8
10	4.2
20	8.3
30	12.5
40	16.7
50	20.8

	24
6	2.4
7	2.8
8	3.2
9	3.6
10	4.0
20	8.0
30	12.0
40	16.0
50	20.0

	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.1	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.65 705̄		9.70 717		0.29 283	9.94 988		60
1	9.65 729	24	9.70 748	31	0.29 252	9.94 982	6	59
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	7	58
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	6	57
4	9.65 804	25	9.70 841	31	0.29 159	9.94 962	7	56
5	9.65 828	24	9.70 873	32	0.29 127	9.94 956	6	55
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	7	54
7	9.65 878	25	9.70 935̄	31	0.29 065	9.94 943	6	53
8	9.65 902	24	9.70 966	31	0.29 034	9.94 936	7	52
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	6	51
10	9.65 952	25	9.71 028	31	0.28 972	9.94 923	7	50
11	9.65 976	24	9.71 059	31	0.28 941	9.94 917	6	49
12	9.66 001	25	9.71 090	31	0.28 910	9.94 911	6	48
13	9.66 025	24	9.71 121	31	0.28 879	9.94 904	7	47
14	9.66 050	25	9.71 153	32	0.28 847	9.94 898	6	46
15	9.66 075̄	25	9.71 184	31	0.28 816	9.94 891	7	45
16	9.66 099	24	9.71 215̄	31	0.28 785	9.94 885̄	6	44
17	9.66 124	25	9.71 246	31	0.28 754	9.94 878	7	43
18	9.66 148	24	9.71 277	31	0.28 723	9.94 871	7	42
19	9.66 173	25	9.71 308	31	0.28 692	9.94 865̄	6	41
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	7	40
21	9.66 221	24	9.71 370	31	0.28 630	9.94 852	6	39
22	9.66 246	25	9.71 401	31	0.28 599	9.94 845	7	38
23	9.66 270	24	9.71 431	30	0.28 569	9.94 839	6	37
24	9.66 295̄	25	9.71 462	31	0.28 538	9.94 832	7	36
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	6	35
26	9.66 343	24	9.71 524	31	0.28 476	9.94 819	7	34
27	9.66 368	25	9.71 555	31	0.28 445̄	9.94 813	6	33
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32
29	9.66 416	24	9.71 617	31	0.28 383	9.94 799	7	31
30	9.66 441	25	9.71 648	31	0.28 352	9.94 793	6	30
31	9.66 465̄	24	9.71 679	31	0.28 321	9.94 786	7	29
32	9.66 489	24	9.71 709	30	0.28 291	9.94 780	6	28
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	7	27
34	9.66 537	24	9.71 771	31	0.28 229	9.94 767	6	26
35	9.66 562	25	9.71 802	31	0.28 198	9.94 760	7	25
36	9.66 586	24	9.71 833	31	0.28 167	9.94 753	7	24
37	9.66 610	24	9.71 863	30	0.28 137	9.94 747	6	23
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	7	22
39	9.66 658	24	9.71 925̄	31	0.28 075	9.94 734	6	21
40	9.66 682	24	9.71 955	30	0.28 045̄	9.94 727	7	20
41	9.66 706	24	9.71 986	31	0.28 014	9.94 720	7	19
42	9.66 731	25	9.72 017	31	0.27 983	9.94 714	6	18
43	9.66 755̄	24	9.72 048	31	0.27 952	9.94 707	7	17
44	9.66 779	24	9.72 078	30	0.27 922	9.94 700	7	16
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	6	15
46	9.66 827	24	9.72 140	31	0.27 860	9.94 687	7	14
47	9.66 851	24	9.72 170	30	0.27 830	9.94 680	7	13
48	9.66 875̄	24	9.72 201	31	0.27 799	9.94 674	6	12
49	9.66 899	24	9.72 231	30	0.27 769	9.94 667	7	11
50	9.66 922	23	9.72 262	31	0.27 738	9.94 660	7	10
51	9.66 946	24	9.72 293	31	0.27 707	9.94 654	6	9
52	9.66 970	24	9.72 323	30	0.27 677	9.94 647	7	8
53	9.66 994	24	9.72 354	31	0.27 646	9.94 640	7	7
54	9.67 018	24	9.72 384	30	0.27 616	9.94 634	6	6
55	9.67 042	24	9.72 415̄	31	0.27 585	9.94 627	7	5
56	9.67 066	24	9.72 445	30	0.27 555̄	9.94 620	7	4
57	9.67 090	24	9.72 476	31	0.27 524	9.94 614	6	3
58	9.67 113	23	9.72 506	30	0.27 494	9.94 607	7	2
59	9.67 137	24	9.72 537	31	0.27 463	9.94 600	7	1
60	9.67 161	24	9.72 567	30	0.27 433	9.94 593	7	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	32	31
6	3.2	3.1
7	3.7	3.6
8	4.3	4.1
9	4.8	4.7
10	5.3	5.2
20	10.7	10.3
30	16.0	15.5
40	21.3	20.7
50	26.7	25.8

	30
6	3.0
7	3.5
8	4.0
9	4.5
10	5.0
20	10.0
30	15.0
40	20.0
50	25.0

	25	24
6	2.5	2.4
7	2.9	2.8
8	3.3	3.2
9	3.8	3.6
10	4.2	4.0
20	8.3	8.0
30	12.5	12.0
40	16.7	16.0
50	20.8	20.0

	23
6	2.3
7	2.7
8	3.1
9	3.5̄
10	3.8
20	7.7
30	11.5
40	15.3
50	19.2

	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.1	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.67 161		9.72 567		0.27 433	9.94 593		60
1	9.67 185	24	9.72 598	31	0.27 402	9.94 587	6	59
2	9.67 208	23	9.72 628	30	0.27 372	9.94 580	7	58
3	9.67 232	24	9.72 659	31	0.27 341	9.94 573	7	57
4	9.67 256	24	9.72 689	30	0.27 311	9.94 567	6	56
5	9.67 280	24	9.72 720	31	0.27 280	9.94 560	7	55
6	9.67 303	23	9.72 750	30	0.27 250	9.94 553	7	54
7	9.67 327	24	9.72 780	30	0.27 220	9.94 546	7	53
8	9.67 350	23	9.72 811	31	0.27 189	9.94 540	6	52
9	9.67 374	24	9.72 841	30	0.27 159	9.94 533	7	51
10	9.67 398	24	9.72 872	31	0.27 128	9.94 526	7	50
11	9.67 421	23	9.72 902	30	0.27 098	9.94 519	7	49
12	9.67 445	24	9.72 932	30	0.27 068	9.94 513	6	48
13	9.67 468	23	9.72 963	31	0.27 037	9.94 506	7	47
14	9.67 492	24	9.72 993	30	0.27 007	9.94 499	7	46
15	9.67 515	23	9.73 023	30	0.26 977	9.94 492	7	45
16	9.67 539	24	9.73 054	31	0.26 946	9.94 485	7	44
17	9.67 562	23	9.73 084	30	0.26 916	9.94 479	6	43
18	9.67 586	24	9.73 114	30	0.26 886	9.94 472	7	42
19	9.67 609	23	9.73 144	30	0.26 856	9.94 465	7	41
20	9.67 633	24	9.73 175	31	0.26 825	9.94 458	7	40
21	9.67 656	23	9.73 205	30	0.26 795	9.94 451	7	39
22	9.67 680	24	9.73 235	30	0.26 765	9.94 445	6	38
23	9.67 703	23	9.73 265	30	0.26 735	9.94 438	7	37
24	9.67 726	23	9.73 295	30	0.26 705	9.94 431	7	36
25	9.67 750	24	9.73 326	31	0.26 674	9.94 424	7	35
26	9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34
27	9.67 796	23	9.73 386	30	0.26 614	9.94 410	7	33
28	9.67 820	24	9.73 416	30	0.26 584	9.94 404	6	32
29	9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31
30	9.67 866	23	9.73 476	30	0.26 524	9.94 390	7	30
31	9.67 890	24	9.73 507	31	0.26 493	9.94 383	7	29
32	9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28
33	9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27
34	9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26
35	9.67 982	23	9.73 627	30	0.26 373	9.94 355	7	25
36	9.68 006	24	9.73 657	30	0.26 343	9.94 349	6	24
37	9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23
38	9.68 052	23	9.73 717	30	0.26 283	9.94 335	7	22
39	9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20
41	9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19
42	9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18
43	9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17
44	9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16
45	9.68 213	23	9.73 927	30	0.26 073	9.94 286	7	15
46	9.68 237	24	9.73 957	30	0.26 043	9.94 279	7	14
47	9.68 260	23	9.73 987	30	0.26 013	9.94 273	6	13
48	9.68 283	23	9.74 017	30	0.25 983	9.94 266	7	12
49	9.68 305	22	9.74 047	30	0.25 953	9.94 259	7	11
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10
51	9.68 351	23	9.74 107	30	0.25 893	9.94 245	7	9
52	9.68 374	23	9.74 137	30	0.25 863	9.94 238	7	8
53	9.68 397	23	9.74 166	29	0.25 834	9.94 231	7	7
54	9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6
55	9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5
56	9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4
57	9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3
58	9.68 512	23	9.74 316	30	0.25 684	9.94 196	7	2
59	9.68 534	22	9.74 345	29	0.25 655	9.94 189	7	1
60	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	31	30
6	3.1	3.0
7	3.6	3.5
8	4.1	4.0
9	4.7	4.5
10	5.2	5.0
20	10.3	10.0
30	15.5	15.0
40	20.7	20.0
50	25.8	25.0

	29
6	2.9
7	3.4
8	3.9
9	4.4
10	4.8
20	9.7
30	14.5
40	19.3
50	24.2

	24	23
6	2.4	2.3
7	2.8	2.7
8	3.2	3.1
9	3.6	3.5
10	4.0	3.8
20	8.0	7.7
30	12.0	11.5
40	16.0	15.3
50	20.0	19.2

	22
6	2.2
7	2.6
8	2.9
9	3.3
10	3.7
20	7.3
30	11.0
40	14.7
50	18.3

	7	6
6	0.7	0.6
7	0.8	0.7
8	0.9	0.8
9	1.1	0.9
10	1.2	1.0
20	2.3	2.0
30	3.5	3.0
40	4.7	4.0
50	5.8	5.0

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.68 557		9.74 375		0.25 625	9.94 182		60
1	9.68 580	23	9.74 405	30	0.25 595	9.94 175	7	59
2	9.68 603	23	9.74 435	30	0.25 565	9.94 168	7	58
3	9.68 625	22	9.74 465	30	0.25 535	9.94 161	7	57
4	9.68 648	23	9.74 494	29	0.25 506	9.94 154	7	56
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	55
6	9.68 694	23	9.74 554	30	0.25 446	9.94 140	7	54
7	9.68 716	22	9.74 583	29	0.25 417	9.94 133	7	53
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52
9	9.68 762	23	9.74 643	30	0.25 357	9.94 119	7	51
10	9.68 784	22	9.74 673	30	0.25 327	9.94 112	7	50
11	9.68 807	23	9.74 702	29	0.25 298	9.94 105	7	49
12	9.68 829	22	9.74 732	30	0.25 268	9.94 098	7	48
13	9.68 852	23	9.74 762	30	0.25 238	9.94 090	8	47
14	9.68 875	23	9.74 791	29	0.25 209	9.94 083	7	46
15	9.68 897	22	9.74 821	30	0.25 179	9.94 076	7	45
16	9.68 920	23	9.74 851	30	0.25 149	9.94 069	7	44
17	9.68 942	22	9.74 880	29	0.25 120	9.94 062	7	43
18	9.68 965	23	9.74 910	30	0.25 090	9.94 055	7	42
19	9.68 987	22	9.74 939	29	0.25 061	9.94 048	7	41
20	9.69 010	23	9.74 969	30	0.25 031	9.94 041	7	40
21	9.69 032	22	9.74 998	29	0.25 002	9.94 034	7	39
22	9.69 055	23	9.75 028	30	0.24 972	9.94 027	7	38
23	9.69 077	22	9.75 058	30	0.24 942	9.94 020	7	37
24	9.69 100	23	9.75 087	29	0.24 913	9.94 012	8	36
25	9.69 122	22	9.75 117	30	0.24 883	9.94 005	7	35
26	9.69 144	22	9.75 146	29	0.24 854	9.93 998	7	34
27	9.69 167	23	9.75 176	30	0.24 824	9.93 991	7	33
28	9.69 189	22	9.75 205	29	0.24 795	9.93 984	7	32
29	9.69 212	23	9.75 235	30	0.24 765	9.93 977	7	31
30	9.69 234	22	9.75 264	29	0.24 736	9.93 970	7	30
31	9.69 256	22	9.75 294	30	0.24 706	9.93 963	7	29
32	9.69 279	23	9.75 323	29	0.24 677	9.93 955	8	28
33	9.69 301	22	9.75 353	30	0.24 647	9.93 948	7	27
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26
35	9.69 345	22	9.75 411	29	0.24 589	9.93 934	7	25
36	9.69 368	23	9.75 441	30	0.24 559	9.93 927	7	24
37	9.69 390	22	9.75 470	29	0.24 530	9.93 920	7	23
38	9.69 412	22	9.75 500	30	0.24 500	9.93 912	8	22
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21
40	9.69 456	22	9.75 558	29	0.24 442	9.93 898	7	20
41	9.69 479	23	9.75 588	30	0.24 412	9.93 891	7	19
42	9.69 501	22	9.75 617	29	0.24 383	9.93 884	7	18
43	9.69 523	22	9.75 647	30	0.24 353	9.93 876	8	17
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16
45	9.69 567	22	9.75 705	29	0.24 295	9.93 862	7	15
46	9.69 589	22	9.75 735	30	0.24 265	9.93 855	7	14
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	8	13
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12
49	9.69 655	22	9.75 822	29	0.24 178	9.93 833	7	11
50	9.69 677	22	9.75 852	30	0.24 148	9.93 826	7	10
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	7	9
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	8	8
53	9.69 743	22	9.75 939	29	0.24 061	9.93 804	7	7
54	9.69 765	22	9.75 969	30	0.24 031	9.93 797	7	6
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	8	5
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4
57	9.69 831	22	9.76 056	29	0.23 944	9.93 775	7	3
58	9.69 853	22	9.76 086	30	0.23 914	9.93 768	7	2
59	9.69 875	22	9.76 115	29	0.23 885	9.93 760	8	1
60	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	30	29	23	22	8	7
6	3.0	2.9	2.3	2.2	0.8	0.7
7	3.5	3.4	2.7	2.6	0.9	0.8
8	4.0	3.9	3.1	2.9	1.1	0.9
9	4.5	4.4	3.5	3.3	1.2	1.1
10	5.0	4.8	3.8	3.7	1.3	1.2
20	10.0	9.7	7.7	7.3	2.7	2.3
30	15.0	14.5	11.5	11.0	4.0	3.5
40	20.0	19.3	15.3	14.7	5.3	4.7
50	25.0	24.2	19.2	18.3	6.7	5.8

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	**60**
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	8	55
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	8	52
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	**50**
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	8	49
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48
13	9.70 180	22	9.76 522	29	0.23 478	9 93 658	8	47
14	9.70 202	22	9.76 551	29	0 23 449	9 93 650	7	46
15	9.70 224	21	9 76 580	29	0.23 420	9 93 643	7	45
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	8	44
17	9.70 267	21	9.76 639	29	0.23 361	9 93 628	7	43
18	9.70 288	22	9 76 668	29	0.23 332	9.93 621	7	42
19	9.70 310	22	9.76 697	28	0.23 303	9 93 614	8	41
20	9.70 332	21	9.76 725	29	0.23 275	9 93 606	7	**40**
21	9.70 353	22	9.76 754	29	0.23 246	9 93 599	8	39
22	9.70 375	21	9.76 783	29	0.23 217	9.93 591	7	38
23	9.70 396	22	9.76 812	29	0.23 188	9 93 584	7	37
24	9.70 418	21	9 76 841	29	0.23 159	9.93 577	8	36
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	35
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	8	34
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33
28	9 70 504	21	9.76 957	29	0.23 043	9.93 547	8	32
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31
30	9.70 547	21	9.77 015	29	0.22 985	9 93 532	7	**30**
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525	8	29
32	9.70 590	21	9.77 073	28	0.22 927	9 93 517	7	28
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	8	27
34	9.70 633	21	9.77 130	29	0 22 870	9.93 502	7	26
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495	8	25
36	9.70 675	22	9.77 188	29	0 22 812	9.93 487	7	24
37	9.70 697	21	9 77 217	29	0.22 783	9.93 480	8	23
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22
39	9.70 739	22	9.77 274	29	0.22 726	9 93 465	8	21
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	**20**
41	9.70 782	21	9.77 332	29	0.22 668	9 93 450	8	19
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18
43	9.70 824	22	9 77 390	28	0.22 610	9.93 435	8	17
44	9.70 846	21	9.77 418	29	0.22 582	9 93 427	7	16
45	9.70 867	21	9 77 447	29	0.22 553	9 93 420	8	15
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14
47	9.70 909	22	9.77 505	28	0.22 495	9.93 405	8	13
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	8	11
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	**10**
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375	8	9
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	8	7
54	9.71 058	21	9.77 706	28	0.22 294	9 93 352	8	6
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	5
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	8	4
57	9.71 121	21	9.77 791	29	0.22 209	9 93 329	7	3
58	9.71 142	21	9 77 820	29	0.22 180	9.93 322	8	2
59	9.71 163	21	9.77 849	28	0.22 151	9 93 314	7	1
60	9.71 184		9.77 877		0.22 123	9.93 307		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	30	29
6	3.0	2.9
7	3.5	3.4
8	4.0	3.9
9	4.5	4.4
10	5.0	4.8
20	10.0	9.7
30	15.0	14.5
40	20.0	19.3
50	25.0	24.2

	28
6	2.8
7	3.3
8	3.7
9	4 2
10	4.7
20	9.3
30	14.0
40	18.7
50	23 3

	22
6	2.2
7	2.6
8	2.9
9	3.3
10	3.7
20	7.3
30	11.0
40	14.7
50	18.3

	21
6	2.1
7	2.5
8	2.8
9	3.2
10	3.5
20	7.0
30	10.5
40	14.0
50	17.5

	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.1
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	60
1	9.71 205	21	9.77 906	29	0.22 094	9.93 299	8	59
2	9.71 226	21	9.77 935	28	0.22 065	9.93 291	7	58
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	8	57
4	9.71 268	21	9.77 992	28	0.22 008	9.93 276	7	56
5	9.71 289	21	9.78 020	29	0.21 980	9.93 269	8	55
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	7	53
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	8	52
9	9.71 373	20	9.78 135	28	0.21 865	9.93 238	8	51
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	7	50
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	8	49
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	7	47
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	8	46
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	45
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	7	44
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	8	43
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	7	41
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	40
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39
22	9.71 643	21	9.78 505	28	0.21 495	9.93 138	7	38
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37
24	9.71 685	20	9.78 562	28	0.21 438	9.93 123	8	36
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	35
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34
27	9.71 747	20	9.78 647	28	0.21 353	9.93 100	8	33
28	9.71 767	21	9.78 675	29	0.21 325	9.93 092	8	32
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	30
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29
32	9.71 850	20	9.78 789	28	0.21 211	9.93 061	8	28
33	9.71 870	21	9.78 817	28	0.21 183	9.93 053	7	27
34	9.71 891	20	9.78 845	29	0.21 155	9.93 046	8	26
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	25
36	9.71 932	20	9.78 902	28	0.21 098	9.93 030	8	24
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	7	22
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	8	21
40	9.72 014	20	9.79 015	28	0.20 985	9.92 999	8	20
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19
42	9.72 055	20	9.79 072	28	0.20 928	9.92 983	7	18
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	15
46	9.72 137	20	9.79 185	28	0.20 815	9.92 952	8	14
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	10
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	5
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4
57	9.72 360	21	9.79 495	28	0.20 505	9.92 866	8	3
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850	8	1
60	9.72 421		9.79 579		0.20 421	9.92 842		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	29	28	21	20	8	7
6	2.9	2.8	2.1	2.0	0.8	0.7
7	3.4	3.3	2.5	2.3	0.9	0.8
8	3.9	3.7	2.8	2.7	1.1	0.9
9	4.4	4.2	3.2	3.0	1.2	1.1
10	4.8	4.7	3.5	3.3	1.3	1.2
20	9.7	9.3	7.0	6.7	2.7	2.3
30	14.5	14.0	10.5	10.0	4.0	3.5
40	19.3	18.7	14.0	13.3	5.3	4.7
50	24.2	23.3	17.5	16.7	6.7	5.8

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	7	56
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	8	55
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40
21	9.72 843	20	9.80 168	27	0.19 832	9.92 675	8	39
22	9.72 863	20	9.80 195	28	0.19 805	9.92 667	8	38
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	9	24
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	8	23
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	9	8
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	8	7
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1
60	9.73 611		9.81 252		0.18 748	9.92 359		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	29	28
6	2.9	2.8
7	3.4	3.3
8	3.9	3.7
9	4.4	4.2
10	4.8	4.7
20	9.7	9.3
30	14.5	14.0
40	19.3	18.7
50	24.2	23.3

	27
6	2.7
7	3.2
8	3.6
9	4.1
10	4.5
20	9.0
30	13.5
40	18.0
50	22.5

	21	20
6	2.1	2.0
7	2.5	2.3
8	2.8	2.7
9	3.2	3.0
10	3.5	3.3
20	7.0	6.7
30	10.5	10.0
40	14.0	13.3
50	17.5	16.7

	19	9
6	1.9	0.9
7	2.2	1.1
8	2.5	1.2
9	2.9	1.4
10	3.2	1.5
20	6.3	3.0
30	9.5	4.5
40	12.7	6.0
50	15.8	7.5

	8	7
6	0.8	0.7
7	0.9	0.8
8	1.1	0.9
9	1.2	1.1
10	1.3	1.2
20	2.7	2.3
30	4.0	3.5
40	5.3	4.7
50	6.7	5.8

Prop. Pts.

57°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.73 611		9.81 252		0.18 748	9.92 359		60
1	9.73 630	19	9.81 279	27	0.18 721	9.92 351	8	59
2	9.73 650	20	9.81 307	28	0.18 693	9.92 343	8	58
3	9.73 669	19	9.81 335	28	0.18 665	9.92 335	8	57
4	9.73 689	20	9.81 362	27	0.18 638	9.92 326	9	56
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	55
6	9.73 727	19	9.81 418	28	0.18 582	9.92 310	8	54
7	9.73 747	20	9.81 445	27	0.18 555	9.92 302	8	53
8	9.73 766	19	9.81 473	28	0.18 527	9.92 293	9	52
9	9.73 785	19	9.81 500	27	0.18 500	9.92 285	8	51
10	9.73 805	20	9.81 528	28	0.18 472	9.92 277	8	50
11	9.73 824	19	9.81 556	28	0.18 444	9.92 269	8	49
12	9.73 843	19	9.81 583	27	0.18 417	9.92 260	9	48
13	9.73 863	20	9.81 611	28	0.18 389	9.92 252	8	47
14	9.73 882	19	9.81 638	27	0.18 362	9.92 244	8	46
15	9.73 901	19	9.81 666	28	0.18 334	9.92 235	9	45
16	9.73 921	20	9.81 693	27	0.18 307	9.92 227	8	44
17	9.73 940	19	9.81 721	28	0.18 279	9.92 219	8	43
18	9.73 959	19	9.81 748	27	0.18 252	9.92 211	8	42
19	9.73 978	19	9.81 776	28	0.18 224	9.92 202	9	41
20	9.73 997	19	9.81 803	27	0.18 197	9.92 194	8	40
21	9.74 017	20	9.81 831	28	0.18 169	9.92 186	8	39
22	9.74 036	19	9.81 858	27	0.18 142	9.92 177	9	38
23	9.74 055	19	9.81 886	28	0.18 114	9.92 169	8	37
24	9.74 074	19	9.81 913	27	0.18 087	9.92 161	8	36
25	9.74 093	19	9.81 941	28	0.18 059	9.92 152	9	35
26	9.74 113	20	9.81 968	27	0.18 032	9.92 144	8	34
27	9.74 132	19	9.81 996	28	0.18 004	9.92 136	8	33
28	9.74 151	19	9.82 023	27	0.17 977	9.92 127	9	32
29	9.74 170	19	9.82 051	28	0.17 949	9.92 119	8	31
30	9.74 189	19	9.82 078	27	0.17 922	9.92 111	8	30
31	9.74 208	19	9.82 106	28	0.17 894	9.92 102	9	29
32	9.74 227	19	9.82 133	27	0.17 867	9.92 094	8	28
33	9.74 246	19	9.82 161	28	0.17 839	9.92 086	8	27
34	9.74 265	19	9.82 188	27	0.17 812	9.92 077	9	26
35	9.74 284	19	9.82 215	27	0.17 785	9.92 069	8	25
36	9.74 303	19	9.82 243	28	0.17 757	9.92 060	9	24
37	9.74 322	19	9.82 270	27	0.17 730	9.92 052	8	23
38	9.74 341	19	9.82 298	28	0.17 702	9.92 044	8	22
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	9	21
40	9.74 379	19	9.82 352	27	0.17 648	9.92 027	8	20
41	9.74 398	19	9.82 380	28	0.17 620	9.92 018	9	19
42	9.74 417	19	9.82 407	27	0.17 593	9.92 010	8	18
43	9.74 436	19	9.82 435	28	0.17 565	9.92 002	8	17
44	9.74 455	19	9.82 462	27	0.17 538	9.91 993	9	16
45	9.74 474	19	9.82 489	27	0.17 511	9.91 985	8	15
46	9.74 493	19	9.82 517	28	0.17 483	9.91 976	9	14
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	8	13
48	9.74 531	19	9.82 571	27	0.17 429	9.91 959	9	12
49	9.74 549	18	9.82 599	28	0.17 401	9.91 951	8	11
50	9.74 568	19	9.82 626	27	0.17 374	9.91 942	9	10
51	9.74 587	19	9.82 653	27	0.17 347	9.91 934	8	9
52	9.74 606	19	9.82 681	28	0.17 319	9.91 925	9	8
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	8	7
54	9.74 644	19	9.82 735	27	0.17 265	9.91 908	9	6
55	9.74 662	18	9.82 762	27	0.17 238	9.91 900	8	5
56	9.74 681	19	9.82 790	28	0.17 210	9.91 891	9	4
57	9.74 700	19	9.82 817	27	0.17 183	9.91 883	8	3
58	9.74 719	19	9.82 844	27	0.17 156	9.91 874	9	2
59	9.74 737	18	9.82 871	27	0.17 129	9.91 866	8	1
60	9.74 756	19	9.82 899	28	0.17 101	9.91 857	9	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	28	27
6	2.8	2.7
7	3.3	3.2
8	3.7	3.6
9	4.2	4.1
10	4.7	4.5
20	9.3	9.0
30	14.0	13.5
40	18.7	18.0
50	23.3	22.5

	20
6	2.0
7	2.3
8	2.7
9	3.0
10	3.3
20	6.7
30	10.0
40	13.3
50	16.7

	19
6	1.9
7	2.2
8	2.5
9	2.9
10	3.2
20	6.3
30	9.5
40	12.7
50	15.8

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	9	8
6	0.9	0.8
7	1.1	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	**60**
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	9	59
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58
3	9.74 812	19	9.82 980	28	0.17 020	9.91 832	9	57
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	9	55
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54
7	9.74 887	19	9.83 089	28	0.16 911	9.91 798	9	53
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	9	51
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	**50**
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	9	48
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	45
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	9	41
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	**40**
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	9	38
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	9	35
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33
28	9.75 276	18	9.83 659	27	0.16 341	9.91 617	9	32
29	9.75 294	19	9.83 686	27	0.16 314	9.91 608	9	31
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	**30**
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	9	26
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	25
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	9	22
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	**20**
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	9	15
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	**10**
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	9	5
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1
60	9.75 859		9.84 523		0.15 477	9.91 336		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	28	27
6	2.8	2.7
7	3.3	3.2
8	3.7	3.6
9	4.2	4.1
10	4.7	4.5
20	9.3	9.0
30	14.0	13.5
40	18.7	18.0
50	23.3	22.5

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	19
6	1.9
7	2.2
8	2.5
9	2.9
10	3.2
20	6.3
30	9.5
40	12.7
50	15.8

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	9	8
6	0.9	0.8
7	1.1	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	′
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60
1	9.75 877	18	9.84 55̄0	26	0.15 450	9.91 328	9	59
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	55
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	8	53
8	9.76 003	18	9.84 738	26	0.15 262	9.91 266	9	52
9	9.76 021	18	9.84 764	27	0.15 236	9.91 257	9	51
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49
12	9.76 075̄	18	9.84 845̄	27	0.15 155	9.91 230	9	48
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47
14	9.76 111	18	9.84 899	26	0.15 101	9.91 212	9	46
15	9.76 129	17	9.84 925	27	0.15 075̄	9.91 203	9	45
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	8	39
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	9	38
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	35
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105̄	9	34
27	9.76 342	18	9.85 247	26	0.14 753	9.91 096	9	33
28	9.76 360	18	9.85 273	27	0.14 727	9.91 087	9	32
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	10	26
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	9	25
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18
43	9.76 625̄	17	9.85 674	26	0.14 326	9.90 951	9	17
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	15
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14
47	9.76 695̄	17	9.85 780	27	0.14 220	9.90 915̄	9	13
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	10	12
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	9	11
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10
51	9.76 765̄	17	9.85 887	26	0.14 113	9.90 878	9	9
52	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8
53	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7
54	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6
55	9.76 835̄	17	9.85 993	27	0.14 007	9.90 842	10	5
56	9.76 852	18	9.86 020	26	0.13 980	9.90 832	9	4
57	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3
58	9.76 887	17	9.86 073	27	0.13 927	9.90 814	9	2
59	9.76 904	18	9.86 100	26	0.13 900	9.90 805̄	9	1
60	9.76 922		9.86 126		0.13 874	9.90 796		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	27	26
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5̄
9	4.1	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	17
6	1.7
7	2.0
8	2.3
9	2.6
10	2.8
20	5.7
30	8.5
40	11.3
50	14.2

	10
6	1.0
7	1.2
8	1.3
9	1.5
10	1.7
20	3.3
30	5.0
40	6.7
50	8.3

	9	8
6	0.9	0.8
7	1.1	0.9
8	1.2	1.1
9	1.4	1.2
10	1.5	1.3
20	3.0	2.7
30	4.5	4.0
40	6.0	5.3
50	7.5	6.7

36°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.76 922		9.86 126		0.13 874	9.90 796		60
1	9.76 939	17	9.86 153	27	0.13 847	9.90 787	9	59
2	9.76 957	18	9.86 179	26	0.13 821	9.90 777	10	58
3	9.76 974	17	9.86 206	27	0.13 794	9.90 768	9	57
4	9.76 991	17	9.86 232	26	0.13 768	9.90 759	9	56
5	9.77 009	18	9.86 259	27	0.13 741	9.90 750	9	55
6	9.77 026	17	9.86 285	26	0.13 715	9.90 741	9	54
7	9.77 043	17	9.86 312	27	0.13 688	9.90 731	10	53
8	9.77 061	18	9.86 338	26	0.13 662	9.90 722	9	52
9	9.77 078	17	9.86 365	27	0.13 635	9.90 713	9	51
10	9.77 095	17	9.86 392	27	0.13 608	9.90 704	9	50
11	9.77 112	17	9.86 418	26	0.13 582	9.90 694	10	49
12	9.77 130	18	9.86 445	27	0.13 555	9.90 685	9	48
13	9.77 147	17	9.86 471	26	0.13 529	9.90 676	9	47
14	9.77 164	17	9.86 498	27	0.13 502	9.90 667	9	46
15	9.77 181	17	9.86 524	26	0.13 476	9.90 657	10	45
16	9.77 199	18	9.86 551	27	0.13 449	9.90 648	9	44
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43
18	9.77 233	17	9.86 603	26	0.13 397	9.90 630	9	42
19	9.77 250	17	9.86 630	27	0.13 370	9.90 620	10	41
20	9.77 268	18	9.86 656	26	0.13 344	9.90 611	9	40
21	9.77 285	17	9.86 683	27	0.13 317	9.90 602	9	39
22	9.77 302	17	9.86 709	26	0.13 291	9.90 592	10	38
23	9.77 319	17	9.86 736	27	0.13 264	9.90 583	9	37
24	9.77 336	17	9.86 762	26	0.13 238	9.90 574	9	36
25	9.77 353	17	9.86 789	27	0.13 211	9.90 565	9	35
26	9.77 370	17	9.86 815	26	0.13 185	9.90 555	10	34
27	9.77 387	17	9.86 842	27	0.13 158	9.90 546	9	33
28	9.77 405	18	9.86 868	26	0.13 132	9.90 537	9	32
29	9.77 422	17	9.86 894	26	0.13 106	9.90 527	10	31
30	9.77 439	17	9.86 921	27	0.13 079	9.90 518	9	30
31	9.77 456	17	9.86 947	26	0.13 053	9.90 509	9	29
32	9.77 473	17	9.86 974	27	0.13 026	9.90 499	10	28
33	9.77 490	17	9.87 000	26	0.13 000	9.90 490	9	27
34	9.77 507	17	9.87 027	27	0.12 973	9.90 480	10	26
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25
36	9.77 541	17	9.87 079	26	0.12 921	9.90 462	9	24
37	9.77 558	17	9.87 106	27	0.12 894	9.90 452	10	23
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22
39	9.77 592	17	9.87 158	26	0.12 842	9.90 434	9	21
40	9.77 609	17	9.87 185	27	0.12 815	9.90 424	10	20
41	9.77 626	17	9.87 211	26	0.12 789	9.90 415	9	19
42	9.77 643	17	9.87 238	27	0.12 762	9.90 405	10	18
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	9	17
44	9.77 677	17	9.87 290	26	0.12 710	9.90 386	10	16
45	9.77 694	17	9.87 317	27	0.12 683	9.90 377	9	15
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	9	14
47	9.77 728	17	9.87 369	26	0.12 631	9.90 358	10	13
48	9.77 744	16	9.87 396	27	0.12 604	9.90 349	9	12
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	10	11
50	9.77 778	17	9.87 448	26	0.12 552	9.90 330	9	10
51	9.77 795	17	9.87 475	27	0.12 525	9.90 320	10	9
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	9	8
53	9.77 829	17	9.87 527	26	0.12 473	9.90 301	10	7
54	9.77 846	17	9.87 554	27	0.12 446	9.90 292	9	6
55	9.77 862	16	9.87 580	26	0.12 420	9.90 282	10	5
56	9.77 879	17	9.87 606	26	0.12 394	9.90 273	9	4
57	9.77 896	17	9.87 633	27	0.12 367	9.90 263	10	3
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	9	2
59	9.77 930	17	9.87 685	26	0.12 315	9.90 244	10	1
60	9.77 946	16	9.87 711	26	0.12 289	9.90 235	9	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	27	26
6	2.7	2.6
7	3.2	3.0
8	3.6	3.5
9	4.1	3.9
10	4.5	4.3
20	9.0	8.7
30	13.5	13.0
40	18.0	17.3
50	22.5	21.7

	18
6	1.8
7	2.1
8	2.4
9	2.7
10	3.0
20	6.0
30	9.0
40	12.0
50	15.0

	17
6	1.7
7	2.0
8	2.3
9	2.6
10	2.8
20	5.7
30	8.5
40	11.3
50	14.2

	16
6	1.6
7	1.9
8	2.1
9	2.4
10	2.7
20	5.3
30	8.0
40	10.7
50	13.3

	10	9
6	1.0	0.9
7	1.2	1.1
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3.0
30	5.0	4.5
40	6.7	6.0
50	8.3	7.5

Prop. Pts.

53°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	60
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	55
6	9.78 047	16	9.87 869	26	0.12 131	9.90 178	10	54
7	9.78 063	17	9.87 895	27	0.12 105	9.90 168	9	53
8	9.78 080	17	9.87 922	26	0.12 078	9.90 159	10	52
9	9.78 097	16	9.87 948	26	0.12 052	9.90 149	10	51
10	9.78 113	17	9.87 974	26	0.12 026	9.90 139	9	50
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	10	49
12	9.78 147	16	9.88 027	26	0.11 973	9.90 120	9	48
13	9.78 163	17	9.88 053	26	0.11 947	9.90 111	10	47
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46
15	9.78 197	16	9.88 105	26	0.11 895	9.90 091	9	45
16	9.78 213	17	9.88 131	27	0.11 869	9.90 082	10	44
17	9.78 230	16	9.88 158	26	0.11 842	9.90 072	9	43
18	9.78 246	17	9.88 184	26	0.11 816	9.90 063	10	42
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	40
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	10	38
23	9.78 329	17	9.88 315	26	0.11 685	9.90 014	9	37
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005	10	36
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	35
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	10	32
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	9	31
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	10	30
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29
32	9.78 478	16	9.88 550	27	0.11 450	9.89 927	9	28
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	10	26
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	25
36	9.78 543	17	9.88 655	26	0.11 345	9.89 888	9	24
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23
38	9.78 576	16	9.88 707	26	0.11 293	9.89 869	10	22
39	9.78 592	17	9.88 733	26	0.11 267	9.89 859	10	21
40	9.78 609	16	9.88 759	27	0.11 241	9.89 849	9	20
41	9.78 625	17	9.88 786	26	0.11 214	9.89 840	10	19
42	9.78 642	16	9.88 812	26	0.11 188	9.89 830	10	18
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	15
46	9.78 707	16	9.88 916	26	0.11 084	9.89 791	10	14
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	10	12
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	9	11
50	9.78 772	16	9.89 020	26	0.10 980	9.89 752	10	10
51	9.78 788	17	9.89 046	27	0.10 954	9.89 742	10	9
52	9.78 805	16	9.89 073	26	0.10 927	9.89 732	10	8
53	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7
54	9.78 837	16	9.89 125	26	0.10 875	9.89 712	10	6
55	9.78 853	16	9.89 151	26	0.10 849	9.89 702	9	5
56	9.78 869	17	9.89 177	26	0.10 823	9.89 693	10	4
57	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3
58	9.78 902	16	9.89 229	26	0.10 771	9.89 673	10	2
59	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1
60	9.78 934		9.89 281		0.10 719	9.89 653		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	27	26	17	16	10	9
6	2.7	2.6	1.7	1.6	1.0	0.9
7	3.2	3.0	2.0	1.9	1.2	1.1
8	3.6	3.5	2.3	2.1	1.3	1.2
9	4.1	3.9	2.6	2.4	1.5	1.4
10	4.5	4.3	2.8	2.7	1.7	1.5
20	9.0	8.7	5.7	5.3	3.3	3.0
30	13.5	13.0	8.5	8.0	5.0	4.5
40	18.0	17.3	11.3	10.7	6.7	6.0
50	22.5	21.7	14.2	13.3	8.3	7.5

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	9	58
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	10	57
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	50
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	40
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	11	32
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	30
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	20
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	11	19
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	10	18
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	11	12
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	10	11
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	10
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	11	2
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	10	1
60	9.79 887		9.90 837		0.09 163	9.89 050		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25
6	2.6	2.5
7	3.0	2.9
8	3.5	3.3
9	3.9	3.8
10	4.3	4.2
20	8.7	8.3
30	13.0	12.5
40	17.3	16.7
50	21.7	20.8

	17
6	1.7
7	2.0
8	2.3
9	2.6
10	2.8
20	5.7
30	8.5
40	11.3
50	14.2

	16	15
6	1.6	1.5
7	1.9	1.8
8	2.1	2.0
9	2.4	2.3
10	2.7	2.5
20	5.3	5.0
30	8.0	7.5
40	10.7	10.0
50	13.3	12.5

	11
6	1.1
7	1.3
8	1.5
9	1.7
10	1.8
20	3.7
30	5.5
40	7.3
50	9.2

	10	9
6	1.0	0.9
7	1.2	1.1
8	1.3	1.2
9	1.5	1.4
10	1.7	1.5
20	3.3	3 [illegible]
30	5.0	4 [illegible]
40	6.7	6.0
50	8.3	7.5

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	60
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	55
6	9.79 981	15	9.90 992	26	0.09 008	9.88 989	11	54
7	9.79 996	16	9.91 018	25	0.08 982	9.88 978	10	53
8	9.80 012	15	9.91 043	26	0.08 957	9.88 968	10	52
9	9.80 027	16	9.91 069	26	0.08 931	9.88 958	10	51
10	9.80 043	15	9.91 095	26	0.08 905	9.88 948	11	50
11	9.80 058	16	9.91 121	26	0.08 879	9.88 937	10	49
12	9.80 074	15	9.91 147	25	0.08 853	9.88 927	10	48
13	9.80 089	16	9.91 172	26	0.08 828	9.88 917	11	47
14	9.80 105	15	9.91 198	26	0.08 802	9.88 906	10	46
15	9.80 120	16	9.91 224	26	0.08 776	9.88 896	10	45
16	9.80 136	15	9.91 250	26	0.08 750	9.88 886	11	44
17	9.80 151	15	9.91 276	25	0.08 724	9.88 875	10	43
18	9.80 166	16	9.91 301	26	0.08 699	9.88 865	10	42
19	9.80 182	15	9.91 327	26	0.08 673	9.88 855	11	41
20	9.80 197	16	9.91 353	26	0.08 647	9.88 844	10	40
21	9.80 213	15	9.91 379	25	0.08 621	9.88 834	10	39
22	9.80 228	16	9.91 404	26	0.08 596	9.88 824	11	38
23	9.80 244	15	9.91 430	26	0.08 570	9.88 813	10	37
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	11	35
26	9.80 290	15	9.91 507	26	0.08 493	9.88 782	10	34
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	11	33
28	9.80 320	16	9.91 559	26	0.08 441	9.88 761	10	32
29	9.80 336	15	9.91 585	25	0.08 415	9.88 751	10	31
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	11	30
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	10	29
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	11	28
33	9.80 397	15	9.91 688	25	0.08 312	9.88 709	10	27
34	9.80 412	16	9.91 713	26	0.08 287	9.88 699	11	26
35	9.80 428	15	9.91 739	26	0.08 261	9.88 688	10	25
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24
37	9.80 458	15	9.91 791	25	0.08 209	9.88 668	11	23
38	9.80 473	16	9.91 816	26	0.08 184	9.88 657	10	22
39	9.80 489	15	9.91 842	26	0.08 158	9.88 647	11	21
40	9.80 504	15	9.91 868	25	0.08 132	9.88 636	10	20
41	9.80 519	15	9.91 893	26	0.08 107	9.88 626	11	19
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	10	18
43	9.80 550	15	9.91 945	26	0.08 055	9.88 605	11	17
44	9.80 565	15	9.91 971	25	0.08 029	9.88 594	10	16
45	9.80 580	15	9.91 996	26	0.08 004	9.88 584	11	15
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	10	14
47	9.80 610	15	9.92 048	25	0.07 952	9.88 563	11	13
48	9.80 625	16	9.92 073	26	0.07 927	9.88 552	10	12
49	9.80 641	15	9.92 099	26	0.07 901	9.88 542	11	11
50	9.80 656	15	9.92 125	25	0.07 875	9.88 531	10	10
51	9.80 671	15	9.92 150	26	0.07 850	9.88 521	11	9
52	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8
53	9.80 701	15	9.92 202	25	0.07 798	9.88 499	10	7
54	9.80 716	15	9.92 227	26	0.07 773	9.88 489	11	6
55	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	5
56	9.80 746	16	9.92 279	25	0.07 721	9.88 468	11	4
57	9.80 762	15	9.92 304	26	0.07 696	9.88 457	10	3
58	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2
59	9.80 792	15	9.92 356	25	0.07 644	9.88 436	11	1
60	9.80 807		9.92 381		0.07 619	9.88 425		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25	16	15	11	10
6	2.6	2.5	1.6	1.5	1.1	1.0
7	3.0	2.9	1.9	1.8	1.3	1.2
8	3.5	3.3	2.1	2.0	1.5	1.3
9	3.9	3.8	2.4	2.3	1.7	1.5
10	4.3	4.2	2.7	2.5	1.8	1.7
20	8.7	8.3	5.3	5.0	3.7	3.3
30	13.0	12.5	8.0	7.5	5.5	5.0
40	17.3	16.7	10.7	10.0	7.3	6.7
50	21.7	20.8	13.3	12.5	9.2	8.3

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.80 807	15	9.92 381	26	0.07 619	9.88 425	10	60
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415	11	59
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	55
6	9.80 897	15	9.92 535	26	0.07 465	9.88 362	11	54
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	50
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	10	49
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	11	48
13	9.81 002	15	9.92 715	25	0.07 285	9.88 287	11	47
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	45
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255	11	44
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	40
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39
22	9.81 136	15	9.92 945	26	0.07 055	9.88 191	11	38
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	35
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33
28	9.81 225	15	9.93 099	25	0.06 901	9.88 126	11	32
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	30
31	9.81 269	15	9.93 175	26	0.06 825	9.88 094	11	29
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	10	26
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	11	25
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	20
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975	11	18
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	15
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14
47	9.81 505	14	9.93 584	26	0.06 416	9.87 920	11	13
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	10
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8
53	9.81 592	15	9.93 738	25	0.06 262	9.87 855	11	7
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	5
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3
58	9.81 665	15	9.93 865	26	0.06 135	9.87 800	11	2
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1
60	9.81 694		9.93 916		0.06 084	9.87 778		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25	15	14	11	10
6	2.6	2.5	1.5	1.4	1.1	1.0
7	3.0	2.9	1.8	1.6	1.3	1.2
8	3.5	3.3	2.0	1.9	1.5	1.3
9	3.9	3.8	2.3	2.1	1.7	1.5
10	4.3	4.2	2.5	2.3	1.8	1.7
20	8.7	8.3	5.0	4.7	3.7	3.3
30	13.0	12.5	7.5	7.0	5.5	5.0
40	17.3	16.7	10.0	9.3	7.3	6.7
50	21.7	20.8	12.5	11.7	9.2	8.3

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	**60**
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54
7	9.81 796	14	9.94 095	25	0.05 905	9.87 701	11	53
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52
9	9.81 825	14	9.94 146	25	0.05 854	9.87 679	11	51
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	**50**
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49
12	9.81 868	14	9.94 222	26	0.05 778	9.87 646	11	48
13	9.81 882	15	9.94 248	25	0.05 752	9.87 635	11	47
14	9.81 897	14	9.94 273	26	0.05 727	9.87 624	11	46
15	9.81 911	15	9.94 299	25	0.05 701	9.87 613	12	45
16	9.81 926	14	9.94 324	26	0.05 676	9.87 601	11	44
17	9.81 940	15	9.94 350	25	0.05 650	9.87 590	11	43
18	9.81 955	14	9.94 375	26	0.05 625	9.87 579	11	42
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	**40**
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39
22	9.82 012	14	9.94 477	26	0.05 523	9.87 535	11	38
23	9.82 026	15	9.94 503	25	0.05 497	9.87 524	11	37
24	9.82 041	14	9.94 528	26	0.05 472	9.87 513	12	36
25	9.82 055	14	9.94 554	25	0.05 446	9.87 501	11	35
26	9.82 069	15	9.94 579	25	0.05 421	9.87 490	11	34
27	9.82 084	14	9.94 604	26	0.05 396	9.87 479	11	33
28	9.82 098	14	9.94 630	25	0.05 370	9.87 468	11	32
29	9.82 112	14	9.94 655	26	0.05 345	9.87 457	11	31
30	9.82 126	15	9.94 681	25	0.05 319	9.87 446	12	**30**
31	9.82 141	14	9.94 706	26	0.05 294	9.87 434	11	29
32	9.82 155	14	9.94 732	25	0.05 268	9.87 423	11	28
33	9.82 169	15	9.94 757	26	0.05 243	9.87 412	11	27
34	9.82 184	14	9.94 783	25	0.05 217	9.87 401	11	26
35	9.82 198	14	9.94 808	26	0.05 192	9.87 390	12	25
36	9.82 212	14	9.94 834	25	0.05 166	9.87 378	11	24
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22
39	9.82 255	14	9.94 910	25	0.05 090	9.87 345	11	21
40	9.82 269	14	9.94 935	26	0.05 065	9.87 334	12	**20**
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19
42	9.82 297	14	9.94 986	26	0.05 014	9.87 311	11	18
43	9.82 311	15	9.95 012	25	0.04 988	9.87 300	12	17
44	9.82 326	14	9.95 037	25	0.04 963	9.87 288	11	16
45	9.82 340	14	9.95 062	26	0.04 938	9.87 277	11	15
46	9.82 354	14	9.95 088	25	0.04 912	9.87 266	11	14
47	9.82 368	14	9.95 113	26	0.04 887	9.87 255	12	13
48	9.82 382	14	9.95 139	25	0.04 861	9.87 243	11	12
49	9.82 396	14	9.95 164	26	0.04 836	9.87 232	11	11
50	9.82 410	14	9.95 190	25	0.04 810	9.87 221	12	**10**
51	9.82 424	15	9.95 215	25	0.04 785	9.87 209	11	9
52	9.82 439	14	9.95 240	26	0.04 760	9.87 198	11	8
53	9.82 453	14	9.95 266	25	0.04 734	9.87 187	12	7
54	9.82 467	14	9.95 291	26	0.04 709	9.87 175	11	6
55	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5
56	9.82 495	14	9.95 342	26	0.04 658	9.87 153	12	4
57	9.82 509	14	9.95 368	25	0.04 632	9.87 141	11	3
58	9.82 523	14	9.95 393	25	0.04 607	9.87 130	11	2
59	9.82 537	14	9.95 418	26	0.04 582	9.87 119	12	1
60	9.82 551		9.95 444		0.04 556	9.87 107		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25	15	14
6	2.6	2.5	1.5	1.4
7	3.0	2.9	1.8	1.6
8	3.5	3.3	2.0	1.9
9	3.9	3.8	2.3	2.1
10	4.3	4.2	2.5	2.3
20	8.7	8.3	5.0	4.7
30	13.0	12.5	7.5	7.0
40	17.3	16.7	10.0	9.3
50	21.7	20.8	12.5	11.7

	12	11
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.7
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	**60**
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59
2	9.82 579	14	9.95 495̄	25	0.04 505	9.87 085̄	12	58
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57
4	9.82 607	14	9.95 545	26	0.04 455̄	9.87 062	12	56
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52
9	9.82 677	14	9.95 672	26	0.04 328	9.87 005̄	12	51
10	9.82 691	14	9.95 698	25	0.04 302	9.86 993	11	**50**
11	9.82 705̄	14	9.95 723	25	0.04 277	9.86 982	12	49
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47
14	9.82 747	14	9.95 799	26	0.04 201	9.86 947	11	46
15	9.82 761	14	9.95 825̄	25	0.04 175	9.86 936	12	45
16	9.82 775̄	13	9.95 850	25	0.04 150	9.86 924	11	44
17	9.82 788	14	9.95 875	26	0.04 125̄	9.86 913	11	43
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	12	42
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	11	41
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	12	**40**
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38
23	9.82 872	13	9.96 028	25	0.03 972	9.86 844	12	37
24	9.82 885	14	9.96 053	25	0.03 947	9.86 832	11	36
25	9.82 899	14	9.96 078	26	0.03 922	9.86 821	12	35
26	9.82 913	14	9.96 104	25	0.03 896	9.86 809	11	34
27	9.82 927	14	9.96 129	26	0.03 871	9.86 798	12	33
28	9.82 941	14	9.96 155̄	25	0.03 845	9.86 786	11	32
29	9.82 955̄	13	9.96 180	25	0.03 820	9.86 775̄	12	31
30	9.82 968	14	9.96 205	26	0.03 795̄	9.86 763	11	**30**
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	12	28
33	9.83 010	13	9.96 281	26	0.03 719	9.86 728	11	27
34	9.83 023	14	9.96 307	25	0.03 693	9.86 717	12	26
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	11	25
36	9.83 051	14	9.96 357	26	0.03 643	9.86 694	12	24
37	9.83 065̄	13	9.96 383	25	0.03 617	9.86 682	12	23
38	9.83 078	14	9.96 408	25	0.03 592	9.86 670	11	22
39	9.83 092	14	9.96 433	26	0.03 567	9.86 659	12	21
40	9.83 106	14	9.96 459	25	0.03 541	9.86 647	12	**20**
41	9.83 120	13	9.96 484	26	0.03 516	9.86 635	11	19
42	9.83 133	14	9.96 510	25	0.03 490	9.86 624	12	18
43	9.83 147	14	9.96 535̄	25	0.03 465	9.86 612	12	17
44	9.83 161	13	9.96 560	26	0.03 440	9.86 600	11	16
45	9.83 174	14	9.96 586	25	0.03 414	9.86 589	12	15
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	12	14
47	9.83 202	13	9.96 636	26	0.03 364	9.86 565	11	13
48	9.83 215	14	9.96 662	25	0.03 338	9.86 554	12	12
49	9.83 229	13	9.96 687	25	0.03 313	9.86 542	12	11
50	9.83 242	14	9.96 712	26	0.03 288	9.86 530	12	**10**
51	9.83 256	14	9.96 738	25	0.03 262	9.86 518	11	9
52	9.83 270	13	9.96 763	25	0.03 237	9.86 507	12	8
53	9.83 283	14	9.96 788	26	0.03 212	9.86 495	12	7
54	9.83 297	13	9.96 814	25	0.03 186	9.86 483	11	6
55	9.83 310	14	9.96 839	25	0.03 161	9.86 472	12	5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	12	4
57	9.83 338	13	9.96 890	25	0.03 110	9.86 448	12	3
58	9.83 351	14	9.96 915̄	25	0.03 085	9.86 436	11	2
59	9.83 365̄	13	9.96 940	26	0.03 060	9.86 425̄	12	1
60	9.83 378		9.96 966		0.03 034	9.86 413		**0**
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25	14	13	12	11
6	2.6	2.5	1.4	1.3	1.2	1.1
7	3.0	2.9	1.6	1.5	1.4	1.3
8	3.5̄	3.3	1.9	1.7	1.6	1.5̄
9	3.9	3.8	2.1	2.0	1.8	1.7
10	4.3	4.2	2.3	2.2	2.0	1.8
20	8.7	8.3	4.7	4.3	4.0	3.7
30	13.0	12.5	7.0	6.5	6.0	5.5
40	17.3	16.7	9.3	8.7	8.0	7.3
50	21.7	20.8	11.7	10.8	10.0	9.2

47°

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60
1	9.83 392	13	9.96 991	25	0.03 009	9.86 401	12	59
2	9.83 405	14	9.97 016	26	0.02 984	9.86 389	12	58
3	9.83 419	13	9.97 042	25	0.02 958	9.86 377	11	57
4	9.83 432	14	9.97 067	25	0.02 933	9.86 366	12	56
5	9.83 446	13	9.97 092	26	0.02 908	9.86 354	12	55
6	9.83 459	14	9.97 118	25	0.02 882	9.86 342	12	54
7	9.83 473	13	9.97 143	25	0.02 857	9.86 330	12	53
8	9.83 486	14	9.97 168	25	0.02 832	9.86 318	12	52
9	9.83 500	13	9.97 193	26	0.02 807	9.86 306	11	51
10	9.83 513	14	9.97 219	25	0.02 781	9.86 295	12	50
11	9.83 527	13	9.97 244	25	0.02 756	9.86 283	12	49
12	9.83 540	14	9.97 269	26	0.02 731	9.86 271	12	48
13	9.83 554	13	9.97 295	25	0.02 705	9.86 259	12	47
14	9.83 567	14	9.97 320	25	0.02 680	9.86 247	12	46
15	9.83 581	13	9.97 345	26	0.02 655	9.86 235	12	45
16	9.83 594	14	9.97 371	25	0.02 629	9.86 223	12	44
17	9.83 608	13	9.97 396	25	0.02 604	9.86 211	11	43
18	9.83 621	13	9.97 421	26	0.02 579	9.86 200	12	42
19	9.83 634	14	9.97 447	25	0.02 553	9.86 188	12	41
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40
21	9.83 661	13	9.97 497	26	0.02 503	9.86 164	12	39
22	9.83 674	14	9.97 523	25	0.02 477	9.86 152	12	38
23	9.83 688	13	9.97 548	25	0.02 452	9.86 140	12	37
24	9.83 701	14	9.97 573	25	0.02 427	9.86 128	12	36
25	9.83 715	13	9.97 598	26	0.02 402	9.86 116	12	35
26	9.83 728	13	9.97 624	25	0.02 376	9.86 104	12	34
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33
28	9.83 755	13	9.97 674	26	0.02 326	9.86 080	12	32
29	9.83 768	13	9.97 700	25	0.02 300	9.86 068	12	31
30	9.83 781	14	9.97 725	25	0.02 275	9.86 056	12	30
31	9.83 795	13	9.97 750	26	0.02 250	9.86 044	12	29
32	9.83 808	13	9.97 776	25	0.02 224	9.86 032	12	28
33	9.83 821	13	9.97 801	25	0.02 199	9.86 020	12	27
34	9.83 834	14	9.97 826	25	0.02 174	9.86 008	12	26
35	9.83 848	13	9.97 851	26	0.02 149	9.85 996	12	25
36	9.83 861	13	9.97 877	25	0.02 123	9.85 984	12	24
37	9.83 874	13	9.97 902	25	0.02 098	9.85 972	12	23
38	9.83 887	14	9.97 927	26	0.02 073	9.85 960	12	22
39	9.83 901	13	9.97 953	25	0.02 047	9.85 948	12	21
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20
41	9.83 927	13	9.98 003	26	0.01 997	9.85 924	12	19
42	9.83 940	14	9.98 029	25	0.01 971	9.85 912	12	18
43	9.83 954	13	9.98 054	25	0.01 946	9.85 900	12	17
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16
45	9.83 980	13	9.98 104	26	0.01 896	9.85 876	12	15
46	9.83 993	13	9.98 130	25	0.01 870	9.85 864	13	14
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	12	13
48	9.84 020	13	9.98 180	26	0.01 820	9.85 839	12	12
49	9.84 033	13	9.98 206	25	0.01 794	9.85 827	12	11
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10
51	9.84 059	13	9.98 256	25	0.01 744	9.85 803	12	9
52	9.84 072	13	9.98 281	26	0.01 719	9.85 791	12	8
53	9.84 085	13	9.98 307	25	0.01 693	9.85 779	13	7
54	9.84 098	14	9.98 332	25	0.01 668	9.85 766	12	6
55	9.84 112	13	9.98 357	26	0.01 643	9.85 754	12	5
56	9.84 125	13	9.98 383	25	0.01 617	9.85 742	12	4
57	9.84 138	13	9.98 408	25	0.01 592	9.85 730	12	3
58	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2
59	9.84 164	13	9.98 458	26	0.01 542	9.85 706	13	1
60	9.84 177		9.98 484		0.01 516	9.85 693		0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26
6	2.6
7	3.0
8	3.5
9	3.9
10	4.3
20	8.7
30	13.0
40	17.3
50	21.7

	25
6	2.5
7	2.9
8	3.3
9	3.8
10	4.2
20	8.3
30	12.5
40	16.7
50	20.8

	14
6	1.4
7	1.6
8	1.9
9	2.1
10	2.3
20	4.7
30	7.0
40	9.3
50	11.7

	13
6	1.3
7	1.5
8	1.7
9	2.0
10	2.2
20	4.3
30	6.5
40	8.7
50	10.8

	12	11
6	1.2	1.1
7	1.4	1.3
8	1.6	1.5
9	1.8	1.7
10	2.0	1.8
20	4.0	3.7
30	6.0	5.5
40	8.0	7.3
50	10.0	9.2

Prop. Pts.

′	L. Sin.	d.	L. Tang.	c. d.	L. Cotg.	L. Cos.	d.	
0	9.84 177		9.98 484		0.01 516	9.85 693		60
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59
2	9.84 203	13	9.98 534	25	0.01 466	9.85 669	12	58
3	9.84 216	13	9.98 560	26	0.01 440	9.85 657	12	57
4	9.84 229	13	9.98 585	25	0.01 415	9.85 645	12	56
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	13	55
6	9.84 255	13	9.98 635	25	0.01 365	9.85 620	12	54
7	9.84 269	14	9.98 661	26	0.01 339	9.85 608	12	53
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	12	52
9	9.84 295	13	9.98 711	25	0.01 289	9.85 583	13	51
10	9.84 308	13	9.98 737	26	0.01 263	9.85 571	12	50
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	12	48
13	9.84 347	13	9.98 812	25	0.01 188	9.85 534	13	47
14	9.84 360	13	9.98 838	26	0.01 162	9.85 522	12	46
15	9.84 373	13	9.98 863	25	0.01 137	9.85 510	12	45
16	9.84 385	12	9.98 888	25	0.01 112	9.85 497	13	44
17	9.84 398	13	9.98 913	25	0.01 087	9.85 485	12	43
18	9.84 411	13	9.98 939	26	0.01 061	9.85 473	12	42
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	13	41
20	9.84 437	13	9.98 989	25	0.01 011	9.85 448	12	40
21	9.84 450	13	9.99 015	26	0.00 985	9.85 436	12	39
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	13	38
23	9.84 476	13	9.99 065	25	0.00 935	9.85 411	12	37
24	9.84 489	13	9.99 090	25	0.00 910	9.85 399	12	36
25	9.84 502	13	9.99 116	26	0.00 884	9.85 386	13	35
26	9.84 515	13	9.99 141	25	0.00 859	9.85 374	12	34
27	9.84 528	13	9.99 166	25	0.00 834	9.85 361	13	33
28	9.84 540	12	9.99 191	25	0.00 809	9.85 349	12	32
29	9.84 553	13	9.99 217	26	0.00 783	9.85 337	12	31
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	13	30
31	9.84 579	13	9.99 267	25	0.00 733	9.85 312	12	29
32	9.84 592	13	9.99 293	26	0.00 707	9.85 299	13	28
33	9.84 605	13	9.99 318	25	0.00 682	9.85 287	12	27
34	9.84 618	13	9.99 343	25	0.00 657	9.85 274	13	26
35	9.84 630	12	9.99 368	25	0.00 632	9.85 262	12	25
36	9.84 643	13	9.99 394	26	0.00 606	9.85 250	12	24
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	13	23
38	9.84 669	13	9.99 444	25	0.00 556	9.85 225	12	22
39	9.84 682	13	9.99 469	25	0.00 531	9.85 212	13	21
40	9.84 694	12	9.99 495	26	0.00 505	9.85 200	12	20
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	13	19
42	9.84 720	13	9.99 545	25	0.00 455	9.85 175	12	18
43	9.84 733	13	9.99 570	25	0.00 430	9.85 162	13	17
44	9.84 745	12	9.99 596	26	0.00 404	9.85 150	12	16
45	9.84 758	13	9.99 621	25	0.00 379	9.85 137	13	15
46	9.84 771	13	9.99 646	25	0.00 354	9.85 125	12	14
47	9.84 784	13	9.99 672	26	0.00 328	9.85 112	13	13
48	9.84 796	12	9.99 697	25	0.00 303	9.85 100	12	12
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	13	11
50	9.84 822	13	9.99 747	25	0.00 253	9.85 074	13	10
51	9.84 835	13	9.99 773	26	0.00 227	9.85 062	12	9
52	9.84 847	12	9.99 798	25	0.00 202	9.85 049	13	8
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	12	7
54	9.84 873	13	9.99 848	25	0.00 152	9.85 024	13	6
55	9.84 885	12	9.99 874	26	0.00 126	9.85 012	12	5
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	13	4
57	9.84 911	13	9.99 924	25	0.00 076	9.84 986	13	3
58	9.84 923	12	9.99 949	25	0.00 051	9.84 974	12	2
59	9.84 936	13	9.99 975	26	0.00 025	9.84 961	13	1
60	9.84 949	13	0.00 000	25	0.00 000	9.84 949	12	0
	L. Cos.	d.	L. Cotg.	c. d.	L. Tang.	L. Sin.	d.	′

Prop. Pts.

	26	25	14	13	12
6	2.6	2.5	1.4	1.3	1.2
7	3.0	2.9	1.6	1.5	1.4
8	3.5	3.3	1.9	1.7	1.6
9	3.9	3.8	2.1	2.0	1.8
10	4.3	4.2	2.3	2.2	2.0
20	8.7	8.3	4.7	4.3	4.0
30	13.0	12.5	7.0	6.5	6.0
40	17.3	16.7	9.3	8.7	8.0
50	21.7	20.8	11.7	10.8	10.0

TABLE III.

NATURAL

SINES, COSINES, TANGENTS, AND COTANGENTS.

° '	N. Sin.	N. Tan.	N. Cot.	N. Cos.		° '	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0 0	.00 000	.00 000	Infinity.	Unity.	90 0	2 30	.04 362	.04 366	22.904	.99 905	87 30
5	145	145	687.55	"	55	35	507	512	22.164	898	25
10	291	291	343 77	"	50	40	653	658	21.470	892	20
15	436	436	229.18	.99 999	45	45	798	803	20.819	885	15
20	582	582	171.89	998	40	50	.04 943	.04 949	20.206	878	10
25	727	727	137.51	997	35	55	.05 088	.05 095	19.627	870	5
30	00 873	.00 873	114.59	.99 996	30	3 0	.05 234	.05 241	19.081	.99 863	87 0
35	.01 018	01 018	98.218	995	25	5	379	387	18.564	855	55
40	164	164	85.940	993	20	10	524	533	18 075	847	50
45	309	309	76.390	991	15	15	669	678	17.611	839	45
50	454	455	68 750	989	10	20	814	824	17.169	831	40
55	600	600	62 499	987	5	25	.05 960	.05 970	16.750	822	35
1 0	.01 745	.01 746	57.290	.99 985	89 0	30	.06 105	.06 116	16.350	.99 813	30
5	.01 891	.01 891	52 882	982	55	35	250	262	15.969	804	25
10	.02 036	02 036	49.104	979	50	40	395	408	.605	795	20
15	181	182	45.829	976	45	45	540	554	15 257	786	15
20	327	328	42.964	973	40	50	685	700	14.924	776	10
25	472	473	40.436	969	35	55	831	847	.606	766	5
30	.02 618	.02 619	38.188	.99 966	30	4 0	.06 976	.06 993	14.301	.99 756	86 0
35	763	764	36.178	962	25	5	.07 121	.07 139	14.008	746	55
40	.02 908	.02 910	34 368	958	20	10	266	285	13.727	736	50
45	.03 054	.03 055	32.730	953	15	15	411	431	.457	725	45
50	199	201	31.242	949	10	20	556	578	13.197	714	40
55	345	346	29.882	944	5	25	701	724	12.947	703	35
2 0	.03 490	.03 492	28.636	.99 939	88 0	30	.07 846	.07 870	12.706	.99 692	30
5	635	638	27.490	934	55	35	.07 991	.08 017	.474	680	25
10	781	783	26.432	929	50	40	.08 136	163	.251	668	20
15	.03 926	.03 929	25.452	923	45	45	281	309	12.035	657	15
20	.04 071	.04 075	24.542	917	40	50	426	456	11.826	644	10
25	217	220	23.695	911	35	55	571	602	.625	632	5
2 30	.04 362	.04 366	22 904	.99 905	87 30	5 0	.08 716	.08 749	11.430	.99 619	85 0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	° '		N. Cos.	N. Cot.	N. Tan.	N. Sin.	° '

° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.		° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
5 0	.08 716	.08 749	11.430	.99 619	**85** 0	**10** 0	.17 365	.17 633	5.6713	.98 481	**80** 0
5	.08 860	.08 895	.242	607	55	5	508	783	.6234	455	55
10	.09 005	.09 042	11.059	594	50	10	651	.17 933	.5764	430	50
15	150	189	10.883	580	45	15	794	.18 083	.5301	404	45
20	295	335	.712	567	40	20	.17 937	233	.4845	378	40
25	440	482	.546	553	35	25	.18 081	384	.4397	352	35
30	.09 585	.09 629	10.385	.99 540	30	30	.18 224	.18 534	5.3955	.98 325	30
35	729	776	.229	526	25	35	367	684	.3521	299	25
40	.09 874	.09 923	10.078	511	20	40	509	835	.3093	272	20
45	.10 019	.10 069	9.9310	497	15	45	652	.18 986	.2672	245	15
50	164	216	.7882	482	10	50	795	.19 136	.2257	218	10
55	308	363	.6493	467	5	55	.18 938	287	.1848	190	5
6 0	.10 453	.10 510	9.5144	.99 452	**84** 0	**11** 0	.19 081	.19 438	5.1446	.98 163	**79** 0
5	597	657	.3831	437	55	5	224	589	.1049	135	55
10	742	805	.2553	421	50	10	366	740	.0658	107	50
15	.10 887	.10 952	.1309	406	45	15	509	.19 891	5.0273	079	45
20	.11 031	.11 099	9.0098	390	40	20	652	.20 042	4.9894	050	40
25	176	246	8.8919	374	35	25	794	194	.9520	.98 021	35
30	.11 320	.11 394	8.7769	.99 357	30	30	.19 937	.20 345	4.9152	.97 992	30
35	465	541	.6648	341	25	35	.20 079	497	.8788	963	25
40	609	688	.5555	324	20	40	222	648	.8430	934	20
45	754	836	.4490	307	15	45	364	800	.8077	905	15
50	.11 898	.11 983	.3450	290	10	50	507	20 952	.7729	875	10
55	.12 043	.12 131	.2434	272	5	55	649	.21 104	.7385	845	5
7 0	.12 187	.12 278	8.1443	.99 255	**83** 0	**12** 0	.20 791	.21 256	4.7046	.97 815	**78** 0
5	331	426	8.0476	237	55	5	.20 933	408	.6712	784	55
10	476	574	7.9530	219	50	10	.21 076	560	.6382	754	50
15	620	722	.8606	200	45	15	218	712	.6057	723	45
20	764	.12 869	.7704	182	40	20	360	.21 864	.5736	692	40
25	.12 908	.13 017	.6821	163	35	25	502	.22 017	.5420	661	35
30	.13 053	.13 165	7.5958	.99 144	30	30	.21 644	.22 169	4.5107	97 630	30
35	197	313	.5113	125	25	35	786	322	.4799	598	25
40	341	461	.4287	106	20	40	21 928	475	.4494	566	20
45	485	609	.3479	087	15	45	.22 070	628	.4194	534	15
50	629	758	.2687	067	10	50	212	781	.3897	502	10
55	773	13 906	.1912	047	5	55	353	.22 934	.3604	470	5
8 0	.13 917	.14 054	7.1154	99 027	**82** 0	**13** 0	.22 495	.23 087	4.3315	.97 437	**77** 0
5	.14 061	202	7.0410	.99 006	55	5	637	240	.3029	404	55
10	205	351	6.9682	98 986	50	10	778	393	.2747	371	50
15	349	499	.8969	965	45	15	.22 920	547	.2468	338	45
20	493	648	.8269	944	40	20	.23 062	700	.2193	304	40
25	637	796	.7584	923	35	25	203	.23 854	.1922	271	35
30	.14 781	.14 945	6.6912	.98 902	30	30	.23 345	.24 008	4.1653	.97 237	30
35	.14 925	.15 094	.6252	880	25	35	486	162	.1388	203	25
40	.15 069	243	.5606	858	20	40	627	316	.1126	169	20
45	212	391	.4971	836	15	45	769	470	.0867	134	15
50	356	540	.4348	814	10	50	.23 910	624	.0611	100	10
55	500	689	.3737	791	5	55	.24 051	778	.0358	065	5
9 0	.15 643	.15 838	6.3138	.98 769	**81** 0	**14** 0	.24 192	.24 933	4.0108	.97 030	**76** 0
5	787	.15 988	.2549	746	55	5	333	.25 087	3.9861	.96 994	55
10	.15 931	.16 137	.1970	723	50	10	474	242	.9617	959	50
15	16 074	286	.1402	700	45	15	615	397	.9375	923	45
20	218	435	.0844	676	40	20	756	552	.9136	887	40
25	361	585	6.0296	652	35	25	.24 897	707	.8900	851	35
30	.16 505	.16 734	5.9758	.98 629	30	30	.25 038	.25 862	3.8667	.96 815	30
35	648	.16 884	.9228	604	25	35	179	.26 017	.8436	778	25
40	792	.17 033	.8708	580	20	40	320	172	.8208	742	20
45	.16 935	183	.8197	556	15	45	460	328	.7983	705	15
50	.17 078	333	.7694	531	10	50	601	483	.7760	667	10
55	222	483	.7199	506	5	55	741	639	.7539	630	5
10 0	.17 365	.17 633	5.6713	.98 481	**80** 0	**15** 0	.25 882	.26 795	3.7321	.96 593	**75** 0
	N. Cos.	N. Cot.	N. Tan.	N Sin.	° ′		N. Cos.	N. Cot.	N. Tan.	N. Sin.	° ′

° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.		° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
15 0	.25 882	.26 795̄	3.7321	.96 593	**75** 0	**20** 0	.34 202	.36 397	2.7475̄	.93 969	**70** 0
5	.26 022	.26 951	.7105̄	555̄	55	5	339	562	.7351	919	55
10	163	.27 107	.6891	517	50	10	475	727	.7228	869	50
15	303	263	.6680	479	45	15	612	.36 892	.7106	819	45
20	443	419	.6470	440	40	20	748	.37 057	.6985	769	40
25	584	576	.6264	402	35	25	.34 884	223	.6865	718	35
30	.26 724	.27 732	3 6059	.96 363	30	30	.35 021	.37 388	2.6746	.93 667	30
35	.26 864	.27 889	.5856	324	25	35	157	554	.6628	616	25
40	.27 004	.28 046	.5656	285̄	20	40	293	720	.6511	565̄	20
45	144	203	.5457	246	15	45	429	.37 887	.6395̄	514	15
50	284	360	.5261	206	10	50	565	.38 053	.6279	462	10
55	424	517	.5067	166	5	55	701	220	.6165̄	410	5
16 0	.27 564	.28 675̄	3.4874	.96 126	**74** 0	**21** 0	.35 837	.38 386	2.6051	.93 358	**69** 0
5	704	832	.4684	086	55	5	.35 973	553	.5938	306	55
10	843	28 990	.4495	046	50	10	.36 108	721	.5826	253	50
15	27 983	.29 147	.4308	.96 005̄	45	15	244	.38 888	.5715̄	201	45
20	.28 123	305	.4124	.95 964	40	20	379	.39 055	.5605̄	148	40
25	262	463	.3941	923	35	25	515̄	223	.5495	095̄	35
30	28 402	.29 621	3.3759	.95 882	30	30	.36 650	.39 391	2.5386	.93 042	30
35	541	780	.3580	841	25	35	785	559	.5279	.92 988	25
40	680	.29 938	.3402	799	20	40	.36 921	727	.5172	935̄	20
45	820	.30 097	.3226	757	15	45	.37 056	.39 896	.5065	881	15
50	.28 959	255	.3052	715	10	50	191	.40 065̄	.4960	827	10
55	.29 098	414	.2879	673	5	55	326	234	.4855̄	773	5
17 0	.29 237	.30 573	3.2709	.95 630	**73** 0	**22** 0	.37 461	.40 403	2.4751	.92 718	**68** 0
5	376	732	.2539	588	55	5	595	572	.4648	664	55
10	515	.30 891	.2371	545	50	10	730	741	.4545	609	50
15	654	.31 051	.2205	502	45	15	865̄	.40 911	.4443	554	45
20	793	210	.2041	459	40	20	.37 999	.41 081	.4342	499	40
25	.29 932	370	.1878	415	35	25	.38 134	251	.4242	444	35
30	.30 071	.31 530	3.1716	.95 372	30	30	.38 268	.41 421	2.4142	.92 388	30
35	209	690	.1556	328	25	35	403	592	.4043	332	25
40	348	.31 850	.1397	284	20	40	537	763	.3945̄	276	20
45	486	.32 010	.1240	240	15	45	671	.41 933	.3847	220	15
50	625̄	171	.1084	195	10	50	805	.42 105̄	.3750	164	10
55	763	331	.0930	150	5	55	.38 939	276	.3654	107	5
18 0	.30 902	.32 492	3.0777	95 106	**72** 0	**23** 0	.39 073	.42 447	2.3559	.92 050	**67** 0
5	.31 040	653	.0625	061	55	5	207	619	.3464	.91 994	55
10	178	814	.0475̄	95 015	50	10	341	791	.3369	936	50
15	316	.32 975	.0326	94 970	45	15	474	.42 963	.3276	879	45
20	454	.33 136	.0178	924	40	20	608	.43 136	.3183	822	40
25	593	298	3.0032	878	35	25	741	308	.3090	764	35
30	.31 730	33 460	2 9887	.94 832	30	30	.39 875̄	.43 481	2.2998	.91 706	30
35	.31 868	621	.9743	786	25	35	.40 008	654	.2907	648	25
40	.32 006	783	.9600	740	20	40	141	.43 828	.2817	590	20
45	144	.33 945	.9459	693	15	45	275̄	.44 001	.2727	531	15
50	282	.34 108	.9319	646	10	50	408	175̄	.2637	472	10
55	419	270	.9180	599	5	55	541	349	.2549	414	5
19 0	.32 557	.34 433	2.9042	.94 552	**71** 0	**24** 0	.40 674	.44 523	2.2460	.91 355̄	**66** 0
5	694	596	.8905	504	55	5	806	697	.2373	295	55
10	832	758	.8770	457	50	10	.40 939	.44 872	.2286	236	50
15	32 969	.34 922	.8636	409	45	15	.41 072	.45 047	.2199	176	45
20	.33 106	.35 085̄	.8502	361	40	20	204	222	.2113	116	40
25	244	248	.8370	313	35	25	337	397	.2028	91 056	35
30	.33 381	.35 412	2.8239	.94 264	30	30	.41 469	.45 573	2.1943	.90 996	30
35	518	576	.8109	215	25	35	602	748	.1859	936	25
40	655̄	740	.7980	167	20	40	734	.45 924	.1775̄	875	20
45	792	.35 904	.7852	118	15	45	866	.46 101	.1692	814	15
50	.33 929	.36 068	.7725	068	10	50	.41 998	277	.1609	753	10
55	.34 065	232	.7600	.94 019	5	55	.42 130	454	.1527	692	5
20 0	.34 202	.36 397	2.7475̄	.93 969	**70** 0	**25** 0	.42 262	.46 631	2.1445	.90 631	**65** 0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	° ′		N. Cos.	N. Cot.	N. Tan.	N. Sin.	° ′

° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.		° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
25 0	.42 262	.46 631	2.1445	.90 631	**65** 0	**30** 0	50 000	.57 735	1.7321	.86 603	**60** 0
5	394	808	.1364	569	55	5	126	.57 929	.7262	530	55
10	525	.46 985	.1283	507	50	10	252	.58 124	.7205	457	50
15	657	.47 163	.1203	446	45	15	377	318	.7147	384	45
20	788	341	.1123	383	40	20	503	513	.7090	310	40
25	.42 920	519	.1044	321	35	25	628	709	.7033	237	35
30	.43 051	.47 698	2.0965	.90 259	30	30	754	.58 905	1.6977	.86 163	30
35	182	.47 876	.0887	196	25	35	.50 879	.59 101	.6920	089	25
40	313	.48 055	.0809	133	20	40	.51 004	297	.6864	.86 015	20
45	445	234	.0732	070	15	45	129	494	.6808	.85 941	15
50	575	414	.0655	.90 007	10	50	254	691	.6753	866	10
55	706	593	.0579	.89 943	5	55	379	.59 888	.6698	792	5
26 0	.43 837	.48 773	2.0503	.89 879	**64** 0	**31** 0	.51 504	.60 086	1.6643	.85 717	**59** 0
5	.43 968	.48 953	.0428	816	55	5	628	284	.6588	642	55
10	.44 098	.49 134	.0353	752	50	10	753	483	.6534	567	50
15	229	315	.0278	687	45	15	.51 877	681	.6479	491	45
20	359	495	.0204	623	40	20	.52 002	.60 881	.6426	416	40
25	490	677	.0130	558	35	25	126	.61 080	.6372	340	35
30	.44 620	49 858	2.0057	.89 493	30	30	.52 250	.61 280	1.6319	.85 264	30
35	750	.50 040	1.9984	428	25	35	374	480	.6265	188	25
40	.44 880	222	.9912	363	20	40	498	681	.6212	112	20
45	.45 010	404	.9840	298	15	45	621	.61 882	.6160	.85 035	15
50	140	587	.9768	232	10	50	745	.62 083	.6107	.84 959	10
55	269	769	.9697	167	5	55	869	285	.6055	882	5
27 0	45 399	.50 953	1.9626	.89 101	**63** 0	**32** 0	.52 992	.62 487	1.6003	.84 805	**58** 0
5	529	.51 136	.9556	.89 035	55	5	.53 115	689	.5952	728	55
10	658	319	.9486	.88 968	50	10	238	.62 892	.5900	650	50
15	787	503	.9416	902	45	15	361	.63 095	.5849	573	45
20	.45 917	688	.9347	835	40	20	484	299	.5798	495	40
25	46 046	.51 872	.9278	768	35	25	607	503	.5747	417	35
30	.46 175	.52 057	1 9210	.88 701	30	30	.53 730	.63 707	1.5697	.84 339	30
35	304	242	.9142	634	25	35	853	.63 912	.5647	261	25
40	433	427	.9074	566	20	40	.53 975	.64 117	.5597	182	20
45	561	613	.9007	499	15	45	.54 097	322	.5547	104	15
50	690	798	.8940	431	10	50	220	528	.5497	.84 025	10
55	819	.52 985	.8873	363	5	55	342	734	.5448	.83 946	5
28 0	.46 947	.53 171	1.8807	.88 295	**62** 0	**33** 0	.54 464	.64 941	1.5399	.83 867	**57** 0
5	.47 076	358	.8741	226	55	5	586	.65 148	.5350	788	55
10	204	545	.8676	158	50	10	708	355	.5301	708	50
15	332	732	.8611	089	45	15	829	563	.5253	629	45
20	460	.53 920	.8546	.88 020	40	20	.54 951	771	.5204	549	40
25	588	.54 107	.8482	.87 951	35	25	.55 072	.65 980	.5156	469	35
30	47 716	.54 296	1.8418	.87 882	30	30	.55 194	.66 189	1.5108	.83 389	30
35	844	484	.8354	812	25	35	315	398	.5061	308	25
40	.47 971	673	.8291	743	20	40	436	608	.5013	228	20
45	48 099	.54 862	.8228	673	15	45	557	.66 818	.4966	147	15
50	226	.55 051	.8165	603	10	50	678	.67 028	.4919	.83 066	10
55	354	241	.8103	532	5	55	799	239	.4872	.82 985	5
29 0	.48 481	.55 431	1.8040	.87 462	**61** 0	**34** 0	.55 919	.67 451	1.4826	.82 904	**56** 0
5	608	621	.7979	391	55	5	.56 040	663	.4779	822	55
10	735	.55 812	.7917	321	50	10	160	.67 875	.4733	741	50
15	862	.56 003	.7856	250	45	15	280	.68 088	.4687	659	45
20	.48 989	194	.7796	178	40	20	401	301	.4641	577	40
25	.49 116	385	.7735	107	35	25	521	514	.4596	495	35
30	.49 242	.56 577	1.7675	.87 036	30	30	641	.68 728	1.4550	.82 413	30
35	369	769	.7615	.86 964	25	35	760	.68 942	.4505	330	25
40	495	.56 962	.7556	892	20	40	.56 880	.69 157	.4460	248	20
45	622	.57 155	.7496	820	15	45	.57 000	372	.4415	165	15
50	748	348	.7437	748	10	50	119	588	.4370	.82 082	10
55	.49 874	541	.7379	675	5	55	238	.69 804	.4326	.81 999	5
30 0	.50 000	.57 735	1.7321	.86 603	**60** 0	**35** 0	.57 358	.70 021	1.4281	.81 915	**55** 0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	° ′		N. Cos.	N. Cot.	N. Tan.	N. Sin.	° ′

° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.		° ′	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
35 0	.57 358	.70 021	1.4281	.81 915	**55** 0	**40** 0	.64 279	.83 910	1.1918	.76 604	**50** 0
5	477	238	.4237	832	55	5	390	.84 158	.1882	511	55
10	596	455	.4193	748	50	10	501	407	.1847	417	50
15	715	673	.4150	664	45	15	612	656	.1812	323	45
20	833	.70 891	.4106	580	40	20	723	.84 906	.1778	229	40
25	.57 952	.71 110	.4063	496	35	25	834	.85 157	.1743	135	35
30	.58 070	.71 329	1.4019	.81 412	30	30	.64 945	.85 408	1.1708	.76 041	30
35	189	549	.3976	327	25	35	.65 055	660	.1674	.75 946	25
40	307	769	.3934	242	20	40	166	.85 912	.1640	851	20
45	425	71 990	.3891	157	15	45	276	.86 166	.1606	756	15
50	543	.72 211	.3848	.81 072	10	50	386	419	.1571	661	10
55	661	432	.3806	.80 987	5	55	496	674	.1538	566	5
36 0	.58 779	.72 654	1.3764	.80 902	**54** 0	**41** 0	.65 606	.86 929	1.1504	.75 471	**49** 0
5	.58 896	.72 877	.3722	816	55	5	716	.87 184	.1470	375	55
10	.59 014	.73 100	.3680	730	50	10	825	441	.1436	280	50
15	131	323	.3638	644	45	15	.65 935	698	.1403	184	45
20	248	547	.3597	558	40	20	.66 044	.87 955	.1369	.75 088	40
25	365	771	.3555	472	35	25	153	.88 214	.1336	.74 992	35
30	.59 482	.73 996	1 3514	.80 386	30	30	.66 262	.88 473	1.1303	.74 896	30
35	599	.74 221	.3473	299	25	35	371	732	.1270	799	25
40	716	447	.3432	212	20	40	480	.88 992	.1237	703	20
45	832	674	.3392	125	15	45	588	.89 253	.1204	606	15
50	.59 949	.74 900	.3351	.80 038	10	50	697	515	.1171	509	10
55	.60 065	.75 128	.3311	.79 951	5	55	805	.89 777	.1139	412	5
37 0	60 182	.75 355	1.3270	.79 864	**53** 0	**42** 0	66 913	.90 040	1.1106	.74 314	**48** 0
5	298	584	.3230	776	55	5	67 021	304	.1074	217	55
10	414	.75 812	.3190	688	50	10	129	569	.1041	120	50
15	529	76 042	.3151	600	45	15	237	.90 834	.1009	.74 022	45
20	645	272	.3111	512	40	20	344	.91 099	.0977	.73 924	40
25	761	502	.3072	424	35	25	452	366	.0945	826	35
30	876	.76 733	1 3032	.79 335	30	30	.67 559	.91 633	1.0913	.73 728	30
35	.60 991	.76 964	.2993	247	25	35	666	.91 901	.0881	629	25
40	.61 107	.77 196	.2954	158	20	40	773	.92 170	.0850	531	20
45	222	428	.2915	.79 069	15	45	880	439	.0818	432	15
50	337	661	.2876	.78 980	10	50	.67 987	709	.0786	333	10
55	451	.77 895	.2838	891	5	55	.68 093	.92 980	.0755	234	5
38 0	.61 566	.78 129	1.2799	.78 801	**52** 0	**43** 0	.68 200	.93 252	1.0724	.73 135	**47** 0
5	681	363	.2761	711	55	5	306	524	.0692	.73 036	55
10	795	598	.2723	622	50	10	412	.93 797	.0661	.72 937	50
15	.61 909	.78 834	.2685	532	45	15	518	.94 071	.0630	837	45
20	.62 024	.79 070	.2647	442	40	20	624	345	.0599	737	40
25	138	306	.2609	351	35	25	730	620	.0569	637	35
30	.62 251	.79 544	1.2572	.78 261	30	30	.68 835	.94 896	1.0538	.72 537	30
35	365	.79 781	.2534	170	25	35	.68 941	.95 173	.0507	437	25
40	479	.80 020	.2497	.78 079	20	40	.69 046	451	.0477	337	20
45	592	258	.2460	.77 988	15	45	151	.95 729	.0446	236	15
50	706	498	.2423	897	10	50	256	.96 008	.0416	136	10
55	819	738	.2386	806	5	55	361	288	.0385	.72 035	5
39 0	.62 932	.80 978	1.2349	.77 715	**51** 0	**44** 0	.69 466	.96 569	1.0355	.71 934	**46** 0
5	.63 045	.81 220	.2312	623	55	5	570	.96 850	.0325	833	55
10	158	461	.2276	531	50	10	675	.97 133	.0295	732	50
15	271	703	.2239	439	45	15	779	416	.0265	630	45
20	383	.81 946	.2203	347	40	20	883	700	.0235	529	40
25	496	.82 190	.2167	255	35	25	.69 987	.97 984	.0206	427	35
30	.63 608	.82 434	1.2131	.77 162	30	30	.70 091	.98 270	1.0176	.71 325	30
35	720	678	.2095	.77 070	25	35	195	556	.0147	223	25
40	832	.82 923	.2059	.76 977	20	40	298	.98 843	.0117	121	20
45	.63 944	.83 169	.2024	884	15	45	401	.99 131	.0088	.71 019	15
50	.64 056	415	.1988	791	10	50	505	420	.0058	.70 916	10
55	167	662	.1953	698	5	55	608	.99 710	.0029	813	5
40 0	.64 279	.83 910	1.1918	.76 604	**50** 0	**45** 0	.70 711	1.00 000	1.0000	.70 711	**45** 0
	N. Cos.	N. Cot	N. Tan.	N. Sin.	° ′		N. Cos.	N. Cot.	N. Tan.	N. Sin.	° ′

Age x	Number Living l_x	Number of Deaths d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x	Age x	Number Living l_x	Number of Deaths d_x	Yearly Probability of Dying q_x	Yearly Probability of Living p_x
10	100,000	749	0.007 490	0.992 510	53	66,797	1091	0.016 333	0.983 667
11	99,251	746	0.007 516	0.992 484	54	65,706	1143	0.017 396	0.982 604
12	98,505	743	0.007 543	0.992 457	55	64,563	1199	0.018 571	0.981 429
13	97,762	740	0.007 569	0.992 431	56	63,364	1260	0.019 885	0.980 115
14	97,022	737	0.007 596	0.992 404	57	62,104	1325	0.021 335	0.978 665
15	96,285	735	0.007 634	0.992 366	58	60,779	1394	0.022 936	0.977 064
16	95,550	732	0.007 661	0.992 339	59	59,385	1468	0.024 720	0.975 280
17	94,818	729	0.007 688	0.992 312	60	57,917	1546	0.026 693	0.973 307
18	94,089	727	0.007 727	0.992 273	61	56,371	1628	0.028 880	0.971 120
19	93,362	725	0.007 765	0.992 235	62	54,743	1713	0.031 292	0.968 708
20	92,637	723	0.007 805	0.992 195	63	53,030	1800	0.033 943	0.966 057
21	91,914	722	0.007 855	0.992 145	64	51,230	1889	0.036 873	0.963 127
22	91,192	721	0.007 906	0.992 094	65	49,341	1980	0.040 129	0.959 871
23	90,471	720	0.007 958	0.992 042	66	47,361	2070	0.043 707	0.956 293
24	89,751	719	0.008 011	0.991 989	67	45,291	2158	0.047 647	0.952 353
25	89,032	718	0.008 065	0.991 935	68	43,133	2243	0.052 002	0.947 998
26	88,314	718	0.008 130	0.991 870	69	40,890	2321	0.056 762	0.943 238
27	87,596	718	0.008 197	0.991 803	70	38,569	2391	0.061 993	0.938 007
28	86,878	718	0.008 264	0.991 736	71	36,178	2448	0.067 665	0.932 335
29	86,160	719	0.008 345	0.991 655	72	33,730	2487	0.073 733	0.926 267
30	85,441	720	0.008 427	0.991 573	73	31,243	2505	0.080 178	0.919 822
31	84,721	721	0.008 510	0.991 490	74	28,738	2501	0.087 028	0.912 972
32	84,000	723	0.008 607	0.991 393	75	26,237	2476	0.094 371	0.905 629
33	83,277	726	0.008 718	0.991 282	76	23,761	2431	0.102 311	0.897 689
34	82,551	729	0.008 831	0.991 169	77	21,330	2369	0.111 064	0.888 936
35	81,822	732	0.008 946	0.991 054	78	18,961	2291	0.120 827	0.879 173
36	81,090	737	0.009 089	0.990 911	79	16,670	2196	0.131 734	0.868 266
37	80,353	742	0.009 234	0.990 766	80	14,474	2091	0.144 466	0.855 534
38	79,611	749	0.009 408	0.990 592	81	12,383	1964	0.158 605	0.841 395
39	78,862	756	0.009 586	0.990 414	82	10,419	1816	0.174 297	0.825 703
40	78,106	765	0.009 794	0.990 206	83	8,603	1648	0.191 561	0.808 439
41	77,341	774	0.010 008	0.989 992	84	6,955	1470	0.211 359	0.788 641
42	76,567	785	0.010 252	0.989 748	85	5,485	1292	0.235 552	0.764 448
43	75,782	797	0.010 517	0.989 483	86	4,193	1114	0.265 681	0.734 319
44	74,985	812	0.010 829	0.989 171	87	3,079	933	0.303 020	0.696 980
45	74,173	828	0.011 163	0.988 837	88	2,146	744	0.346 692	0.653 308
46	73,345	848	0.011 562	0.988 438	89	1,402	555	0.395 863	0.604 137
47	72,497	870	0.012 000	0.988 000	90	847	385	0.454 545	0.545 455
48	71,627	896	0.012 509	0.987 491	91	462	246	0.532 466	0.467 534
49	70,731	927	0.013 106	0.986 894	92	216	137	0.634 259	0.365 741
50	69,804	962	0.013 781	0.986 219	93	79	58	0.734 177	0.265 823
51	68,842	1011	0.014 541	0.985 459	94	21	18	0.857 143	0.142 857
52	67,841	1044	0.015 389	0.984 611	95	3	3	1.000 000	0.000 000

Table V.—Commutation Columns, Single Premiums, and Annuities Due, American Experience Table, 3½ Per Cent

Age x	D_x	N_x	C_x	M_x	$1 + a_x$	A_x
10	70891.9	1575 535.	513.02	17612.9	22.2245	0.24845
11	67981.5	1504 643.	493.69	17099.9	22.1331	0.25154
12	65189.0	1436 662.	475.08	16606.2	22.0384	0.25474
13	62509.4	1371 473.	457.16	16131.1	21.9403	0.25806
14	59938.4	1308 963.	439.91	15674.0	21.8385	0.26151
15	57471.6	1249 025.	423.88	15234.1	21.7329	0.26508
16	55104.2	1191 553.	407.87	14810.2	21.6236	0.26877
17	52832.9	1136 449.	392.47	14402.3	21.5102	0.27261
18	50653.9	1083 616.	378.15	14009.8	21.3926	0.27659
19	48562.8	1032 962.	364.36	13631.7	21.2707	0.28071
20	46556.2	984 400.	351.07	13267.3	21.1443	0.28497
21	44630.8	937 843.	338.73	12916.3	21.0134	0.28940
22	42782.8	893 213.	326.82	12577.5	20.8779	0.29399
23	41009.2	850 430.	315.33	12250.7	20.7375	0.29873
24	39307.1	809 421.	304.24	11935.4	20.5922	0.30365
25	37673.6	770 113.	293.55	11631.1	20.4417	0.30873
26	36106.1	732 440.	283.62	11337.6	20.2858	0.31401
27	34601.5	696 334.	274.03	11054.0	20.1244	0.31947
28	33157.4	661 732.	264.76	10779.9	19.9573	0.32512
29	31771.3	628 575.	256.16	10515.2	19.7843	0.33097
30	30440.8	596 804.	247.85	10259.0	19.6054	0.33702
31	29163.5	566 363.	239.797	10011.2	19.4202	0.34328
32	27937.5	537 199.	232.331	9771.38	19.2286	0.34976
33	26760.5	509 262.	225.406	9539.04	19.0304	0.35646
34	25630.1	482 501.	218.683	9313.64	18.8256	0.36339
35	24544.7	456 871.	212.157	9094.96	18.6138	0.37055
36	23502.5	432 326.	206.383	8882.80	18.3949	0.37795
37	22501.4	408 824.	200.757	8676.42	18.1688	0.38560
38	21539.7	386 323.	195.798	8475.66	17.9354	0.39349
39	20615.5	364 783.	190.945	8279.86	17.6946	0.40163
40	19727.4	344 167.	186.684	8088.92	17.4461	0.41003
41	18873.6	324 440.	182.493	7902.23	17.1901	0.41869
42	18052.9	305 566.	178.828	7719.74	16.9262	0.42762
43	17263.6	287 513.	175.421	7540.91	16.6543	0.43681
44	16504.4	270 250.	172.680	7365.49	16.3744	0.44628
45	15773.6	253 745.	170.127	7192.81	16.0867	0.45600
46	15070.0	237 972.	168.345	7022.68	15.7911	0.46600
47	14392.1	222 902.	166.872	6854.34	15.4878	0.47626
48	13738.5	208 510.	166.047	6687.47	15.1770	0.48677
49	13107.9	194 771.	165.983	6521.42	14.8591	0.49752
50	12498.6	181 663.	166.424	6355.44	14.5346	0.50849
51	11909.6	169 165.	167.316	6189.01	14.2041	0.51967
52	11339.5	157 252.	168.601	6021.70	13.8679	0.53104

Table V.—Commutation Columns, Single Premiums, and Annuities Due, American Experience Table, 3½ Per Cent

Age x	D_x	N_x	C_x	M_x	$1 + a_x$	A_x
53	10787.4	145916.	170.234	5853.10	13.5264	0.54258
54	10252.4	135128.	172.317	5682.86	13.1801	0.55430
55	9733.40	124876.	174.646	5510.54	12.8296	0.56615
56	9229.60	115142.	177.325	5335.90	12.4753	0.57813
57	8740.17	105912.8	180.168	5158.57	12.1179	0.59022
58	8264.44	97172.6	183.139	4978.40	11.7579	0.60239
59	7801.82	88908.2	186.340	4795.27	11.3958	0.61463
60	7351.65	81106.4	189.604	4608.93	11.0324	0.62692
61	6913.44	73754.7	192.909	4419.32	10.6683	0.63924
62	6486.75	66841.3	196.117	4226.41	10.3043	0.65155
63	6071.27	60354.5	199.109	4030.30	9.9410	0.66383
64	5666.85	54283.3	201.887	3831.19	9.5791	0.67607
65	5273.33	48616.4	204.457	3629.30	9.2193	0.68824
66	4890.55	43343.1	206.522	3424.84	8.8626	0.70030
67	4518.65	38452.5	208.022	3218.32	8.5097	0.71223
68	4157.82	33933.9	208.903	3010.30	8.1615	0.72401
69	3808.32	29776.1	208.858	2801.40	7.8187	0.73560
70	3470.67	25967.7	207.881	2592.54	7.4820	0.74698
71	3145.43	22497.1	205.639	2384.66	7.1523	0.75813
72	2833.42	19351.6	201.851	2179.02	6.8298	0.76904
73	2535.75	16518.2	196.436	1977.17	6.5141	0.77972
74	2253.57	13982.5	189.491	1780.73	6.2046	0.79018
75	1987.87	11728.9	181.253	1591.24	5.9002	0.80048
76	1739.39	9741.02	171.940	1409.99	5.6002	0.81062
77	1508.63	8001.63	161.889	1238.05	5.3039	0.82064
78	1295.73	6493.00	151.2646	1076.158	5.0111	0.83054
79	1100.647	5197.27	140.0891	924.894	4.7220	0.84032
80	923.338	4096.62	128.8801	784.805	4.4368	0.84997
81	763.234	3173.29	116.9588	655.924	4.1577	0.85940
82	620.465	2410.05	104.4881	538.966	3.8843	0.86865
83	494.995	1789.59	91.6152	434.478	3.6154	0.87774
84	386.641	1294.59	78.9565	342.862	3.3483	0.88677
85	294.610	907.95	67.0490	263.906	3.0819	0.89578
86	217.598	613.34	55.8566	196.857	2.8187	0.90468
87	154.383	395.74	45.1992	141.000	2.5634	0.91332
88	103.963	241.36	34.82426	95.8011	2.3216	0.92149
89	65.6231	137.398	25.09929	60.9768	2.0937	0.92920
90	38.3047	71.775	16.82244	35.8775	1.8738	0.93664
91	20.18692	33.4700	10.385393	19.05509	1.6580	0.94393
92	9.11888	13.2831	5.588150	8.66970	1.4567	0.95074
93	3.22236	4.16420	2.285484	3.08155	1.2923	0.95630
94	0.827611	0.94184	0.685393	0.79576	1.1380	0.96152
95	0.114232	0.114232	0.110369	0.110369	1.0000	0.96618

TABLE VI.—AMOUNT OF 1

$s = (1 + i)^n$

n	1%	1½%	2%	3%	n
1	1.0100 0000	1.0150 0000	1.0200 0000	1.0300 0000	1
2	1.0201 0000	1.0302 2500	1.0404 0000	1.0609 0000	2
3	1.0303 0100	1.0456 7838	1.0612 0800	1.0927 2700	3
4	1.0406 0401	1.0613 6355	1.0824 3216	1.1255 0881	4
5	1.0510 1005	1.0772 8400	1.1040 8080	1.1592 7407	5
6	1.0615 2015	1.0934 4326	1.1261 6242	1.1940 5230	6
7	1.0721 3535	1.1098 4491	1.1486 8567	1.2298 7387	7
8	1.0828 5671	1.1264 9259	1.1716 5938	1.2667 7008	8
9	1.0936 8527	1.1433 8998	1.1950 9257	1.3047 7318	9
10	1.1046 2213	1.1605 4083	1.2189 9442	1.3439 1638	10
11	1.1156 6835	1.1779 4894	1.2433 7431	1.3842 3387	11
12	1.1268 2503	1.1956 1817	1.2682 4179	1.4257 6089	12
13	1.1380 9328	1.2135 5244	1.2936 0663	1.4685 3371	13
14	1.1494 7421	1.2317 5573	1.3194 7876	1.5125 8972	14
15	1.1609 6896	1.2502 3207	1.3458 6834	1.5579 6742	15
16	1.1725 7864	1.2689 8555	1.3727 8571	1.6047 0644	16
17	1.1843 0443	1.2880 2033	1.4002 4142	1.6528 4763	17
18	1.1961 4748	1.3073 4064	1.4282 4625	1.7024 3306	18
19	1.2081 0895	1.3269 5075	1.4568 1117	1.7535 0605	19
20	1.2201 9004	1.3468 5501	1.4859 4740	1.8061 1123	20
21	1.2323 9194	1.3670 5783	1.5156 6634	1.8602 9457	21
22	1.2447 1586	1.3875 6370	1.5459 7967	1.9161 0341	22
23	1.2571 6302	1.4083 7715	1.5768 9926	1.9735 8651	23
24	1.2697 3465	1.4295 0281	1.6084 3725	2.0327 9411	24
25	1.2824 3200	1.4509 4535	1.6406 0599	2.0937 7793	25
26	1.2952 5631	1.4727 0953	1.6734 1811	2.1565 9127	26
27	1.3082 0888	1.4948 0018	1.7068 8648	2.2212 8901	27
28	1.3212 9097	1.5172 2218	1.7410 2421	2.2879 2768	28
29	1.3345 0388	1.5399 8051	1.7758 4469	2.3565 6551	29
30	1.3478 4892	1.5630 8022	1.8113 6158	2.4272 6247	30
31	1.3613 2740	1.5865 2642	1.8475 8882	2.5000 8035	31
32	1.3749 4068	1.6103 2432	1.8845 4059	2.5750 8276	32
33	1.3886 9009	1.6344 7918	1.9222 3140	2.6523 3524	33
34	1.4025 7699	1.6589 9637	1.9606 7603	2.7319 0530	34
35	1.4166 0276	1.6838 8132	1.9998 8955	2.8138 6245	35
36	1.4307 6878	1.7091 3954	2.0398 8734	2.8982 7833	36
37	1.4450 7647	1.7347 7663	2.0806 8509	2.9852 2668	37
38	1.4595 2724	1.7607 9828	2.1222 9879	3.0747 8348	38
39	1.4741 2251	1.7872 1025	2.1647 4477	3.1670 2698	39
40	1.4888 6373	1.8140 1841	2.2080 3966	3.2620 3779	40
41	1.5037 5237	1.8412 2868	2.2522 0046	3.3598 9893	41
42	1.5187 8989	1.8688 4712	2.2972 4447	3.4606 9589	42
43	1.5339 7779	1.8968 7982	2.3431 8936	3.5645 1677	43
44	1.5493 1757	1.9253 3302	2.3900 5314	3.6714 5227	44
45	1.5648 1075	1.9542 1301	2.4378 5421	3.7815 9584	45
46	1.5804 5885	1.9835 2621	2.4866 1129	3.8950 4372	46
47	1.5962 6344	2.0132 7910	2.5363 4351	4.0118 9503	47
48	1.6122 2608	2.0434 7829	2.5870 7039	4.1322 5188	48
49	1.6283 4834	2.0741 3046	2.6388 1179	4.2562 1944	49
50	1.6446 3182	2.1052 4242	2.6915 8803	4.3839 0602	50

TABLE VI.—AMOUNT OF 1

$$s = (1 + i)^n$$

n	3½%	4%	5%	6%	*n*
1	1.0350 0000	1.0400 0000	1.0500 0000	1.0600 0000	1
2	1.0712 2500	1.0816 0000	1.1025 0000	1.1236 0000	2
3	1.1087 1788	1.1248 6400	1.1576 2500	1.1910 1600	3
4	1.1475 2300	1.1698 5856	1.2155 0625	1.2624 7696	4
5	1.1876 8631	1.2166 5290	1.2762 8156	1.3382 2558	5
6	1.2292 5533	1.2653 1902	1.3400 9564	1.4185 1911	6
7	1.2722 7926	1.3159 3178	1.4071 0042	1.5036 3026	7
8	1.3168 0904	1.3685 6905	1.4774 5544	1.5938 4807	8
9	1.3628 9735	1.4233 1181	1.5513 2822	1.6894 7896	9
10	1.4105 9876	1.4802 4428	1.6288 9463	1.7908 4770	10
11	1.4599 6972	1.5394 5406	1.7103 3936	1.8982 9856	11
12	1.5110 6866	1.6010 3222	1.7958 5633	2.0121 9647	12
13	1.5639 5606	1.6650 7351	1.8856 4914	2.1329 2826	13
14	1.6186 9452	1.7316 7645	1.9799 3160	2.2609 0396	14
15	1.6753 4883	1.8009 4351	2.0789 2818	2.3965 5819	15
16	1.7339 8604	1.8729 8125	2.1828 7459	2.5403 5168	16
17	1.7946 7555	1.9479 0050	2.2920 1832	2.6927 7279	17
18	1.8574 8920	2.0258 1652	2.4066 1923	2.8543 3915	18
19	1.9225 0132	2.1068 4918	2.5269 5020	3.0255 9950	19
20	1.9897 8886	2.1911 2314	2.6532 9771	3.2071 3547	20
21	2.0594 3147	2.2787 6807	2.7859 6259	3.3995 6360	21
22	2.1315 1158	2.3699 1879	2.9252 6072	3.6035 3742	22
23	2.2061 1448	2.4647 1554	3.0715 2376	3.8197 4966	23
24	2.2833 2849	2.5633 0416	3.2250 9994	4.0489 3464	24
25	2.3632 4498	2.6658 3633	3.3863 5494	4.2918 7072	25
26	2.4459 5856	2.7724 6978	3.5556 7269	4.5493 8296	26
27	2.5315 6711	2.8833 6858	3.7334 5632	4.8223 4594	27
28	2.6201 7196	2.9987 0332	3.9201 2914	5.1116 8670	28
29	2.7118 7798	3.1186 5145	4.1161 3560	5.4183 8790	29
30	2.8067 9370	3.2433 9751	4.3219 4238	5.7434 9117	30
31	2.9050 3148	3.3731 3341	4.5380 3949	6.0881 0064	31
32	3.0067 0759	3.5080 5875	4.7649 4147	6.4533 8668	32
33	3.1119 4235	3.6483 8110	5.0031 8854	6.8405 8988	33
34	3.2208 6033	3.7943 1634	5.2533 4797	7.2510 2528	34
35	3.3335 9045	3.9460 8899	5.5160 1537	7.6860 8679	35
36	3.4502 6611	4.1039 3255	5.7918 1614	8.1472 5200	36
37	3.5710 2543	4.2680 8986	6.0814 0694	8.6360 8712	37
38	3.6960 1132	4.4388 1345	6.3854 7729	9.1542 5235	38
39	3.8253 7171	4.6163 6599	6.7047 5115	9.7035 0749	39
40	3.9592 5972	4.8010 2063	7.0399 8871	10.2857 1794	40
41	4.0978 3381	4.9930 6145	7.3919 8815	10.9028 6101	41
42	4.2412 5799	5.1927 8391	7.7615 8756	11.5570 3267	42
43	4.3897 0202	5.4004 9527	8.1496 6693	12.2504 5463	43
44	4.5433 4160	5.6165 1508	8.5571 5028	12.9854 8191	44
45	4.7023 5855	5.8411 7568	8.9850 0779	13.7646 1083	45
46	4.8669 4110	6.0748 2271	9.4342 5818	14.5904 8748	46
47	5.0372 8404	6.3178 1562	9.9059 7109	15.4659 1673	47
48	5.2135 8898	6.5705 2824	10.4012 6965	16.3938 7173	48
49	5.3960 6459	6.8333 4937	10.9213 3313	17.3775 0403	49
50	5.5849 2686	7.1066 8335	11.4673 9979	18.4201 5427	50

Table VII.—Present Value of 1

$$v^n = (1 + i)^{-n}$$

n	1%	1½%	2%	3%	n
1	0.9900 9901	0.9852 2167	0.9803 9216	0.9708 7379	1
2	0.9802 9605	0.9706 6175	0.9611 6878	0.9425 9591	2
3	0.9705 9015	0.9563 1699	0.9423 2233	0.9151 4166	3
4	0.9609 8034	0.9421 8423	0.9238 4543	0.8884 8705	4
5	0.9514 6569	0.9282 6033	0.9057 3081	0.8626 0878	5
6	0.9420 4524	0.9145 4219	0.8879 7138	0.8374 8426	6
7	0.9327 1805	0.9010 2679	0.8705 6018	0.8130 9151	7
8	0.9234 8322	0.8877 1112	0.8534 9037	0.7894 0923	8
9	0.9143 3982	0.8745 9224	0.8367 5527	0.7664 1673	9
10	0.9052 8695	0.8616 6723	0.8203 4830	0.7440 9391	10
11	0.8963 2372	0.8489 3323	0.8042 6304	0.7224 2128	11
12	0.8874 4923	0.8363 8742	0.7884 9318	0.7013 7988	12
13	0.8786 6260	0.8240 2702	0.7730 3253	0.6809 5134	13
14	0.8699 6297	0.8118 4928	0.7578 7502	0.6611 1781	14
15	0.8613 4947	0.7998 5150	0.7430 1473	0.6418 6195	15
16	0.8528 2126	0.7880 3104	0.7284 4581	0.6231 6694	16
17	0.8443 7749	0.7763 8526	0.7141 6256	0.6050 1645	17
18	0.8360 1731	0.7649 1159	0.7001 5937	0.5873 9461	18
19	0.8277 3992	0.7536 0747	0.6864 3076	0.5702 8603	19
20	0.8195 4447	0.7424 7042	0.6729 7133	0.5536 7575	20
21	0.8114 3017	0.7314 9795	0.6597 7582	0.5375 4928	21
22	0.8033 9621	0.7206 8763	0.6468 3904	0.5218 9250	22
23	0.7954 4179	0.7100 3708	0.6341 5592	0.5066 9175	23
24	0.7875 6613	0.6995 4392	0.6217 2149	0.4919 3374	24
25	0.7797 6844	0.6892 0583	0.6095 3087	0.4776 0557	25
26	0.7720 4796	0.6790 2052	0.5975 7928	0.4636 9473	26
27	0.7644 0392	0.6689 8574	0.5858 6204	0.4501 8906	27
28	0.7568 3557	0.6590 9925	0.5743 7455	0.4370 7675	28
29	0.7493 4215	0.6493 5887	0.5631 1231	0.4243 4636	29
30	0.7419 2292	0.6397 6243	0.5520 7089	0.4119 8676	30
31	0.7345 7715	0.6303 0781	0.5412 4597	0.3999 8715	31
32	0.7273 0411	0.6209 9292	0.5306 3330	0.3883 3703	32
33	0.7201 0307	0.6118 1568	0.5202 2873	0.3770 2625	33
34	0.7129 7334	0.6027 7407	0.5100 2817	0.3660 4490	34
35	0.7059 1420	0.5938 6608	0.5000 2761	0.3553 8340	35
36	0.6989 2495	0.5850 8974	0.4902 2351	0.3450 3243	36
37	0.6920 0490	0.5764 4309	0.4806 1093	0.3349 8294	37
38	0.6851 5337	0.5679 2423	0.4711 8719	0.3252 2615	38
39	0.6783 6967	0.5595 3126	0.4619 4822	0.3157 5355	39
40	0.6716 5314	0.5512 6232	0.4528 9042	0.3065 5684	40
41	0.6650 0311	0.5431 1559	0.4440 1021	0.2976 2800	41
42	0.6584 1892	0.5350 8925	0.4353 0413	0.2889 5922	42
43	0.6518 9992	0.5271 8153	0.4267 6875	0.2805 4294	43
44	0.6454 4546	0.5193 9067	0.4184 0074	0.2723 7178	44
45	0.6390 5492	0.5117 1494	0.4101 9680	0.2644 3862	45
46	0.6327 2764	0.5041 5265	0.4021 5373	0.2567 3653	46
47	0.6264 6301	0.4967 0212	0.3942 6836	0.2492 5876	47
48	0.6202 6041	0.4893 6170	0.3865 3761	0.2419 9880	48
49	0.6141 1921	0.4821 2975	0.3789 5844	0.2349 5029	49
50	0.6080 3882	0.4750 0468	0.3715 2788	0.2281 0708	50

$v^n = (1 + i)^{-n}$

n	3½%	4%	5%	6%	n
1	0.9661 8357	0.9615 3846	0.9523 8095	0.9433 9623	1
2	0.9335 1070	0.9245 5621	0.9070 2948	0.8899 9644	2
3	0.9019 4271	0.8889 9636	0.8638 3760	0.8396 1928	3
4	0.8714 4223	0.8548 0419	0.8227 0247	0.7920 9366	4
5	0.8419 7317	0.8219 2711	0.7835 2617	0.7472 5817	5
6	0.8135 0064	0.7903 1453	0.7462 1540	0.7049 6054	6
7	0.7859 9096	0.7599 1781	0.7106 8133	0.6650 5711	7
8	0.7594 1156	0.7306 9021	0.6768 3936	0.6274 1237	8
9	0.7337 3097	0.7025 8674	0.6446 0892	0.5918 9846	9
10	0.7089 1881	0.6755 6417	0.6139 1325	0.5583 9478	10
11	0.6849 4571	0.6495 8093	0.5846 7929	0.5267 8753	11
12	0.6617 8330	0.6245 9705	0.5568 3742	0.4969 6936	12
13	0.6394 0415	0.6005 7409	0.5303 2135	0.4688 3902	13
14	0.6177 8179	0.5774 7508	0.5050 6795	0.4423 0096	14
15	0.5968 9062	0.5552 6450	0.4810 1710	0.4172 6506	15
16	0.5767 0591	0.5339 0818	0.4581 1152	0.3936 4628	16
17	0.5572 0378	0.5133 7325	0.4362 9669	0.3713 6442	17
18	0.5383 6114	0.4936 2812	0.4155 2065	0.3503 4379	18
19	0.5201 5569	0.4746 4242	0.3957 3396	0.3305 1301	19
20	0.5025 6588	0.4563 8695	0.3768 8948	0.3118 0473	20
21	0.4855 7090	0.4388 3360	0.3589 4236	0.2941 5540	21
22	0.4691 5063	0.4219 5539	0.3418 4987	0.2775 0510	22
23	0.4532 8563	0.4057 2633	0.3255 7131	0.2617 9726	23
24	0.4379 5713	0.3901 2147	0.3100 6791	0.2469 7855	24
25	0.4231 4699	0.3751 1680	0.2953 0277	0.2329 9863	25
26	0.4088 3767	0.3606 8923	0.2812 4073	0.2198 1003	26
27	0.3950 1224	0.3468 1657	0.2678 4832	0.2073 6795	27
28	0.3816 5434	0.3334 7747	0.2550 9364	0.1956 3014	28
29	0.3687 4815	0.3206 5141	0.2429 4632	0.1845 5674	29
30	0.3562 7841	0.3083 1867	0.2313 7745	0.1741 1013	30
31	0.3442 3035	0.2964 6026	0.2203 5947	0.1642 5484	31
32	0.3325 8971	0.2850 5794	0.2098 6617	0.1549 5740	32
33	0.3213 4271	0.2740 9417	0.1998 7254	0.1461 8622	33
34	0.3104 7605	0.2635 5209	0.1903 5480	0.1379 1153	34
35	0.2999 7686	0.2534 1547	0.1812 9029	0.1301 0522	35
36	0.2898 3272	0.2436 6872	0.1726 5741	0.1227 4077	36
37	0.2800 3161	0.2342 9685	0.1644 3563	0.1157 9318	37
38	0.2705 6194	0.2252 8543	0.1566 0536	0.1092 3885	38
39	0.2614 1250	0.2166 2061	0.1491 4797	0.1030 5552	39
40	0.2525 7247	0.2082 8904	0.1420 4568	0.0972 2219	40
41	0.2440 3137	0.2002 7793	0.1352 8160	0.0917 1905	41
42	0.2357 7910	0.1925 7493	0.1288 3962	0.0865 2740	42
43	0.2278 0590	0.1851 6820	0.1227 0440	0.0816 2962	43
44	0.2201 0231	0.1780 4635	0.1168 6133	0.0770 0908	44
45	0.2126 5924	0.1711 9841	0.1112 9651	0.0726 5007	45
46	0.2054 6787	0.1646 1386	0.1059 9668	0.0685 3781	46
47	0.1985 1968	0.1582 8256	0.1009 4921	0.0646 5831	47
48	0.1918 0645	0.1521 9476	0.0961 4211	0.0609 9840	48
49	0.1853 2024	0.1463 4112	0.0915 6391	0.0575 4566	49
50	0.1790 5337	0.1407 1262	0.0872 0373	0.0542 8836	50

$$s_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}$$

n	1%	1½%	2%	3%	n
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1
2	2.0100 0000	2.0150 0000	2.0200 0000	2.0300 0000	2
3	3.0301 0000	3.0452 2500	3.0604 0000	3.0909 0000	3
4	4.0604 0100	4.0909 0338	4.1216 0800	4.1836 2700	4
5	5.1010 0501	5.1522 6693	5.2040 4016	5.3091 3581	5
6	6.1520 1506	6.2295 5093	6.3081 2096	6.4684 0988	6
7	7.2135 3521	7.3229 9419	7.4342 8338	7.6624 6218	7
8	8.2856 7056	8.4328 3911	8.5829 6905	8.8923 3605	8
9	9.3685 2727	9.5593 3169	9.7546 2843	10.1591 0613	9
10	10.4622 1254	10.7027 2167	10.9497 2100	11.4638 7931	10
11	11.5668 3467	11.8632 6249	12.1687 1542	12.8077 9569	11
12	12.6825 0301	13.0412 1143	13.4120 8973	14.1920 2956	12
13	13.8093 2804	14.2368 2960	14.6803 3152	15.6177 9045	13
14	14.9474 2132	15.4503 8205	15.9739 3815	17.0863 2416	14
15	16.0968 9554	16.6821 3778	17.2934 1692	18.5989 1389	15
16	17.2578 6449	17.9323 6984	18.6392 8525	20.1568 8130	16
17	18.4304 4314	19.2013 5539	20.0120 7096	21.7615 8774	17
18	19.6147 4757	20.4893 7572	21.4123 1238	23.4144 3537	18
19	20.8108 9504	21.7967 1636	22.8405 5863	25.1168 6844	19
20	22.0190 0399	23.1236 6710	24.2973 6980	26.8703 7449	20
21	23.2391 9403	24.4705 2211	25.7833 1719	28.6764 8572	21
22	24.4715 8598	25.8375 7994	27.2989 8354	30.5367 8030	22
23	25.7163 0183	27.2251 4364	28.8449 6321	32.4528 8370	23
24	26.9734 6485	28.6335 2080	30.4218 6247	34.4264 7022	24
25	28.2431 9950	30.0630 2361	32.0302 9972	36.4592 6432	25
26	29.5256 3150	31.5139 6896	33.6709 0572	38.5530 4225	26
27	30.8208 8781	32.9866 7850	35.3443 2383	40.7096 3352	27
28	32.1290 9669	34.4814 7867	37.0512 1031	42.9309 2252	28
29	33.4503 8766	35.9987 0085	38.7922 3451	45.2188 5020	29
30	34.7848 9153	37.5386 8137	40.5680 7921	47.5754 1571	30
31	36.1327 4045	39.1017 6159	42.3794 4079	50.0026 7818	31
32	37.4940 6785	40.6882 8801	44.2270 2961	52.5027 5852	32
33	38.8690 0853	42.2986 1233	46.1115 7020	55.0778 4128	33
34	40.2576 9862	43.9330 9152	48.0338 0160	57.7301 7652	34
35	41.6602 7560	45.5920 8789	49.9944 7763	60.4620 8181	35
36	43.0768 7836	47.2759 6921	51.9943 6719	63.2759 4427	36
37	44.5076 4714	48.9851 0874	54.0342 5453	66.1742 2259	37
38	45.9527 2361	50.7198 8538	56.1149 3962	69.1594 4927	38
39	47.4122 5085	52.4806 8366	58.2372 3841	72.2342 3275	39
40	48.8863 7336	54.2678 9391	60.4019 8318	75.4012 5973	40
41	50.3752 3709	56.0819 1232	62.6100 2284	78.6632 9753	41
42	51.8789 8946	57.9231 4100	64.8622 2330	82.0231 9645	42
43	53.3977 7936	59.7919 8812	67.1594 6777	85.4838 9234	43
44	54.9317 5715	61.6888 6794	69.5026 5712	89.0484 0911	44
45	56.4810 7472	63.6142 0096	71.8927 1027	92.7198 6139	45
46	58.0458 8547	65.5684 1398	74.3305 6447	96.5014 5723	46
47	59.6263 4432	67.5519 4018	76.8171 7576	100.3965 0095	47
48	61.2226 0777	69.5652 1929	79.3535 1927	104.4083 9598	48
49	62.8348 3385	71.6086 9758	81.9405 8966	108.5406 4785	49
50	64.4631 8218	73.6828 2804	84.5794 0145	112.7968 6729	50

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

n	3½%	4%	5%	6%	n
1	1.0000 0000	1.0000 0000	1.0000 0000	1.0000 0000	1
2	2.0350 0000	2.0400 0000	2.0500 0000	2.0600 0000	2
3	3.1062 2500	3.1216 0000	3.1525 0000	3.1836 0000	3
4	4.2149 4288	4.2464 6400	4.3101 2500	4.3746 1600	4
5	5.3624 6588	5.4163 2256	5.5256 3125	5.6370 9296	5
6	6.5501 5218	6.6329 7546	6.8019 1281	6.9753 1854	6
7	7.7794 0751	7.8992 9448	8.1420 0845	8.3938 3765	7
8	9.0516 8677	9.2142 2626	9.5491 0888	9.8974 6791	8
9	10.3684 9581	10.5827 9531	11.0265 6432	11.4913 1598	9
10	11.7313 9316	12.0061 0712	12.5778 9254	13.1807 9494	10
11	13.1419 9192	13.4863 5141	14.2067 8716	14.9716 4264	11
12	14.6019 6164	15.0258 0546	15.9171 2652	16.8699 4120	12
13	16.1130 3030	16.6268 3768	17.7129 8285	18.8821 3767	13
14	17.6769 8636	18.2919 1119	19.5986 3199	21.0150 6593	14
15	19.2956 8088	20.0235 8764	21.5785 6359	23.2759 6988	15
16	20.9710 2971	21.8245 3114	23.6574 9177	25.6725 2808	16
17	22.7050 1575	23.6975 1239	25.8403 6636	28.2128 7976	17
18	24.4996 9130	25.6454 1288	28.1323 8467	30.9056 5255	18
19	26.3571 8050	27.6712 2940	30.5390 0391	33.7599 9170	19
20	28.2796 8181	29.7780 7858	33.0659 5410	36.7855 9120	20
21	30.2694 7068	31.9692 1072	35.7192 5181	39.9927 2668	21
22	32.3289 0215	34.2479 6979	38.5052 1440	43.3922 9028	22
23	34.4604 1373	36.6178 8858	41.4304 7512	46.9958 2769	23
24	36.6665 2821	39.0826 0412	44.5019 9887	50.8155 7735	24
25	38.9498 5669	41.6459 0829	47.7270 9882	54.8645 1200	25
26	41.3131 0168	44.3117 4462	51.1134 5376	59.1563 8272	26
27	43.7590 6024	47.0842 1440	54.6691 2645	63.7057 6568	27
28	46.2906 2734	49.9675 8298	58.4025 8277	68.5281 1162	28
29	48.9107 9930	52.9662 8630	62.3227 1191	73.6397 9832	29
30	51.6226 7728	56.0849 3775	66.4388 4750	79.0581 8622	30
31	54.4294 7098	59.3283 3526	70.7607 8988	84.8016 7739	31
32	57.3345 0247	62.7014 6867	75.2988 2937	90.8897 7803	32
33	60.3412 1005	66.2095 2742	80.0637 7084	97.3431 6471	33
34	63.4531 5240	69.8579 0851	85.0669 5938	104.1837 5460	34
35	66.6740 1274	73.6522 2486	90.3203 0735	111.4347 7987	35
36	70.0076 0318	77.5983 1385	95.8363 2272	119.1208 6666	36
37	73.4578 6930	81.7022 4640	101.6281 3886	127.2681 1866	37
38	77.0288 9472	85.9703 3626	107.7095 4580	135.9042 0578	38
39	80.7249 0604	90.4091 4971	114.0950 2309	145.0584 5813	39
40	84.5502 7775	95.0255 1570	120.7997 7424	154.7619 6562	40
41	88.5095 3747	99.8265 3633	127.8397 6295	165.0476 8356	41
42	92.6073 7128	104.8195 9778	135.2317 5110	175.9505 4457	42
43	96.8486 2928	110.0123 8169	142.9933 3866	187.5075 7724	43
44	101.2383 3130	115.4128 7696	151.1430 0559	199.7580 3188	44
45	105.7816 7290	121.0293 9204	159.7001 5587	212.7435 1379	45
46	110.4840 3145	126.8705 6772	168.6851 6366	226.5081 2462	46
47	115.3509 7255	132.9453 9043	178.1194 2185	241.0986 1210	47
48	120.3882 5659	139.2632 0604	188.0253 9294	256.5645 2882	48
49	125.6018 4557	145.8337 3429	198.4266 6259	272.9584 0055	49
50	130.9979 1016	152.6670 8366	209.3479 9572	290.3359 0458	50

$$a_{\overline{n}|} = \frac{(1 - v^n)}{i}$$

n	1%	1½%	2%	3%	n
1	0.9900 9901	0.9852 2167	0.9803 9216	0.9708 7379	1
2	1.9703 9506	1.9558 8342	1.9415 6094	1.9134 6970	2
3	2.9409 8521	2.9122 0042	2.8838 8327	2.8286 1135	3
4	3.9019 6555	3.8543 8465	3.8077 2870	3.7170 9840	4
5	4.8534 3124	4.7826 4497	4.7134 5951	4.5797 0719	5
6	5.7954 7647	5.6971 8717	5.6014 3089	5.4171 9144	6
7	6.7281 9453	6.5982 1396	6.4719 9107	6.2302 8296	7
8	7.6516 7775	7.4859 2508	7.3254 8144	7.0196 9219	8
9	8.5660 1758	8.3605 1732	8.1622 3671	7.7861 0892	9
10	9.4713 0453	9.2221 8455	8.9825 8501	8.5302 0284	10
11	10.3676 2825	10.0711 1779	9.7868 4805	9.2526 2411	11
12	11.2550 7747	10.9075 0521	10.5753 4122	9.9540 0399	12
13	12.1337 4007	11.7315 3222	11.3483 7375	10.6349 5533	13
14	13.0037 0304	12.5433 8150	12.1062 4877	11.2960 7314	14
15	13.8650 5252	13.3432 3301	12.8492 6350	11.9379 3509	15
16	14.7178 7378	14.1312 6405	13.5777 0931	12.5611 0203	16
17	15.5622 5127	14.9076 4931	14.2918 7188	13.1661 1847	17
18	16.3982 6858	15.6725 6089	14.9920 3125	13.7535 1308	18
19	17.2260 0850	16.4261 6837	15.6784 6201	14.3237 9911	19
20	18.0455 5297	17.1686 3879	16.3514 3334	14.8774 7486	20
21	18.8569 8313	17.9001 3673	17.0112 0916	15.4150 2414	21
22	19.6603 7934	18.6208 2437	17.6580 4820	15.9369 1664	22
23	20.4558 2113	19.3308 6145	18.2922 0412	16.4436 0839	23
24	21.2433 8726	20.0304 0537	18.9139 2560	16.9355 4212	24
25	22.0231 5570	20.7196 1120	19.5234 5647	17.4131 4769	25
26	22.7952 0366	21.3986 3172	20.1210 3576	17.8768 4242	26
27	23.5596 0759	22.0676 1746	20.7068 9780	18.3270 3147	27
28	24.3164 4316	22.7267 1671	21.2812 7236	18.7641 0823	28
29	25.0657 8530	23.3760 7558	21.8443 8466	19.1884 5459	29
30	25.8077 0822	24.0158 3801	22.3964 5555	19.6004 4135	30
31	26.5422 8537	24.6461 4582	22.9377 0152	20.0004 2849	31
32	27.2695 8947	25.2671 3874	23.4683 3482	20.3887 6553	32
33	27.9896 9255	25.8789 5442	23.9885 6355	20.7657 9178	33
34	28.7026 6589	26.4817 2849	24.4985 9172	21.1318 3668	34
35	29.4085 8009	27.0755 9458	24.9986 1933	21.4872 2007	35
36	30.1075 0504	27.6606 8431	25.4888 4248	21.8322 5250	36
37	30.7995 0994	28.2371 2740	25.9694 5341	22.1672 3544	37
38	31.4846 6330	28.8050 5163	26.4406 4060	22.4924 6159	38
39	32.1630 3298	29.3645 8288	26.9025 8883	22.8082 1513	39
40	32.8346 8611	29.9158 4520	27.3554 7924	23.1147 7197	40
41	33.4996 8922	30.4589 6079	27.7994 8945	23.4123 9997	41
42	34.1581 0814	30.9940 5004	28.2347 9358	23.7013 5920	42
43	34.8100 0806	31.5212 3157	28.6615 6233	23.9819 0213	43
44	35.4554 5352	32.0406 2223	29.0799 6307	24.2542 7392	44
45	36.0945 0844	32.5523 3718	29.4901 5987	24.5187 1254	45
46	36.7272 3608	33.0564 8983	29.8923 1360	24.7754 4907	46
47	37.3536 9909	33.5531 9195	30.2865 8196	25.0247 0783	47
48	37.9739 5949	34.0425 5365	30.6731 1957	25.2667 0664	48
49	38.5880 7871	34.5246 8339	31.0520 7801	25.5016 5693	49
50	39.1961 1753	34.9996 8807	31.4236 0589	25.7297 6401	50

$$a_{\overline{n}|} = \frac{(1 - v^n)}{i}$$

n	3½%	4%	5%	6%	n
1	0.9661 8357	0.9615 3846	0.9523 8095	0.9433 9623	1
2	1.8996 9428	1.8860 9467	1.8594 1043	1.8333 9267	2
3	2.8016 3698	2.7750 9103	2.7232 4803	2.6730 1195	3
4	3.6730 7921	3.6298 9522	3.5459 5050	3.4651 0561	4
5	4.5150 5238	4.4518 2233	4.3294 7667	4.2123 6379	5
6	5.3285 5302	5.2421 3686	5.0756 9206	4.9173 2433	6
7	6.1145 4398	6.0020 5467	5.7863 7340	5.5823 8144	7
8	6.8739 5554	6.7327 4487	6.4632 1276	6.2097 9381	8
9	7.6076 8651	7.4353 3161	7.1078 2168	6.8016 9227	9
10	8.3166 0532	8.1108 9578	7.7217 3493	7.3600 8705	10
11	9.0015 5104	8.7604 7671	8.3064 1422	7.8868 7458	11
12	9.6633 3433	9.3850 7376	8.8632 5164	8.3838 4394	12
13	10.3027 3849	9.9856 4785	9.3935 7299	8.8526 8296	13
14	10.9205 2028	10.5631 2293	9.8986 4094	9.2949 8393	14
15	11.5174 1090	11.1183 8743	10.3796 5804	9.7122 4899	15
16	12.0941 1681	11.6522 9561	10.8377 6956	10.1058 9527	16
17	12.6513 2059	12.1656 6885	11.2740 6625	10.4772 5969	17
18	13.1896 8173	12.6592 9697	11.6895 8690	10.8276 0348	18
19	13.7098 3742	13.1339 3940	12.0853 2086	11.1581 1649	19
20	14.2124 0330	13.5903 2634	12.4622 1034	11.4699 2122	20
21	14.6979 7420	14.0291 5995	12.8211 5271	11.7640 7662	21
22	15.1671 2484	14.4511 1533	13.1630 0258	12.0415 8172	22
23	15.6204 1047	14.8568 4167	13.4885 7388	12.3033 7898	23
24	16.0583 6760	15.2469 6314	13.7986 4179	12.5503 5753	24
25	16.4815 1459	15.6220 7994	14.0939 4457	12.7833 5616	25
26	16.8903 5226	15.9827 6918	14.3751 8530	13.0031 6619	26
27	17.2853 6451	16.3295 8575	14.6430 3362	13.2105 3414	27
28	17.6670 1885	16.6630 6322	14.8981 2726	13.4061 6428	28
29	18.0357 6700	16.9837 1463	15.1410 7358	13.5907 2102	29
30	18.3920 4541	17.2920 3330	15.3724 5103	13.7648 3115	30
31	18.7362 7576	17.5884 9356	15.5928 1050	13.9290 8599	31
32	19.0688 6547	17.8735 5150	15.8026 7667	14.0840 4339	32
33	19.3902 0818	18.1476 4567	16.0025 4921	14.2302 2961	33
34	19.7006 8423	18.4111 9776	16.1929 0401	14.3681 4[illegible]14	34
35	20.0006 6110	18.6646 1323	16.3741 9429	14.4982 4636	35
36	20.2904 9381	18.9082 8195	16.5468 5171	14.6209 8713	36
37	20.5705 2542	19.1425 7880	16.7112 8734	14.7367 8031	37
38	20.8410 8736	19.3678 6423	16.8678 9271	14.8460 1916	38
39	21.1024 9987	19.5844 8484	17.0170 4067	14.9490 7468	39
40	21.3550 7234	19.7927 7388	17.1590 8635	15.0462 9687	40
41	21.5991 0371	19.9930 5181	17.2943 6796	15.1380 1592	41
42	21.8348 8281	20.1856 2674	17.4232 0758	15.2245 4332	42
43	22.0626 8870	20.3707 9494	17.5459 1198	15.3061 7294	43
44	22.2827 9102	20.5488 4129	17.6627 7331	15.3831 8202	44
45	22.4954 5026	20.7200 3970	17.7740 6982	15.4558 3209	45
46	22.7009 1813	20.8846 5356	17.8800 6650	15.5243 6990	46
47	22.8994 3780	21.0429 3612	17.9810 1571	15.5890 2821	47
48	23.0912 4425	21.1951 3088	18.0771 5782	15.6500 2661	48
49	23.2765 6450	21.3414 7200	18.1687 2173	15.7075 7227	49
50	23.4556 1787	21.4821 8462	18.2559 2546	15.7618 6064	50

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