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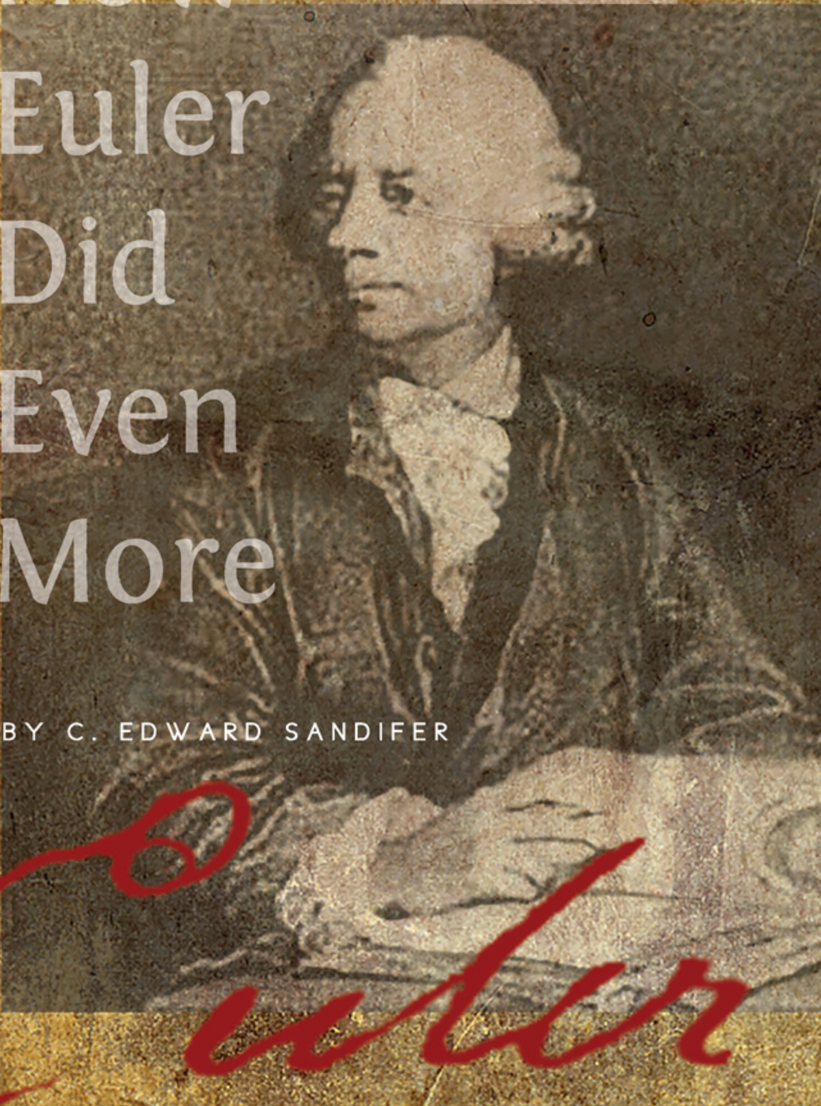
How Euler Did Even More

BY C. EDWARD SANDIFER



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How Euler Did Even More

C. Edward Sandifer
Western Connecticut State University

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Preface

In the first years of this century, the mathematics community was gearing up for Leonhard Euler's tercentenary. I see the official kick-off as being a special session sponsored by the Mathematical Association of America (MAA) at the 2001 Joint Mathematics Meeting called "Mathematics in the Age of Euler," organized by William Dunham and V. Frederick Rickey. At the same meeting, The Euler Society was first conceived of; it would come into existence the next year and begin holding annual meetings devoted to Euler studies. As 2007 approached, the MAA planned a special five-volume book series called *The MAA Tercentenary Euler Celebration*. The MAA invited the Euler Society to make the 2007 MathFest in San Jose, CA, a joint meeting of the two societies. MathFest was the climax of the tercentenary celebrations, featuring a plenary address from Bill Dunham, the official release of the last two volumes of the *Tercentenary* book series, and a presentation of a copy of the entire series to the Swiss consul, who traveled from the consulate in San Francisco to be a part of the celebration.

Ed Sandifer was the driving force behind much of this activity. I first met Ed in 1999, when he spoke at my university in the Pohle Colloquium on the History of Mathematics on—who else?—Leonhard Euler. Even though the big anniversary was still eight years off, Ed had already begun travelling the country, sharing the excitement of reading Euler in the original, and encouraging faculty and students alike to translate Euler's many papers and books into English.

Two years later, at the Joint Mathematics Meeting in New Orleans, Ed discussed the formation of an academic society devoted to Euler, his work, and his intellectual community, with Ronald Calinger and John Glaus. And so the Euler Society was born. Ed served as the secretary of the society from its inception until long after the tercentenary celebrations. As secretary, he tirelessly recruited members, promoted the society's annual meetings and made proceedings available in electronic form.

At about the same time that he helped found The Euler Society, Ed began work on a book that would eventually appear as the first volume in the MAA *Tercentenary* book series, titled *The Early Mathematics of Leonhard Euler*. It is the only comprehensive survey of Euler's groundbreaking mathematical papers from the early years of his career, before his move to Berlin in 1741.

In late 2003, Ed began writing a series of monthly columns that appeared on MAA Online called *How Euler Did It*. Each column was a self-contained exploration of one of Euler's theorems, or some particular facet of his work, illustrated with figures and excerpts from Euler's books and papers. The first 40 of these columns, through February 2007, were collected in the third volume of the MAA *Tercentenary* book series, appropriately titled *How Euler Did It*.

As the tercentenary drew nearer, Ed and I began editing two volumes of papers on Euler, both of which appeared in 2007. The first of these—*Leonard Euler: Life, Work and Legacy*—was published by Elsevier. The second collection, for which Lawrence D'Antonio also served as editor, became the fifth and final volume of the MAA *Tercentenary* book series, called *Euler at 300: An Appreciation*. Of the five volumes in the series, Ed was responsible for two and a third!

After the tercentenary celebrations, Ed devoted some of his seemingly boundless energy to other projects, such as our annotated translation of Cauchy's *Cours d'analyse*, published by Springer in 2009. He also continued his monthly online column for the MAA. By the summer of 2009, he had made plans to wind down *How Euler Did It*, and possibly start a new on-line column for the MAA. He determined that his swan song would be a two-part column in January and February of 2010, on Euler as a teacher. This would mean a full three years' worth of additional material, which he hoped would be collected in a second volume of *How Euler Did It*.

Sadly, Ed suffered a stroke in August of 2009. It happened just days after he attended MathFest in Portland, OR, where he gave the keynote address "Prove it Again, Sam" at the Opening Banquet and pitched a new book proposal to Birkhäuser. Those who know Ed, and especially those of us who ran with him, understand the irony that someone who was so ridiculously healthy and fit (he completed 37 consecutive Boston Marathons, beginning in 1973) should have suffered from blockage of the carotid artery. The stroke was severe. Ed was initially unable to read, write or talk and he suffered significant loss of mobility on his right side. He took to the various therapies that were prescribed for him with the same energy and dedication that he applies to everything he does. Within a year, he was able to read again. He can now type with his left hand and his speech has gradually improved over the years so that it is now, as Ed himself might say, "good, but not great." Ed's many friends and colleagues were happy to see him attend the 2012 Joint Mathematics Meeting in Boston, MA, and the 2013 MathFest in Hartford, CT.

Ed had already submitted his August 2009 *How Euler Did It* column at the time of the stroke. He also had his final two-part column in a reasonably good draft form, which I assisted him in editing in December of that year. But what to do about the autumn months of 2009? With Ed's consent, I submitted a couple of guest columns to the MAA that fall. Dominic Klyve, a member of the Euler Society's executive committee, also contributed a guest piece, so the flow of columns was almost without noticeable interruption. In 2002, when Dominic was in his first year of graduate studies, Ed visited Dartmouth (his alma mater) and gave a talk on Euler. Dominic and his fellow graduate student Lee Stemkoski were inspired by Ed's presentation and attended the first meeting of the Euler Society later that year. Not long thereafter, the two of them created the Euler Archive, a vital online resource for Euler studies. It's particularly fitting that someone whom Ed so deeply inspired was able to contribute a column in Ed's time of need.

The final 35 *How Euler Did It* columns are all collected in this book. 32 of them were written by Ed. They are lightly edited versions of the columns as they actually appeared on MAA Online between March 2007 and February 2010. The order of the articles as presented here is thematic, not chronological. As with the first volume of columns, we have assigned most of them into sections on Geometry, Number Theory, Combinatorics and Analysis. (Euler being Euler, the Analysis section is the largest.) There is also a section on Applied Mathematics, which opens with a column about Euler's study of saws, that also includes some of Ed's general observations about Euler's applied work. The final section, which we have dubbed "Euleriana," consists of papers devoted, at least in part, to Euler himself, including misattributions, mistakes, and a light-hearted column about "Euler and the Pirates."

I had the pleasure of working closely with Ed during the time that he wrote these columns. More than once at the end of a long day working on some other project, we would relax over a beer and he would tell me of some new Eulerian gem that he had uncovered in his reading, outlining his plans to feature it in an upcoming column. It's a great pleasure to know that all of Ed's columns will soon be in print, and a great privilege to have worked with him in making this book a reality.

Rob Bradley
Adelphi University
September 16, 2013

Part I

Geometry

1

The Euler Line

(January 2009)

A hundred years ago, if you'd asked people why Leonhard Euler was famous, those who had an answer would very likely have mentioned his discovery of the Euler line, the remarkable property that the orthocenter, the center of gravity and the circumcenter of a triangle are collinear. But times change, and so do fashions and the standards by which we interpret history.

At the end of the 19th century, triangle geometry was regarded as one of the crowning achievements of mathematics, and the Euler line was one of its finest jewels. Mathematicians who neglected triangle geometry to study exotic new fields like logic, abstract algebra or topology were taking brave risks to their professional careers. Now it would be the aspiring triangle geometer taking the risks.

Still, the late H. S. M. Coxeter made a long and distinguished career without straying far from the world of triangles, and he introduced hundreds of others to their delightful properties, especially the Euler line. This month, we look at how Euler discovered the Euler line and what he was trying to do when he discovered it. We will find that the discovery was rather incidental to the problem he was trying to solve, and that the problem itself was otherwise rather unimportant.

This brings us to the 325th paper in Gustav Eneström's index of Euler's published work, "Solutio facilis problematum quorumdam geometricorum difficillimorum" (An easy solution to a very difficult problem in geometry) [E325]. Euler wrote the paper in 1763 when he lived in Berlin and worked at the academy of Frederick the Great. The Seven Years War, which extended from 1756 to 1763, was just ending. Late in the war, Berlin had been occupied by foreign troops. Euler and the other academicians had lived the last years in fear for their own safety and that of their families, but a dramatic turn of events enabled Frederic to snatch victory from the jaws of defeat and win the war. When he returned to Berlin, he tried to manage his Academy of Sciences the same way he had managed his troops. In just three years, he had completely alienated Euler and Euler left for St. Petersburg, Russia to work at the academy of Catherine the Great. The paper was published in 1767 in the journal of the St. Petersburg Academy.

Euler begins his paper by reminding us that a triangle has four particularly interesting points:

1. the intersection of its three altitudes, which he denotes by E. Since about 1870, people have called this point the *orthocenter* and before that it was called the *Archimedian point*. Euler does not use either term;
2. the intersection of its median lines. Euler labels this point F and, as we do today, calls it the center of gravity;
3. the intersection of its angle bisectors. Euler labels this point G and calls it the center of the inscribed circle. Since about 1890, people have been calling this point the *incenter*;
4. the intersection of the perpendicular bisectors of the sides. Euler labels this point H and notes that it is the center of the circumscribed circle. Since about 1890, people have called it the *circumcenter*.

Modern texts usually use different letters to denote these same points, but as usual we will follow Euler's notation.

Then he announces what he regards as the main results of this paper: If these four points do not coincide, then the triangle is determined. If any two coincide, then all four coincide, and the triangle is equilateral, but it could be any size.

To prepare for his analysis, he defines some notation. He calls his triangle ABC, and lets its sides be of lengths a, b and c, where the side of length a is opposite vertex A, etc. Euler also denotes the area of the triangle by A, and trusts the reader to keep track of whether he is talking about the point A or the area A. He knows Heron's formula, though he doesn't know it by that name. Here, as in [E135], it is just a formula that he assumes we know about. He gives it in two forms:

$$AA = \frac{1}{16}(a+b+c)(a+b-c)(b+c-a)(c+a-b)$$
$$= \frac{1}{16}(2aabb + 2aacc + 2bbcc - a^4 - b^4 - c^4),$$

where, if you've been keeping track like we told you to, you know that AA denotes the square of the area of triangle ABC.

With the notation established, Euler sets out to give the locations of each of the centers, E, F, G and H, in terms of the lengths of the sides, a, b and c, and relative to the point A as an origin and the side AB as an axis. He begins with the orthocenter, E.

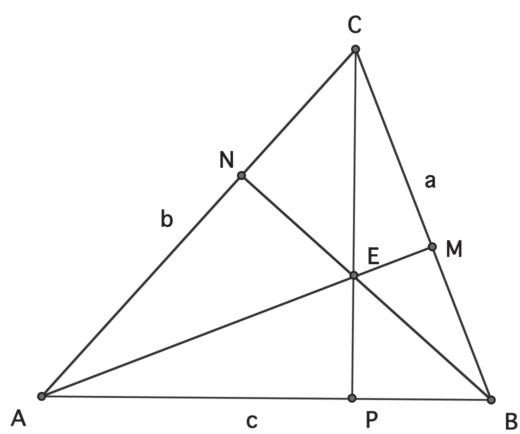

Fig. 1

Let P be the point where the line through C perpendicular to AB intersects AB (see Fig. 1). Then AP serves as a kind of x-coordinate of the point E, and EP acts as a y-coordinate. Likewise, he takes MA to be the perpendicular to BC and NB to be the perpendicular to AC, but these lines do not play a role as coordinates.

Euler tells us that

$$AP = \frac{cc + bb - aa}{2c}.$$

He gives us no reason, but it is easy algebra if you use the law of cosines to write $a^2 = b^2 + c^2 - 2bc \cos A$ and observe that $\cos A = \frac{AP}{b}$. That takes care of the x-coordinate of the point E.

Likewise, $BM = \frac{aa+cc-bb}{2a}$. Also, the triangle area formula gives us $A = \frac{1}{2} AM \cdot BC$, so that $AM = \frac{2A}{a}$. Triangles ABM and AEP are similar (because they are both right triangles and they share angle B), so $AM : BM = AP : EP$, and this easily leads to the second coordinate of the point E, namely

$$EP = \frac{(cc + bb - aa)(aa + cc - bb)}{8cA}.$$

Euler repeats similar analysis for each of the other centers. He introduces the points Q, R and S as the points on side AB corresponding to the x-coordinates of the centers F, G and H respectively. For the coordinates of the center of gravity, F, he finds

$$AQ = \frac{3cc + bb - aa}{6c} \quad \text{and} \quad QF = \frac{2A}{3c}.$$

For the coordinates of G, the center of the inscribed circle, he gets

$$AR = \frac{c + b - a}{2} \quad \text{and} \quad RG = \frac{2A}{a + b + c}.$$

Finally, for the H, the center of the circumscribed circle, he finds

$$AS = \frac{1}{2}c \quad \text{and} \quad SH = \frac{c(aa + bb - cc)}{8A}.$$

We leave to the reader the pleasant task of checking these calculations. Some are trickier than others.

This concludes the first part of Euler's paper. He has located his four centers in terms of the lengths of the three sides of the triangle. This has taken him about five pages of this 21-page paper.

There are six pairwise distances among these points:

$$EF^2 = (AP - AQ)^2 + (PE + QF)^2$$
$$EG^2 = (AP - AR)^2 + (PE - RG)^2$$
$$EH^2 = (AP - AS)^2 + (PE - SH)^2$$
$$FG^2 = (AQ - AR)^2 + (QF - RG)^2$$
$$FH^2 = (AQ - AS)^2 + (QF - SH)^2$$
$$GH^2 = (AR - AS)^2 + (RG - SH)^2,$$

where EF^2 denotes the square of the length of segment EF, etc.

To investigate these distances, it will be convenient to take

$$a + b + c = p,\ ab + ac + bc = q \text{ and } abc = r.$$

Later it will be important that this definition of p, q, r makes the lengths of the sides, a, b and c, equal to the roots of the cubic equation

$$z^3 - pzz + qz - r = 0.$$

Then it will also be useful to know that

$$aa + bb + cc = pp - 2q$$
$$aabb + aacc + bbcc = qq - 2pr$$
$$a^4 + b^4 + c^4 = p^4 - 4ppq + 2qq + 4pr$$

and that the area A can be expressed as

$$AA = \frac{1}{16}p(-p^3 + 4pq - 8r) = \frac{-p^4 + 4ppq - 8pr}{16}.$$

Six pages of rather tedious and straightforward calculations lead Euler to relations among the six distances between pairs of these four points. Referring to his Fig. 5, he eventually gets

I. $EF^2 = \dfrac{rr}{4AA} - \dfrac{4}{9}(pp - 2q)$

II. $EG^2 = \dfrac{rr}{4AA} - pp + 3q - \dfrac{4r}{p}$

III. $EH^2 = \dfrac{9rr}{16AA} - pp + 2q$

IV. $FG^2 = -\dfrac{1}{9}pp + \dfrac{5}{9}q - \dfrac{2r}{p}$

V. $FH^2 = \dfrac{rr}{16AA} - \dfrac{1}{9}(pp - 2q)$

VI. $GH^2 = \dfrac{rr}{16AA} - \dfrac{r}{p}$

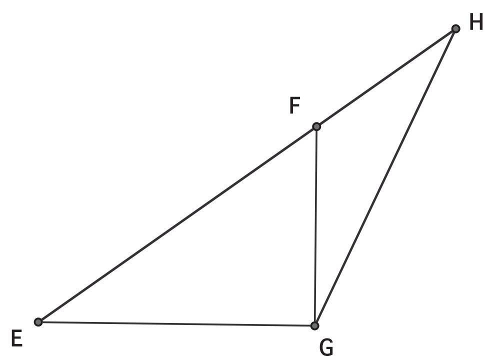

Fig. 5

This looks like just another list of formulas, but there is a gem hidden here. Euler sees that $EH = \frac{3}{2}EF$ and $FH = \frac{1}{2}EF$. He remarks that this implies that if the points E and F are known, then the point H can be found on the straight line through E and F. He does not specifically mention that because $EF + FH = EH$, the three points are collinear.

Nothing in Euler's presentation suggests that he thought this was very important or even very interesting. He only mentions that he can find H given E and F, and not that E or F could be found knowing the other two.

In more modern terms, and with modern emphasis, we give this result by saying that the orthocenter E, the center of gravity F and the circumcenter H are collinear, with $EH = \frac{3}{2}EF$, and we call the line through the three points the Euler line. Moreover, Euler seems almost equally interested in another harder-to-see and surely less important consequence of the same equations, that

$$4GH^2 + 2EG^2 = 3EF^2 + 6FG^2.$$

But Euler doesn't dwell on this. His problem is not to discover the properties of these various centers of the triangle, but to try to reconstruct the triangle given these centers. Towards this end, he introduces three new values, P, Q and R, defined in terms of p, q and r by

$$\frac{rr}{ps} = R, \quad \frac{r}{p} = Q \quad \text{and} \quad pp = P.$$

Then he rewrites the relations given in formulas I to VI in terms of P, Q and R. He only ends up using three of these formulas:

I. $GH^2 = \frac{1}{4}R - Q$

II. $FH^2 = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2QQ}{9R}$

III. $FG^2 = \frac{1}{36}P - \frac{8}{9}Q + \frac{5QQ}{9R}$

Finally, Euler is ready to state and solve the problem that is his reason for writing this paper:

Problem. Given these four points related to a triangle, to construct the triangle.

For reasons he doesn't make exactly clear, Euler divides the problem into two cases. The first case is where the point G does not lie on the Euler line, or as Euler says it, the case where the points F, G and H form a triangle, as shown in his Fig. 6. In the second case, all four lines are on the same line.

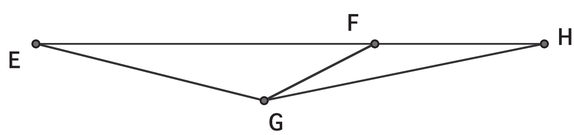

Fig. 6

Euler begins by taking

$$GH = f, \quad FH = g \quad \text{and} \quad FG = h.$$

Then, from the statement, such as it is, of the Euler line theorem, and from the remark that follows it, we get

$$EF = 2g, \quad EH = 3g \quad \text{and} \quad EG = \sqrt{6gg + 3hh - 2ff},$$

and formulas I, II and III can be rewritten as

I. $ff = \frac{1}{4}R - Q$

II. $gg = \frac{1}{4}R - \frac{1}{18}P + \frac{4}{9}Q + \frac{2QQ}{9R}$

III. $hh = \frac{1}{36}P - \frac{8}{9}Q + \frac{5QQ}{9R}$

Solving these for P, Q and R gives

$$R = \frac{4f^4}{3gg + 6hh - 2ff}, \quad Q = \frac{3ff(ff - gg - 2hh)}{3gg + 6hh - 2ff},$$
$$P = \frac{27f^4}{3gg + 6hh - 2ff} - 12ff - 15gg + 6hh,$$

and these make

$$\frac{QQ}{R} = \frac{9(ff - gg - 2hh)^2}{4(3gg + 6hh - 2ff)}.$$

Now he writes p, q and r in terms of P, Q and R (which means that they are also known in terms of f, g and h) and gets

$$p = \sqrt{P}, \quad q = \frac{1}{4}P + 2Q + \frac{QQ}{R} \quad \text{and} \quad r = Q\sqrt{P}.$$

Finally, he reminds us that the three sides of the triangle are the three roots of the cubic equation

$$z^3 - pzz + qz - r = 0.$$

In case we're not sure how to use Euler's solution to solve the problem, he does an example. First, like a good teacher, he designs the problem so it will have an easy answer. He considers a triangle with sides $a = 5$, $b = 6$ and $c = 7$. In a sense, this is the simplest acute scalene triangle. He uses the first version of his formulas I to VI to find that for this triangle,

$$ff = \frac{35}{32}, \quad gg = \frac{155}{288} \quad \text{and} \quad hh = \frac{1}{9}.$$

Now he pretends he doesn't know a, b and c and that he's only given ff, gg and hh. The formulas for P, Q and R in terms of f, g and h give

$$R = \frac{1225}{24}, \quad Q = \frac{35}{3}, \quad P = 324 \quad \text{and} \quad \frac{QQ}{R} = \frac{24}{9} = \frac{8}{3}.$$

These give p, q and r as

$$p = \sqrt{P} = 18, \quad q = 107 \quad \text{and} \quad r = \frac{35}{3} \cdot 18 = 5 \cdot 6 \cdot 7 = 210.$$

The cubic equation is then

$$z^3 - 18zz + 107z - 210 = 0.$$

As expected, the three roots of this equation are 5, 6 and 7.

Euler considers the case separately where all four centers lie on one line. He finds that the cubic has a double root and that this gives an isosceles triangle. Perhaps he thought that the slight differences between cubic equations with three distinct roots and those with a double root were enough to merit distinguishing between the cases. He doesn't give any details of why the triangle must be equilateral if all four centers coincide.

In some ways, Euler's discovery of the Euler line is analogous to Columbus's "discovery" of America. Both made their discoveries while looking for something else. Columbus was trying to find China. Euler was trying to find a way to reconstruct a triangle, given the locations of some of its various centers. Neither named his discovery. Columbus never called it "America" and Euler never called it "the Euler line."

Both misunderstood the importance of their discoveries. Columbus believed he had made a great and wonderful discovery, but he thought he'd discovered a better route from Europe to the Far East. Euler knew what he'd discovered, but didn't realize how important it would turn out to be.

Finally, Columbus made several more trips to the New World, but Euler, as with his polyhedral formula and the Königsberg bridge problem, made an important discovery but never went back to study it further.

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A Forgotten Fermat Problem

(December 2008)

The early French mathematician Pierre de Fermat (1601–1665) is well known for his misnamed “Last Theorem,” the conjecture that if $n > 2$, then the equation

$$x^n + y^n = z^n$$

has no nonzero integer solutions. Many of us also know “Fermat’s Little Theorem,” that if p is a prime number that does not divide a number a, then p divides $a^{p-1} - 1$.

It is less well known that Fermat left dozens, perhaps even hundreds of problems and conjectures for his successors. Typically, Fermat would state a problem in a letter to a friend. Sometimes he would claim that he had a solution or a partial solution. Other times he made no such claim. Many of Fermat’s problems fell to the new methods of calculus discovered by Newton and Leibniz in the decade after Fermat died. Most of the rest of them were solved in the 18th century, many by Euler. The so-called “Last Theorem” remained unsolved until the 1990s.

This month we look at one of Fermat’s other problems, a problem in geometry that Euler solved in 1747 or 1748. Euler had apparently been reading a collection of Fermat’s letters, [Fermat 1658, Fermat 1894, pp. 402–408] for he writes “A certain proposition is found in Fermat’s collected letters, which he proposed to be demonstrated by geometers,”[1] and that “up to now, no demonstration of it has ever been provided.” Euler claims that a proof “by analysis,” by which he means analytic geometry, is easy, but that he will give an old-fashioned geometric proof. Indeed, his proof, and even the style of his writing, is more like the 17th century than his own 18th century.

As stated in undated letter (probably June 1658) from Fermat to Kenelm Digby, the problem is this. Let ANB be a semicircle with diameter AB and where N bisects the

[1] Here and elsewhere we usually follow the fine 2005 translation by Adam Glover, then a student of Homer White at Georgetown College and available at EulerArchive.org.

semicircumference (See Fig. 1, taken from [Fermat 1896, p. 406]). Let AD and BC be perpendicular to the diameter, of length AN and BN respectively

This makes $AD = BC = \frac{AB}{\sqrt{2}}$. Let E be an arbitrary point on the circumference. Draw lines EC and ED and let V and O respectively be the points where these lines cut the diameter.[2] Then Fermat claims and Euler proves that $AV^2 + BO^2 = AB^2$. This sounds very Pythagorean.

Before Euler even states Fermat's conjecture, he proves a lemma.

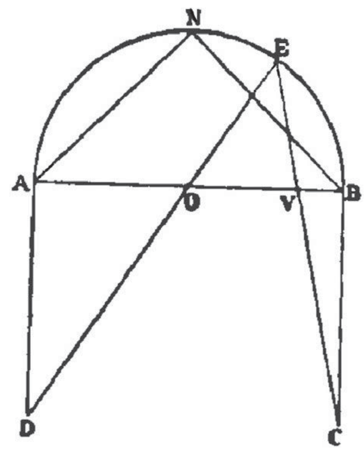

Fig. 1. Fermat's diagram

Lemma. *If a straight line AB (Fig. 2) is cut at points R and S, then the rectangle on the whole segment AB in the middle part RS along with the rectangle on the extreme parts AR and BS equals the rectangle from the parts AS and BR, that is*

$$AB \cdot RS + AR \cdot BS = AS \cdot BR.$$

Fig. 2

Note that Euler speaks of the "rectangles" and not of products. He uses the old vocabulary, describing the product of AB and RS as "AB in RS," in the style of the Renaissance geometers. He is determined to solve Fermat's geometry problem in the spirit of 17th century geometry. His proof is straightforward.

Proof.

$$AB = AS + BS$$

Multiply by RS (he writes "drawing in RS")

$$AB \cdot RS = AS \cdot RS + BS \cdot RS$$

Add

$$AR \cdot BS = AR \cdot BS$$

and get

$$AB \cdot RS + AR \cdot BS = AS \cdot RS + BS \cdot RS + AR \cdot BS.$$

The last two terms on the right can be rewritten as

$$BS \cdot RS + AR \cdot BS = BS(RS + AR) = BS \cdot AS.$$

[2] Breaking with tradition, here O is not the center of the circle.

Similarly,

$$AS \cdot RS + BS \cdot AS = AS(RS + BS) = AS \cdot BR.$$

Consequently we have

$$AB \cdot RS + AR \cdot BS = AS \cdot BR. \qquad \text{Q.E.D.}$$

The next to last step could have been clearer, but it does work out.

In case that proof was still too algebraic for his more traditionalist readers, Euler offers a second proof in what he calls a scholion. This word "scholion" has largely fallen from the modern mathematical vocabulary, so people are often perplexed when they see it. To Euler, a scholion is an application of an idea or a technique that isn't as specific as an "example" would be. In this case, he seems to be giving an example of more classical geometrical thinking, and gives us a less algebraic proof.

Second proof: Make a copy of the line $ARSB$ given above and call it *arsB*. Use it to form the square $ABab$ shown in Fig. 3. Name the points c, d, e, f, g and h as shown in the diagram, and denote by $\square Ae$ the area of the rectangle with diagonal Ae, etc. Euler also uses $\square Af$ to denote the area of the rectangle with diagonal Af. In the illustration, this looks suspiciously like a square, but it isn't, except in the unlikely event that $AS = BR$. In his translation Glover replaces all the rectangles with squares. Here we repeat Euler's use of squares and rectangles, even though it seems misleading.

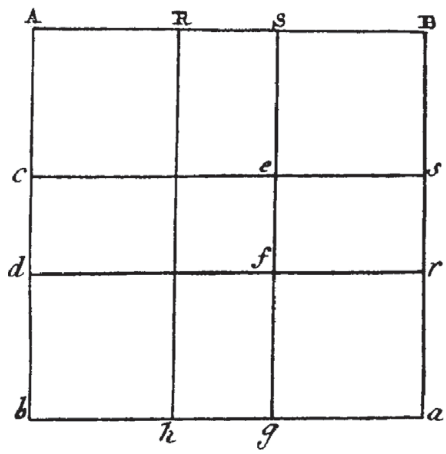

Fig. 3. Euler's second proof of his lemma

Euler calculates.
It is given that

$$Bs = BS, \quad sr = SR \quad \text{and} \quad ar = AR.$$

Then

$$\square AE = \square ae.$$

Adding $\square cf$ to both rectangles gives

$$\square Ae + \square cf = \square ae + \square cf,$$

$$\text{or} \quad \square Af = \square ae + \square cf,$$

$$\text{But} \quad \square ae = \square af + \square er$$

$$\text{and so} \quad \square Af = \square af + \square er + \square cf$$

$$= \square af + \square cr.$$

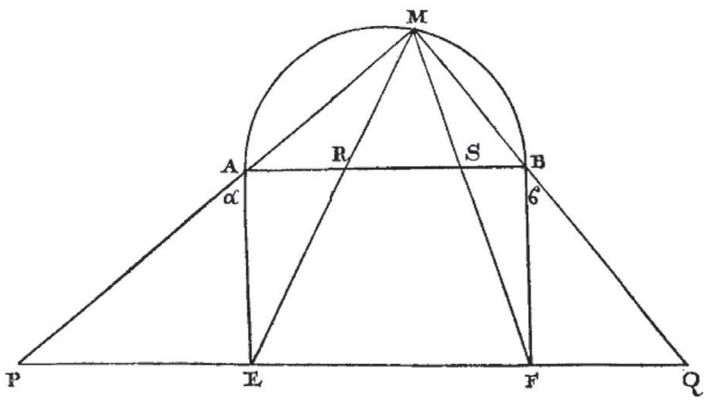

Fig. 4. Euler's diagram for Fermat's conjecture

Substituting segments for sides of rectangles and squares, we have

$$\square Af = AS \cdot Br = AS \cdot BR$$

$$\square af = ar \cdot BS = AR \cdot BS$$

$$\text{and} \quad \square cr = AB \cdot rs = AB \cdot RS.$$

Putting these together, we get

$$AS \cdot BR = AR \cdot BS + AB \cdot RS$$

which proves the lemma.

Now Euler is ready to prove Fermat's theorem. We restate it using Euler's notation, and give it with Euler's version of Fig. 1, including the auxiliary lines that Euler will add.

"Fermat's theorem: If upon the diameter AB of the semicircle AMB (Fig. 4) there is a rectangle $ABFE$, whose side AE or BF is equal to the chord of a quarter of that circle, that is to the side of its inscribed square, and from the points E and F there are drawn two lines EM and FM to any point on the circumference, then the diameter AB will be cut at points R and S such that $AS^2 + BR^2 = AB^2$."

Proof. From the point M, draw lines MA and MB, and extend them to where they meet the base EF at points P and Q respectively.

Easily, we have a number of similar triangles, PEA, BFQ (and AMB), so $PE{:}AE = BF{:}\ QF$ and so $PE \cdot QF = AE \cdot BF = EF^2$, and so $2PE \cdot QF = 2AE^2$. Because AE is the chord of the quadrant, $2AE^2 = AB^2 = EF^2$, and so it follows that $2PE \cdot QF = EF^2$.

The base PQ is cut by the points E and F (do you see the lemma coming) in such a way that the double of the rectangle on the extreme parts PE and QF is equal to the square of the middle part EF, that is to say $2PE \cdot QF = EF^2$. The segment AB is cut by points R and S in a similar way (i.e., preserving proportions), so the same is true there, that is to say that $2AR \cdot BS = RS^2$. Because $AS + BR = AB + RS$, we have

$$AS^2 + BR^2 + 2AS \cdot BR = AB^2 + RS^2 + 2AB \cdot RS.$$

Taking for RS^2 its value $2AR \cdot BS$ we get

$$AS^2 + BR^2 + 2AS \cdot BR = AB^2 + 2AB \cdot RS + 2AR \cdot BS.$$

Now, by the preceding lemma,

$$AB \cdot RS + AR \cdot BS = AS \cdot BR,$$

which makes

$$2AB \cdot RS + 2AR \cdot BS = 2AS \cdot BR.$$

Substituting this value into the previous equation gives rise to

$$AS^2 + BR^2 + 2AS \cdot BR = AB^2 + 2AS \cdot BR.$$

Eliminating common parts from both sides leaves

$$AS^2 + BR^2 = AB^2.$$ Q.E.D.

Euler wouldn't be Euler if he only solved the problem and didn't give us anything more than we asked for. He keeps going. But first he gives us three lemmas, this time disguised as theorems:

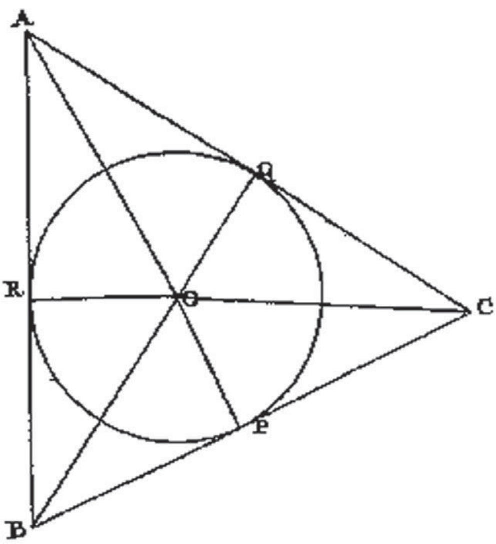

Fig. 5

Theorem. *The area of any triangle ABC (Fig. 5) is equal to the rectangle from the half-sum of the sides in the radius of the inscribed circle, that is*

$$\text{area } \Delta ABC = \frac{1}{2}(AB + AC + BC)OP.$$

Euler's proof is the usual one. He takes O to be the center of the inscribed circle. He dissects the triangle into three smaller triangles, ΔAOB, ΔAOC and ΔBOC, and he takes *OR*, *OQ* and *OP* to be the respective altitudes of these triangles. They are all radii of the inscribed circle. His illustration misleadingly suggests that *AOP*, *BOQ* and *COR* form straight lines. In general, they do not.

Now, these three triangles each have area half their base times their height, so the area of the whole triangle is half the sum of the three bases times their common height, that is the half-sum of their sides times the radius of the inscribed circle.

Theorem. *In Fig. 5, if we denote the half-sum of the sides by S, then* $AR + BP + CQ = S$.

This is a little surprising, because the points *P*, *Q* and *R* are not necessarily the midpoints of their respective sides. Recall, though, from the construction of the center *O*, that segments

AO, BO and CO are the bisectors of their respective angles. Because the angles at P, Q and R are right angles, it follows from the congruence theorems for right triangles that $AR = AQ$, $BR = BP$ and $CP = CQ$. Their sum is the perimeter of the triangle, and the half sum is $AR + BP + CQ$.

Euler's third lemma involves volumes instead of areas. Note that in Fig. 6, points A, B, C, P, Q, R and O are described as they were in Fig. 5, though the triangle is not so nearly equilateral and the inscribed circle is not shown. In this version, it is much more clear that AOP is not a straight line.

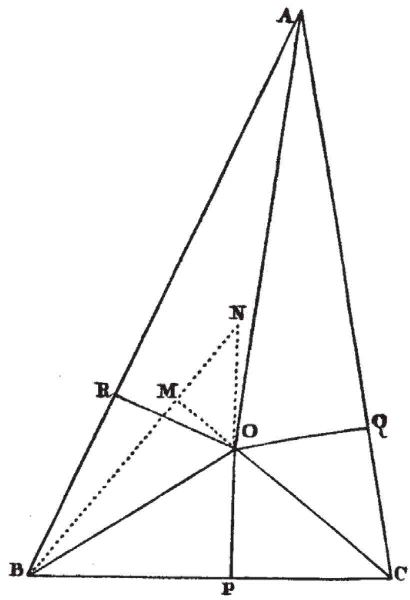

Fig. 6

Theorem. *"If, as before (Fig. 6), we let perpendicular lines OP, OQ and OR be drawn from the center O of the circle inscribed in triangle ABC to each side, we get the product contained under the parts $AR \cdot BP \cdot CQ$ is equal to the solid from the half-sum S of the sides and the square of the radius of the inscribed circle, OP, that is*

$$AR \cdot BP \cdot CQ = S \cdot OP^2.$$

Euler's proof is rather computational, and it uses the previous theorem as a lemma. We leave its proof as an exercise for the enthusiastic reader, with the hints that the angles at *M* are right angles and that *COM* and *PON* are, as they appear, straight lines.

With these three theorems added to his toolbox, Euler is ready to give a new proof to a famous result, what we know as Heron's formula. A beautiful exposition of Heron's own proof is in [Dunham 1990]. Euler does not mention Heron, and instead calls it a "common rule for finding the area of a triangle given its three sides."

Theorem. *The area of any triangle ABC is given if from the half-sum of its sides (which is S), each side is successively subtracted, and the product of these three remainders is multiplied by S itself and the square root of the product is taken. That is, the area of triangle ABC is*

$$\sqrt{S(S - AB)(S - AC)(S - BC)}.$$

Proof. From the first of his three lemma-theorems, we know that the area of triangle ABC is the half-perimeter S times the radius of the inscribed circle, OP, that is Area $=$ $S \cdot OP$. From the third of the lemma-theorems we know that $S \cdot OP^2 = AR \cdot BP \cdot CQ$. Multiplying both sides by S gives

$$S^2 \cdot OP^2 = S \cdot AR \cdot BP \cdot CQ.$$

This is the square of the area, so

$$\text{Area} = S \cdot OP = \sqrt{S \cdot AR \cdot BP \cdot CQ}.$$

Now, the second of Euler's lemma-theorems tells us that

$$AR = S - BC, \quad BP = S - AC \quad \text{and} \quad CQ = S - AB.$$

By simple substitution, we get the result immediately,

$$\text{Area } \Delta ABC = \sqrt{S(S - AB)(S - AC)(S - BC)}.$$

Euler continues in the same vein with a proof of what is often called Brahmagupta's formula, giving the area of a quadrilateral inscribed in a circle in terms of the lengths of its four sides. Euler describes a then-recent proof of the formula by Philip Naudé as "not only exceedingly intricate, with a multitude of lines covering the figure so that it cannot be understood without the greatest attentions, but also the clear vestiges of analytical calculus everywhere create too much of a problem." This is the same Philip Naudé who several years earlier had posed a problem in partitions that now bears his name. [Sandifer 2005b] Euler had praised Naudé back then, but clearly didn't think much of his skills at geometry. Indeed, Euler's diagram (Fig. 7) is very simple, but his proof, including all his new lemmas, is six pages long, and we omit it. Curious readers can refer to Adam Glover's fine translation on line at EulerArchive.org.

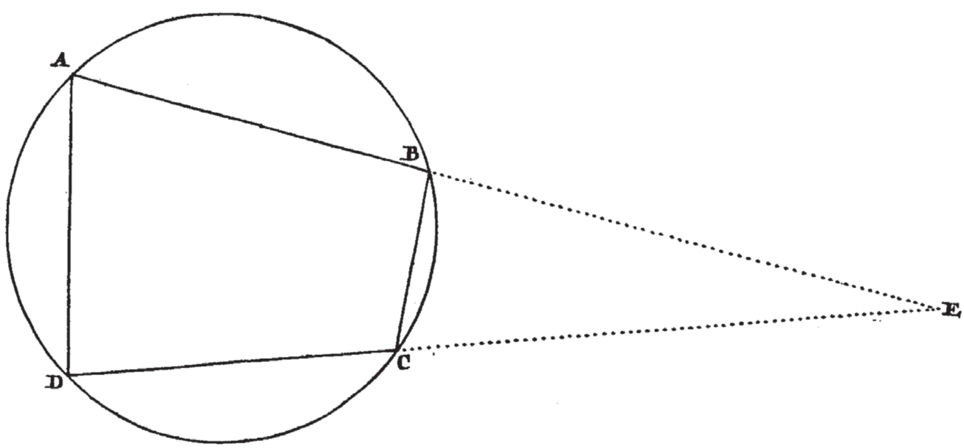

Fig. 7

Thus far, beyond his proof of Fermat's conjecture, Euler has given us only new proofs of old results. Euler closes his paper, though, with "another theorem, which up to now has neither been proposed or demonstrated. It involves a remarkable property of general quadrilaterals." Indeed, it was this theorem that first attracted me to this paper, and it was the subject of an earlier column titled "The Euler-Pythagoras Theorem." [Sandifer 2005a] It has been pleasant to revisit it.

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[Fermat 1658] Fermat, Pierre de, *Commercium epistolicum de quaestionibus quibusdam mathematicis etc.*, Oxford, 1658.

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[Sandifer 2005a] Sandifer, Ed, "The Euler-Pythagoras Theorem", *How Euler Did It*, January 2005. Reprinted in *How Euler Did It*, MAA, Washington, DC, 2007. 16

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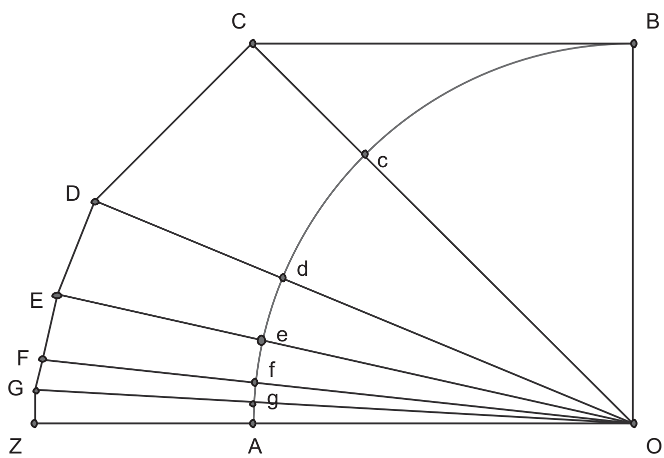
3

A Product of Secants

(May 2008)

When an interesting illustration catches our eye, we sometimes stop to figure out what it is. But when I first saw this illustration I was in a hurry. I resolved to come back to it "later." Now that later has finally arrived, I'm glad I remembered to go back.

The picture that caught my eye was the squarish-looking spiral below. It was part of the *Summarium* of [E275], "Notes on a certain passage of Descartes for looking at the quadrature of the circle." The *Summarium* is a summary of an article, usually written by the editor of the journal, that is printed at the beginning of the volume. This time, the *Summarium* was four pages long, and the article itself was twelve.



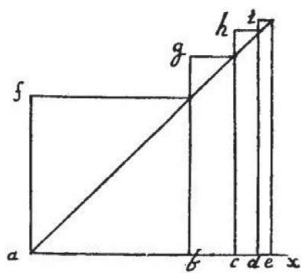
The *Summarium* gives us a bit of history that is not included in the article itself. The Editor tells us that "the circumference of a circle is incommensurable with its diameter," or, as we would say it now, p is an irrational number. He goes on to tell us that Archimedes approximated the ratio as 7 to 22 and Metius gave us 113 to 355.

A bit later, and with only this discussion of Archimedes and Metius as motivation, the Editor asks us to let q be "the length of the quadrant of a circle whose radius is equal to 1," what we would denote $\frac{\pi}{4}$. Then

$$q = \sec \frac{1}{2}q \cdot \sec \frac{1}{4}q \cdot \sec \frac{1}{8}q \cdot \sec \frac{1}{16}q \cdot \sec \frac{1}{32}q \cdot \text{etc.}$$

A minute with Maple® confirms this, at least to ten decimal places, and the Editor leads us to believe that the illustration should help to convince us that it is true. There is no mention of Descartes in the *Summarium*.

Euler begins the article itself describing a very different construction and with a different illustration. He tells us that the method is due to Descartes and that it "indicates brilliantly the insightful character of its discoverer."[1] As we go through Descartes' construction, it is helpful to note that Descartes describes a rectangle or a square by telling us two diagonally opposite corners of the shape. So, in the figure below, he calls the large square bf, and the rectangle next to it is cg.



Using the figure above, Descartes gives a procedure that begins with the length ab and the square on that length, bf. Then he constructs a new length ax. He claims (and Euler agrees) that the length ax forms the diameter of a circle that has the same circumference as the square bf. Hence, if $ab = 1$, then $ax = \frac{4}{\pi}$, about 1.2732. Here's how the construction works.

Take ao to be the ray containing the diagonal of square bf. Beside this, construct rectangle cg so that its area is 1/4 the area of square bf, and so that its (unnamed) corner lies on the ray ao.

Beside this, construct another rectangle dh, with its area 1/4 the area of rectangle bf, and again with its corner on the ray ao.

Continue constructing rectangles, each with 1/4 the area of the previous one, and each with its corner on the ray. It is easy to see that the sum of the bases of these rectangles,

$$ab + bc + cd + de + \cdots$$

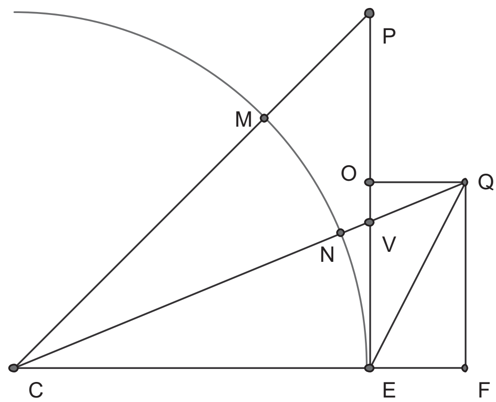
converges to some length, call it ax, but it is not so easy to see how ax is related to ab, or how this has anything to do with the circumference of a circle. Descartes, in the style of his times, doesn't tell us. Euler, though, sets out to prove it, and he shares the details of his proof with us.

[1] Here and elsewhere, when we quote from the text of E275, we usually follow the translation of Jordan Bell, available at EulerArchive.org and at the arXiv. Thank you, Jordan, for your many fine translations of Euler's work.

But first, Euler proposes to solve the following:

Problem. Given a circle around which a regular polygon is circumscribed, to find another circle, about which if a regular polygon with twice as many sides is circumscribed, the perimeter of the first polygon will be equal to the perimeter of the second one.

It is not yet clear what this has to do with Descartes' construction, or with the product of secants and the squarish spiral we saw in the *Summarium*, but those of us who remember how Archimedes approximated p as 22/7 will recognize how this problem is related to the value of p. Euler uses the figure below to solve his problem.



Here, MNE is an arc of the first circle. That circle has center C and radius CE. The segment PE is half of one side of the first polygon, circumscribed about MNE.

The new circle will be larger than the original one, so if we let CF be the radius of the new circle, we will have $CE < CF$.

Let FQ be a half-side of the new polygon. Since the new polygon has the same circumference, but twice as many sides as the original, we have EP = 2 FQ. Likewise, $\angle$ECP = 2$\angle$FCQ.

Let O be the midpoint of PE. Then QO || EF. Also, let V be the point where the radius CQ intersects PE. Note that V is between O and E, that is to say, EV < EO = ½PE.

Now Euler leads us through some triangle geometry. Some steps are easy, but one involves knowing something that is largely forgotten today.

First, EV : CE = FQ : CF because ?CEV ~ ?CFQ. That was easy.

Second, EV : CE = EP : CE + CP.This is not so easy. Here's how I figured it out. I compared ?PCE with ?VCE. Recall that $\angle$VCE = ½$\angle$PCE. I saw that the ratios CE : EV, CE : EP and CP : EP involved cotangents and cosecants of these angles, so I looked up the half-angle formula for cotangent and found it to be

$$\cot\frac{\theta}{2} = \cot\theta + \sec\theta.$$

I think it was the first time in my life that I'd ever *used* the half angle formula for cotangents, but I thank my high school geometry teacher, Ken Solem, for teaching me that there is one, so I'd know to look it up when I needed it. There may be a simpler way, but this was quick.

Third, by combining the first two steps, we get FQ : CF = EP : CE + CP.

Because FQ = ½ CF, this last proportion tells us that

$$CF = \frac{1}{2}(CE + CP).$$

Subtracting CE from both sides gives

$$EF = \frac{1}{2}(CP - CE)'.$$

Multiplying these last two together gives

$$CF \cdot EF = \frac{1}{4}(CP^2 - CE^2)$$

$$= \frac{1}{4}EP^2.$$

The first equality is just algebra and the second line is an application of the Pythagorean theorem to the right triangle CEP.

This solves Euler's problem because the point F is now defined so that the rectangle with sides CF and EF has as its area one-fourth the area of the rectangle with sides EP and FQ. That, in turn, equals the area of the square with sides FQ.

This result is a little awkward to use, so Euler "cleans it up" with four corollaries:

Corollary 1. *Because $CF \cdot EF = FQ^2$, we have $CF : FQ = FQ : EF$. This means we have similar triangles* CFQ ∼ FQE*, so that* ∠FCQ = ∠FQE.

Corollary 2. *From Corollary* 1*,* CE : EV = EO : EF*, so that point F can be defined by drawing from the point* O *a straight line perpendicular to the line* CV *extended, and finding where that new line intersects the base line* CE.

Corollary 3. *If the polygon circumscribing circle ENM has n sides, then $\angle ECP = \frac{\pi}{n}$, and $\angle FCQ = \frac{\pi}{2n}$. If we let the radius* CE = r*, then*

$$EP = r \tan\frac{\pi}{n} \quad \text{and} \quad FQ = \frac{1}{2}r\tan\frac{\pi}{n}.$$

Corollary 4. *Because $\angle FQE = \frac{\pi}{2n}$,*

$$EF = FQ\tan\frac{\pi}{2n} = \frac{1}{2}r\tan\frac{\pi}{n}\tan\frac{\pi}{2n}.$$

If we let CF = s, then we have

$$FQ = s\tan\frac{\pi}{2n},$$

and because

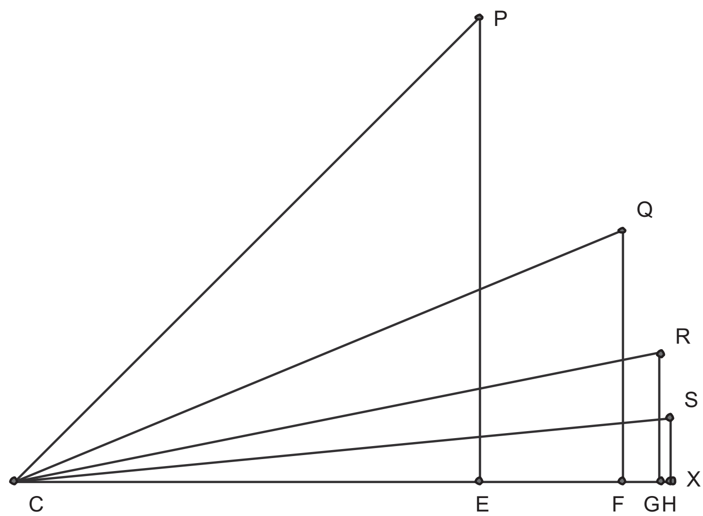
$$FQ = \frac{1}{2}r\tan\frac{\pi}{n},$$

we have

$$s = \frac{1}{2}r\tan\frac{\pi}{n}\cot\frac{\pi}{2n}.$$

Thus, we have a direct means of finding the length CF from the original length CE and the number of sides *n*.

Now Euler is ready to prove that Descartes' construction does what is claimed. This requires a new figure:



Here, we let CE be the radius of a circle inscribed in a square, CF that of an octagon, CG of a hexadecagon, CH, etc., and let EP, FQ, GR, HS be the corresponding half-sides. As we saw before,

$$FQ = \frac{1}{2}EP, \quad GR = \frac{1}{2}FQ = \frac{1}{4}EP, \quad HS = \frac{1}{2}GR = \frac{1}{4}FQ = \frac{1}{8}EP, \quad \text{etc.}$$

From the preceding problem,

$$CF \cdot EF = \frac{1}{4}EP^2 = FQ^2.$$

Similarly,

$$CG \cdot EF = \frac{1}{4}FQ^2 = \frac{1}{4}CF \cdot EF = GR^2,$$

$$CH \cdot GH = \frac{1}{4}GR^2 = \frac{1}{4}CG \cdot FG = HS^2, \quad \text{etc.}$$

With the points F, G, H, etc. determined in this way, we get Descartes' construction. Moreover, the points E, F, G, H, etc. "ultimately approach" the point *x*, the radius C*x* will be the radius of the circle the circumference of which is approached by the corresponding polygons.

Thus, the construction of Descartes is proved. The construction leads to a means of approximating p that Euler describes in another corollary:

Corollary 1. *If we take* CE $= a$, CF $= b$, CG $= c$, CH $= d$, etc., *we have* EP $= a$, *and we get the recursive sequence*

$$b(b-a) = \frac{1}{4}aa, \quad c(c-b) = \frac{1}{4}b(b-a), \quad d(d-c) = \frac{1}{4}c(c-b), \quad \text{etc.}$$

From this, quadratic formula gives us

$$b = \frac{a+\sqrt{2aa}}{2}, \quad c = \frac{b+\sqrt{2bb-ab}}{2}, \quad d = \frac{c+\sqrt{2cc-bc}}{2}, \quad \text{etc.}$$

and these quantities, taken to infinity, give the radius of the circle with perimeter equal to 8a.

Indeed, if we take $a = 1$, then the first several values of this sequence are

$$a = 1.00000$$
$$b = 1.20711$$
$$c = 1.25683$$
$$d = 1.26915$$
$$e = 1.27222$$
$$f = 1.27298$$
$$g = 1.27318$$

and these seem to be converging towards the required value of $4/p \sim 1.27323954$. Indeed, they agree to these eight decimal places on the 14th step (taking $a = 1$ as step 1).

Let us pause to take stock of what has happened so far in this article. Its title promised that we would learn about a method of Descartes for approximating p. We have done that. However, the *Summarium*, as well as the title I chose for the column, advertised an infinite product of secants. We haven't seen such an infinite product, nor have we seen anything of that spiral illustration that caught my eye in the first place. It's time to see what we can do about that. Euler begins a new problem.

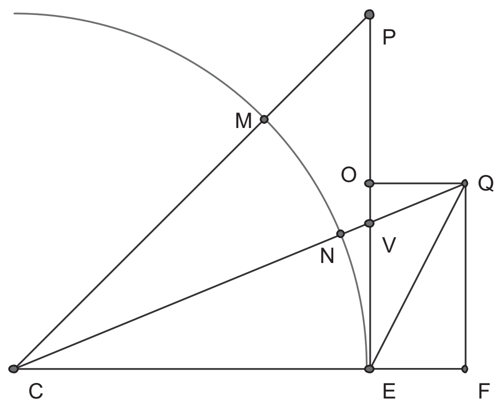
Problem. Taking φ to be any arc of a circle of radius 1, to find the sum of the infinite series

$$\tan\varphi + \frac{1}{2}\tan\frac{1}{2}\varphi + \frac{1}{4}\tan\frac{1}{4}\varphi + \frac{1}{8}\tan\frac{1}{8}\varphi + \frac{1}{16}\tan\frac{1}{16}\varphi + \text{etc.}$$

To solve this problem, Euler brings back the figure from his first problem. This time he lets $\angle ECP = \varphi$ be any angle, and $\angle FCQ = \frac{1}{2}\varphi$. He scales his drawing so that FQ = 1, which makes EP = 2. Then $CE = 2\cot\varphi$, $CF = \cot\frac{1}{2}\varphi$ and $EF = \tan\frac{1}{2}\varphi$. This last formula requires that we recall from Corollary 1 of the first problem that ? FQE $\sim$? FCO

Now CE = CF − EF, so

$$2\cot\varphi = \cot\tfrac{1}{2}\varphi - \tan\tfrac{1}{2}\varphi \quad\text{and}\quad \tan\tfrac{1}{2}\varphi = \cot\tfrac{1}{2}\varphi - 2\cot\varphi.$$



In the same way, $\tan\varphi = \cot\varphi - 2\cot 2\varphi$. We can apply these identities to get every term of the series in the problem, and we find that

$$\begin{aligned}
\tan\varphi &= \cot\varphi - 2\cot 2\varphi,\\
\tfrac{1}{2}\tan\tfrac{1}{2}\varphi &= \tfrac{1}{2}\cot\tfrac{1}{2}\varphi - \cot\varphi,\\
\tfrac{1}{4}\tan\tfrac{1}{4}\varphi &= \tfrac{1}{4}\cot\tfrac{1}{4}\varphi - \tfrac{1}{2}\cot\tfrac{1}{2}\varphi,\\
\tfrac{1}{8}\tan\tfrac{1}{8}\varphi &= \tfrac{1}{8}\cot\tfrac{1}{8}\varphi - \tfrac{1}{4}\cot\tfrac{1}{4}\varphi,\\
&\textit{etc.}
\end{aligned}$$

Bravely adding both sides of these together, and in characteristic Eulerian form, taking n to be an infinite number, we see that on the left we get exactly the series we are trying to sum, and on the right a riot of cancellation from which the only terms that survive are

$$-2\cot 2\varphi + \frac{1}{n}\cot\frac{1}{n}\varphi.$$

The second term is subject to l'Hôpital's rule, and becomes just $\frac{1}{\varphi}$, so the sum of Euler's series and the solution to the latest problem is

$$\frac{1}{\varphi} - 2\cot 2\varphi.$$

From here, Euler gives a few different paths to his product of secants. We'll describe my favorite. Start with

$$\tan\varphi + \frac{1}{2}\tan\frac{1}{2}\varphi + \frac{1}{4}\tan\frac{1}{4}\varphi + \frac{1}{8}\tan\frac{1}{8}\varphi + + \frac{1}{16}\tan\frac{1}{16}\varphi + \mathrm{L} = \frac{1}{\varphi} - 2\cot 2\varphi.$$

Integrate both sides to get

$$\begin{aligned}
&-\ln\cos\varphi - \ln\cos\frac{1}{2}\varphi - \ln\cos\frac{1}{4}\varphi - \ln\cos\frac{1}{8}\varphi - \ln\cos\frac{1}{16}\varphi - \mathrm{L}\\
&\quad = \ln\varphi - \ln\sin 2\varphi + \text{ Const.}
\end{aligned}$$

Taking $\varphi = 0$ leads to finding that the constant is $\ln 2$. Also, $-\ln\cos\theta = \ln\sec\theta$, so, by the laws of logarithms we get

$$\frac{1}{\cos\varphi.\cos\frac{1}{2}\varphi.\cos\frac{1}{4}\varphi.\cos\frac{1}{8}\varphi\cdot\cos\frac{1}{16}\varphi\cdots} = \frac{2\varphi}{\sin 2\varphi},$$

or, what amounts to the same thing,

$$\sec\varphi.\sec\frac{1}{2}\varphi.\sec\frac{1}{4}\varphi\cdot\sec\frac{1}{8}\varphi\cdot\sec\frac{1}{16}\varphi\cdots = \frac{2\varphi}{\sin 2\varphi}.$$

The product given in the *Summarium* is the special case of this formula where $\varphi = \frac{\pi}{4}$.

And what does Euler say of the pretty spiral that started it all? Nothing. He leaves that to us. Take AB = OB = 1. Then we might begin by noting that ?OBC is a right triangle and $\angle BOC = \frac{\pi}{4}$. So, $\frac{OC}{OB} = \sec\frac{\pi}{4}$. Since OB = 1, this makes $OC = \sec\frac{\pi}{4}$.

Then ?OCD is a right triangle and $\angle COD = \frac{\pi}{8}$. So $\frac{OD}{OC} = \sec\frac{\pi}{8}$.We know OC from the previous step. The result follows by repeating this process infinitely many times.

So, a pretty picture leads to a pleasing result.

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4

Curves and Paradox

(October 2008)

In the two centuries between Descartes (1596–1650) and Dirichlet (1805–1859), the mathematics of curves gradually shifted from the study of the means by which the curves were constructed to a study of the functions that define those curves. Indeed, Descartes' great insight, achieved around 1637, was that curves, at least the curves he knew about, had associated equations, and some properties of the curves could be revealed by studying those equations. Almost exactly 200 years later, in 1837, Dirichlet gave his famous example of a function defined on the closed interval [0, 1] that is discontinuous at every point, namely

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

The roles of the two objects had been reversed and mathematics had become far more interested in the study of functions than in the study of curves.

Euler came roughly half way between Descartes, both in years and in the evolution of mathematical ideas. He was instrumental in the early development of the modern idea of a *function* used the concept to lead mathematics away from its geometric foundations and replace them with analytic, i.e., symbolic manipulations, but he could not foresee how general and abstract the idea of a function could eventually become.

In 1756, Euler was devoting much of his intellectual powers to using differential equations to study the world. As we will see in Chapter 29, he used them to model fluid flow. A future column will be devoted to how he used differential equations to design more efficient saws. It should be no surprise that he also used differential equations to get new results about curves.

When Euler wrote *Exposition de quelques paradoxes dans le calcul integral*, (Explanation of certain paradoxes in integral calculus), [E236], he posed four problems with a distinctly 17th century flavor, as we shall see. Then he used his new tools of differential equations to solve the problems. He dwelt as much on his clever technique for solving the problems as he did on the geometry itself. Indeed, he was so intrigued by his technique that he dubbed it a "paradox" and used that word in the title of his article. In this column, we will look at the first of his four problems, see why I've described it as having a 17th century flavor, and then look at Euler's clever solution.

Euler describes the problem as follows:[1]

Problem 1. *Given point A, find the curve EM such that the perpendicular AV, derived from point A onto some tangent of the curve MV, is the same size everywhere.* (*Fig. 1*)

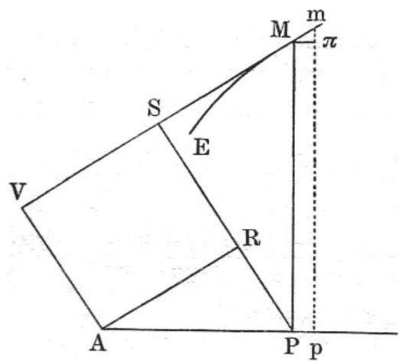

Fig. 1

This is a little confusing, and the problem is neither the clarity of Euler's French nor the quality of Andrew Fabian's translation. Both are excellent. Rather, it is that some of the vocabulary of geometry has changed in the last 250 years. Let us untangle it as we go along.

First, let A be the given point, and take the line AP to be an axis. The curve we want to find is EM, and a little later on, Euler will assume that m is another point on the same curve infinitely close to M. The special property that Euler wants his curve to have involves the line MV, which is tangent to the curve EM at the point M. He wants the distance between this line MV and the given point A to be the same, for every line tangent to the curve EM.

Before proceeding to Euler's answer to his own question, let's see why we described this problem as having the flavor of the century before Euler.

In ancient times, Euclid studied some of the properties of the lines tangent to a circle. His main result was that the tangent lines are perpendicular to their corresponding radii. The next century, Apollonius determined the lines tangent to a parabola by determining the point at which the tangent to a particular point intersects the axis of the parabola.

There was little new in the world of tangents until the early 1600s, when Descartes and Fermat (1601–1665) each found algebraic ways to find things we now recognize as being equivalent to tangents. In each case, the object they used was either a line segment or the length of a line segment. In Fig. 2, let TN be an axis and let EMm be a curve. If the line TM is tangent to the curve EMm at the point M and if T is the point where that tangent intersects the axis, then the segment MT, or its length, is what Descartes or Fermat would have called the *tangent*. The projection of the tangent onto the axis, that is the segment PT, was called the *subtangent*. The other two objects involved the line perpendicular to the curve EMm at the point M. Taking N to be the point where that perpendicular intersects the

[1] Here and elsewhere, when we quote Euler's article, we use the fine translation by Andrew Fabian.

axis, the segment MN was the *normal* and its projection onto the axis, that is the segment PN, was the *subnormal*.

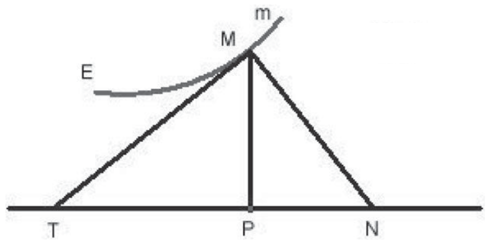

Fig. 2

Note that as modern readers, we easily recognize these line segments that are closely related to the derivative of a function at a point, but in the 1600s, hardly anyone had taken even Calculus I, so recognizing this relationship would have been much more difficult for them.

One of the founders of the Paris Academy, Claude Perrault, (1613–1688) recognized a physical problem related to the length of the tangent segment. Let us imagine a boy walking along the axis AP and pulling a wagon, assumed to be not on the axis, with a rope of fixed length a. People knew enough about resolution of forces to know that the direction the wagon moved would always be the same as the direction of the rope, and because the length of the rope was constant and the boy stays on the axis, the rope itself forms the segment we called the Tangent. Hence, the curve traced by the wagon is the curve for which the length of the tangent segment is always equal to the fixed length a. Perrault named this curve the *tractrix*, and to find an equation for the tractrix became one of the important unsolved problems of the late 1600s.

Another important problem was posed by Florimond de Beaune, (1601–1652) one of the people who helped Frans van Schooten (1615–1660) translate Descartes' *Geometrie* into Latin and then to write commentary that helped other scientists understand it better. De Beaune asked to find a curve for which the subtangent had a fixed length a. De Beaune did not give this curve a name, but we now know it to be an exponential curve.

Christian Huygens (1629–1695) solved the problem of the tractrix in 1693, and G. W. Leibniz (1646–1716) solved de Beaune's problem in 1684 his first paper on calculus, "Nova Methodus pro Maximis et Minimis."

There are two other obvious problems in this same vein, to find curves for which the normal segments have a fixed length, and for which the subnormal segments have a fixed length. Though these are both easily found using calculus, it turns out that their solutions do not *require* calculus, so whoever solved them first didn't become famous for their solutions.

Now we can re-phrase Euler's Problem 1 in the language of the 1600s: to find a curve EM for which every tangent segment MV passes a fixed distance a from the origin A.

As usual, Euler begins his solution of the problem by assigning variables. Again, see Fig. 1. He takes the line AP as his axis, and takes the length $AP = x$. The corresponding ordinate is $PM = y$. From the description of the problem, we know that $AV = a$, where AV is perpendicular to the tangent line MV. Euler adds calculus by introducing an infinitesimal arc element $Mm = ds$. Then the corresponding changes in x and y are $dx = Pp = Mp$

and $pm = dy$, where the segment Mp is taken to be parallel to the axis AP. Moreover, all this makes $ds = \sqrt{dx^2 + dy^2}$.

Euler plans to use similar triangles, so he needs to build some more triangles. He introduces a new segment, SP, perpendicular at S to the tangent line VM, and a second segment AR, perpendicular to SP at R. Note that this makes $a = AV = RS = PS - PR$.

We have three similar triangles, ΔPMS, ΔAPR and the differential triangle ΔMmp, and ratios of corresponding parts give us

$$PS = \frac{M\pi \cdot PM}{Mm} = \frac{ydx}{ds} \quad \text{and} \quad PR = \frac{m\pi \cdot AP}{Mm} = \frac{xdy}{ds}.$$

Substituting this into our observation that $a = PS - PR$, we get

$$ydx - xdy = ads = a\sqrt{dx^2 + dy^2}.$$

Euler tells us that this equation *exprimera la nature de la courbe cherchée*, "will express the nature of the curve being sought."

He sets out to solve this problem, squaring both sides and performing a well-choreographed sequence of substitutions. It is messy. Three pages and 17 display equations later, Euler leads us to see that the differential equation has an infinite family of solutions

$$y = \frac{n}{2}(a + x) + \frac{1}{2n}(a - x),$$

where n is an arbitrary constant, and the equation also has a singular solution, a family of one, namely

$$xx + yy = aa.$$

Readers familiar with differential equations will recognize that it is not a coincidence that the lines in the infinite family are the tangent lines to the circle $xx + yy = aa$. Note that Euler has overlooked the vertical line that corresponds to the case $n = 0$.

So, Euler has solved the problem, and it was a little bit interesting, but there seems to be nothing to suggest the "paradox" mentioned in the title of the article. We must read on to learn what Euler found paradoxical in this problem.

He continues by offering us an easier way to solve this same problem. He begins his second solution by introducing a new function, p, defined by the equation

$$dy = pdx.$$

A modern reader immediately recognizes p as the first derivative, but in Euler's day, the fundamental tools of differential calculus were differentials, and when they needed what we now call a derivative function, they had to define it as a quotient of differentials. The modern concept and notation was introduced by Lagrange in 1797. This definition of p makes

$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1 + p^2}$$

in general, and it lets us rewrite the equation that "expresses the nature of the curve" as

$$y - px = a\sqrt{1 + pp} \quad \text{or} \quad y = px + a\sqrt{1 + pp}. \tag{1}$$

Note that they differentials seem to have disappeared here, though they are actually hiding in the definition of the variable p.

Here comes the "paradox." "[I]nstead of integrating this differential equation, I differentiate it" and get

$$dy = pdx + xdp + \frac{apdp}{\sqrt{1+pp}}.$$

The "paradox" in this was more dramatic to Euler's readers because in his time, chapters on differential equations often bore titles like "On the *integration* of equations" rather than "On solving differential equations." Differential equations were ones that one resolved by integration, and the word "solving" was reserved for easier problems.

So, now we know what the paradox is. How does it work? We have $dy = pdx$, and subtracting this from the previous equation gives

$$0 = xdp + \frac{apdp}{\sqrt{1+pp}}. \tag{2}$$

First, divide this by dp (later, we'll need to remember that we divided by dp) and solve for x to get

$$x = -\frac{ap}{\sqrt{1+pp}}. \tag{3}$$

Then substitute this into equation (1) to get

$$y = -\frac{app}{\sqrt{1+pp}} + a\sqrt{1+pp} \quad \text{or} \quad y = \frac{a}{\sqrt{1+pp}}. \tag{4}$$

Viewed as a differential equation, equation (4) would take a good deal of work to solve, but viewed in conjunction with equation (3), we see a parameterization of the curve being sought in terms of the parameter p. We have

$$x = -\frac{ap}{\sqrt{1+pp}} \quad \text{and} \quad y = \frac{a}{\sqrt{1+pp}}. \tag{5}$$

Squaring these and adding them together gives

$$xx + yy = \frac{aapp + aa}{1+pp} = aa,$$

the equation of a circle, which is one of the solutions to the given problem.

But what about all those other solutions, the infinite family of straight lines? This method does not seem to provide us with them. But let's take a closer look at equation (2), and recall that we divided it by dp. In doing so, we may overlook solutions corresponding to the cases when $dp = 0$. In Euler's words, equation (2) also "contains" the solutions $dp = 0$. This would mean that p is a constant, and Euler chooses to call that constant n. Substituting $p = n$ into equation (1) gives us the infinite family of lines lines,

$$y = nx + a\sqrt{1+nn}.$$

With this, Euler has solved his differential equation by differentiating, rather than by integrating. Euler has a good deal more to say about other geometrically inspired problems that also lead to "paradoxical" differential equations, but for this column, this one will have to suffice. I hope that this account has whetted the readers' appetites for such problems, because Hieu Nguyen, professor and sometime chair of Mathematics at Rowan University in Glassboro, New Jersey and his student, Andrew Fabian have translated all of E236 from French into English and made it available through The Euler Archive. They are preparing an article for publication that describes E236 in considerably more depth than we have here, and have found a plethora of fascinating insights and new problems. Hieu Nguyen spoke on their results at the 2008 annual meeting of The Euler Society when they met in New York in July.

It is part of the magic of mathematics in general and of the works of great mathematicians like Euler in particular, that so often there are new things to be found in old mathematics. Watch for their article in the not-too-distant future.

Special thanks to Andrew Fabian for his English translation of E236 and to Hieu Nguyen for drawing this paper and Andrew Fabian's translation to my attention.

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5

Did Euler Prove Cramer's Rule?

(November 2009—A Guest Column by Rob Bradley)

The 200th anniversary of Euler's death took place in September 1983. The milestone was marked in a reasonable number of publications, although fewer than the number that celebrated the tercentenary of his birth two years ago. The MAA devoted an entire issue of the *Mathematics Magazine* to Euler's life and work [Vol. 56, no. 4, Nov. 1983] and there were at least two edited volumes of essays published to mark the event.

Among the many essays included in [Burckhardt 1983] is a piece by Pierre Speziali on Euler and Gabriel Cramer (1704–1752), the same Cramer whose name is attached to the famous rule for solving linear systems. Most of Speziali's article is a survey of the correspondence between Euler and Cramer. The Euler-Cramer correspondence will soon be readily available, because it will be included in the forthcoming Volume 7 of Series IVA of Euler's *Opera Omnia*, scheduled to be published in 2010. Siegfried Bodenmann, a co-editor of IVA.7, discusses the correspondence as an important example of 18th scientific correspondence in his chapter in [Henry 2007].

The correspondence consists of 19 letters in perfect alternation. The first one was a brief letter from Euler, written in 1743. Its contents and tone make it clear that there had previously been no direct contact between the two men. The final letter was written by Euler in late 1751, just a few weeks before Cramer's death. However, the 1975 catalog of Euler's correspondence lists only 17 of these letters. One of the two missing documents was Cramer's final letter to Euler. Although its whereabouts remain a mystery, its contents were known to Speziali and will be included in the *Opera Omnia*, because Cramer's draft survives in the archives of the public library in Geneva, where Cramer lived and taught. The other missing letter was Euler's third to Cramer. It was written at some point between Cramer's letters of September 30 and November 11, 1744, and was entirely unknown to Speziali in 1983.

Speziali pays particular attention to the letter of November 11, 1744, in which Cramer gave Euler a complete description of his rule for solving systems of linear equations. This is noteworthy, because Cramer's Rule would not appear in print until six years later, where it was an appendix in his very influential book *Introduction to the Analysis of Algebraic*

Curves [Cramer 1750]. What's even more interesting is that the passage in Cramer's letter is virtually identical, word for word, to a three-page passage in the *Introduction* [Cramer 1750, pp. 657–659].

Speziali further notes that immediately before describing his famous rule, Cramer says to Euler "your remark cannot but strike me as quite correct, because it agrees entirely with my own thoughts on the subject." Speziali goes on to say that it is "very regrettable that Euler's letter is lost because—who knows?—it might have revealed to us a rule similar to Cramer's, or an original idea that had inspired the latter." That is, he speculates that Euler might have discovered Cramer's Rule independently of Cramer, or that he might even deserve priority for it, by communicating a result in his lost letter that led Cramer to the discovery of his rule. If only that third letter to Cramer hadn't been lost . . .

The lost letter became known to Euler scholars at the meeting of the Euler Society in August 2003. At some point in the 20th century, it found its way into the private collection of Bern Dibner (1897–1988). Dibner was an engineer, entrepreneur and philanthropist, as well as a historian of science. Over the course of his long life, he amassed an impressive private collection of rare books, manuscripts and letters. He donated about a quarter of this collection to the Smithsonian in 1974 and Euler's missing letter of October 20, 1744, was part of that gift. Mary Lynn Doan, professor of mathematics at Victor Valley Community College, had contacted the Dibner Library of the Smithsonian Institution in the summer of 2003 and had learned that they have a small collection of documents by Leonhard Euler [Euler Papers]. She visited the Library on her way to the Euler Society's meeting that summer and brought a photocopy of the letter with her. I was able to identify the addressee as Cramer and shortly afterwards I brought the letter to the attention of Andreas Kleinert, co-editor of the forthcoming volume IVA.7 of the *Opera Omnia*. Thanks to Mary Lynn and the excellent archivists at the Smithsonian, Euler's *Opera Omnia* will now include the complete correspondence with Cramer.

Does Euler's letter, now cataloged as R.461a in the *Opera Omnia*, show that he knew Cramer's Rule before Cramer did? Not in the least. In fact, one of the things it tells us is that when Cramer wrote in November that Euler's remark "agrees entirely with my own thoughts," Euler had actually been talking about *Cramer's Paradox*, not Cramer's Rule.

Cramer's Paradox was the subject of an earlier *How Euler Did It* column [Sandifer 2004]. The simplest case of Cramer's Paradox involves two curves of the third degree. On one hand, two curves of degree m and n can intersect in as many as mn points. This theorem, named after Etienne Bézout (1730–1783), implies that two cubic curves may intersect in nine places. On the other hand, an equation of degree n has $\frac{(n+1)(n+2)}{2} = \frac{n^2+3n}{2} + 1$ coefficients. Because an equation can be multiplied by an arbitrary constant without affecting its graph, $\frac{n^2+3n}{2}$ points typically determine a curve of order n. Thus nine points in general position *uniquely* determine a cubic curve, and yet *two* such curves can typically intersect in nine points. In R.461a, Euler proposed the following resolution of the paradox:

> "I say, then, that although it is indeed true that a line of order n be determined by $\frac{nn+3n}{2}$ points, this rule is nevertheless subject to certain exceptions. . . . it may happen that such a number of equations, which we draw from the same number of given points, is not sufficient for this effect: this is evident, when two or several of these equations become identical. . . . I conceive therefore, that this inconvenience will

> take place when the nine points, which ought to determine a line of the 3rd order, are disposed such that two curved lines of this order may be drawn through them. In this case, the nine given points, because they include[1] two identical equations, are worth but 8, ... From this, one easily understands that if the nine points, from which one ought to draw a line of the third order, are at the same time the intersections of two curved lines of this order, then, after having completed all of the calculations, there must remain in the general equation for this order an undetermined coefficient, and beginning from this case not only two, but an infinity of lines of the 3rd order may be drawn from the same nine points."

Euler and Cramer agreed that this was the correct resolution of the paradox. Euler described it in his article [E147], which is discussed at length in [Sandifer 2004]. Cramer also gave his account in [Cramer 1750]. Agnes Scott judged that "Euler's resolution of the paradox ... agrees with that of Cramer, and goes just as far, or a little bit further" [Scott 1898, p. 263], but it was Cramer's name that became attached to it.

In modern terms, the question of determining the equation of a cubic curve reduces to solving a system of 9 homogeneous linear equations in 10 unknowns. The unknowns are the coefficients of the general equation of the third degree

$$\alpha x^3 + \beta x^2 y + \gamma x y^2 2 + \delta y^3 + \varepsilon x^2 + \zeta x y + \eta y^2 + \theta x + \iota y + \kappa = 0. \qquad (1)$$

Every time we plug the coordinates (x_0, y_0) of a particular point into equation (1), we have a homogeneous linear equation in the 10 unknowns. If the nine given points lead to a system of rank 9, then there is a unique solution, up to scalar multiplicity. However, when the nine points in question are not generic, but happen to be the points of intersection of two cubic curves, then the rank of the resulting linear system is no greater than 8. Cramer and Euler did not have these definitions and concepts of modern linear algebra at their disposal, but they certainly understood that the question reduced to the solution of linear systems, and this is why Cramer described his famous Rule to Euler in his reply of November 1744.

Euler's lost letter contains more than just a discussion of Cramer's Paradox. In fact, it contained something of a bombshell: Euler announced that he had just discovered a simple curve that exhibited something called a *cusp of the second kind* or a *ramphoid* cusp. A *cusp of first kind* or *keratoid* cusp is illustrated in figure (1)—the figure is from [E169]. The curve consists of two branches, AM and AN, with a cusp at A and a common tangent AL at that point. The two branches have opposite concavity with respect to the common tangent. A simple example of such a point is the origin in the graph of $y = x^{2/3}$, where the common tangent is the y-axis. We note that this is a cubic curve, because the equation can be re-written in the form of equation (1) as $y^3 - x^2 = 0$.

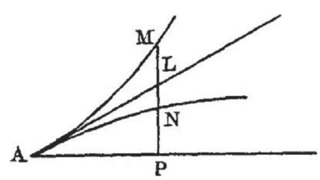

Figure 1. Cusp of the First Kind

[1] Euler used the French verb *renfermer*, which means both to include and to hide. This is particularly appropriate, because the "two identical equations" can only be derived through considerable calculation.

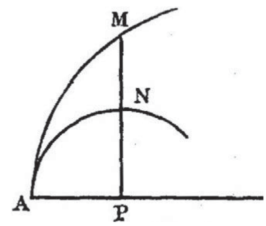

Figure 2. Cusp of the Second Kind

Figure (2), also from [E169], illustrates a *cusp of second kind*. Once again, there are of two branches, a cusp at A, and a common tangent at that point. This time, however, the two branches have the same concavity with respect to the tangent. L'Hôpital (1661–1704) is responsible for defining these two types of cusps. In 1740, Jean-Paul de Gua de Malves (1713–1785) published a proof that no algebraic curve could have a cusp of the second kind in [Gua de Malves 1740].

Euler was familiar with Gua de Malves' work and had initially accepted his result, but in 1744 he discovered that there was a subtle flaw in the supposed proof. In R.461a, he wrote to Cramer that "even in the fourth order there is a curved line of this kind, whose equation is $y^4 - 2xy^2 + xx = x^3 + 4yxx$, which simplifies to $y = \sqrt{x} \pm \sqrt[4]{x^3}$."

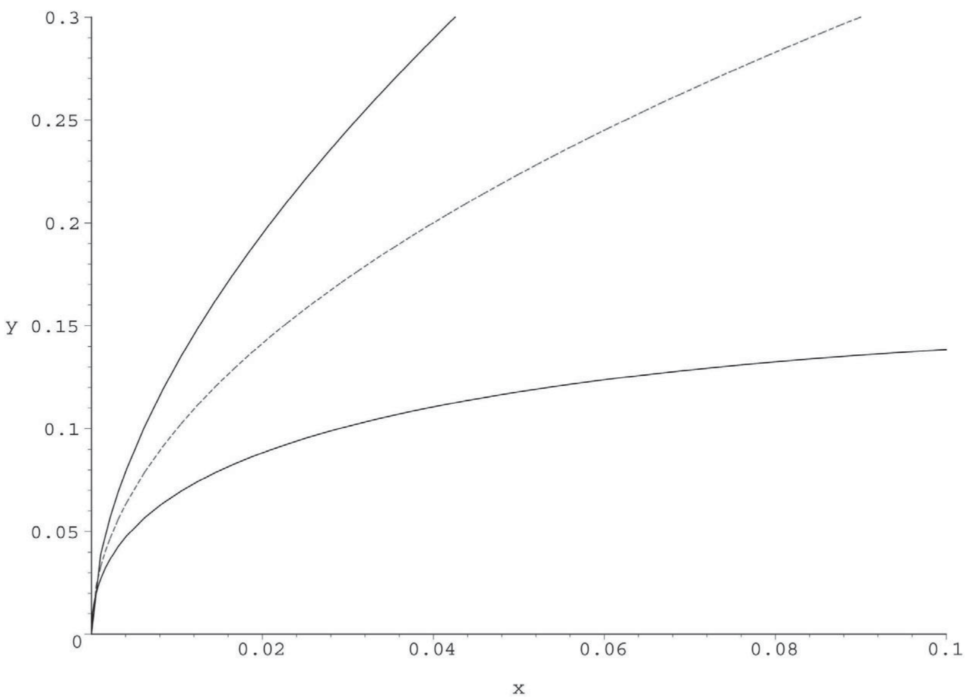

Figure 3. Euler's example

It's much easier to graph Euler's example in the form $y = \sqrt{x} \pm x^{3/4}$ than as an equation of the fourth degree. In figure (3), the curve is illustrated as a solid line, with the dotted graph of of $y = \sqrt{x}$ added for reference. The branch $y = \sqrt{x} + x^{3/4}$ lies above the square root and the other branch $y = \sqrt{x} - x^{3/4}$ lies below. This picture makes it clear why Euler referred to this curve as "a bird's beak" in [E169].

What may be less clear is that the two forms of the equation given by Euler are equivalent. To see this, begin by adding $4xy^2$ to both sides of $y^4 - 2xy^2 + xx = x^3 + 4yxx$. We then have

$$y^4 + 2xy^2 + x^2 = x(x^2 + 4xy + 4y^2),$$

or

$$(y^2 + x)^2 = x(x + 2y)^2.$$

Let's observe that we must have $x \geq 0$, because the two squares are nonnegative. Taking square roots, we have

$$y^2 + x = \pm\sqrt{x}(x + 2y), \tag{2}$$

or

$$y^2 \mp 2\sqrt{xy} + x = \pm x^{3/2}. \tag{3}$$

The left side of equation (3) is a perfect square, so we reject the case $-x^{3/2}$ on the right side. That means that equation (2) reduces to $y^2 + x = \sqrt{x}(x + 2y)$. Subtracting the term $2\sqrt{xy}$ from both sides, we have

$$(y - \sqrt{x})^2 = x^{3/2} \quad \text{and so}$$

$$y - \sqrt{x} = \pm x^{3/4}, \text{ or}$$

$$y = \sqrt{x} \pm x^{3/4}.$$

Euler wrote R.461a shortly after completing his *Introductio in analysin infinitorum*. The second volume of the *Introductio* [E102] is a very thorough treatment of analytic geometry, including a classification of cubic and quartic curves and their equations. [Cramer 1750] dealt with many of the same topics. In the Euler-Cramer correspondence, we have the opportunity to see two giants of the theory of equations in a free exchange of ideas.

Letters in Euler's hand were prized by collectors in the 19th and 20th century. Because of this, there are quite a few more letters to Euler in the *Opera Omnia* than from him. Every so often, missing letters like R.461a resurface and add to our knowledge of Euler's achievements. It was tantalizing to think that Euler might have scooped Cramer on his Rule, but the real story is no less captivating.

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Guest columnist, Rob Bradley (bradley@adelphi.edu), is Professor of Mathematics at Adelphi University in Garden City, NY, and president of the Euler Society. Rob and Ed's first book, *Leonhard Euler: Life, Work and Legacy*, was published by Elsevier in 2007. With Larry d'Antonio, they edited *Euler at 300: An Appreciation*, which was published by the MAA in 2007 as part of the celebration of Euler's tercentennial. Rob and Ed's third book, *Cauchy's Cours d'analyse: An Annotated Translation*, was published by Springer in 2009.

Part II

Number Theory

6

Factoring F_5

(March 2007)

Two names stand large in the history of number theory, Pierre de Fermat (1601–1665) and Leonhard Euler (1707–1783). Fermat, sometimes called The Great Amateur, was a part-time mathematician, a contemporary and rival of Descartes. His "real job" was as a judge in Toulouse, France. At the time, judges were expected to avoid the company of people on whom they might be required to pass judgment, so Fermat lived in comparative isolation, away from the people of Toulouse, with plenty of time to work on his mathematics. He kept in touch with current developments through his correspondence with Marin Mersenne.

Fermat worked on many of the same problems as René Descartes (1596–1650). They independently discovered analytic geometry, but since Fermat seldom published anything, *Cartesian coordinates* bear the name of Descartes, not Fermat. Both tried to "restore" the lost books of Apollonius, and when Fermat discovered a pair of amicable numbers, Descartes retaliated by finding another pair.[1] Both discovered techniques for finding the line tangent to a given curve at a given point, and Fermat showed how to find the area under a curve given by the equation $y = x^n$, as long as n was not equal to -1. All of this was very important in setting the stage for the discovery of calculus, later in the 1600's. Fermat and Descartes did not like each other very much. In fact, some people describe their relationship as a "feud," but it seems that Descartes resented Fermat more than Fermat disliked Descartes. They probably never met.

Fermat followed many of the mathematical traditions of the 17th century. Rather than provide proofs, he more often simply announced results and left the proofs of his claims as challenges to his readers. Others he wrote in the margins of his books. Those margin notes became known to the mathematical world when his son, Samuel, published his notes and papers in 1666, just after Pierre's death. This is how the conjecture we call Fermat's Last Theorem[2] became known to the rest of the mathematical world. Most of Fermat's

[1] To be fair, both pairs had been known to Arab mathematicians for a few hundred years, but the discoveries were new to Europe.

[2] The conjecture was that $x^n + y^n = z^n$ has no nontrivial integer solutions if $n > 2$. Its proof eluded mathematicians for over 300 years until Andrew Wiles solved it in the 1990's.

conjectures were neither as difficult nor as influential as the Last Theorem. For example, in another of his margin notes, he claimed

Theorem. *No triangular number other than* 1 *can be a fourth power.*

This is interesting, but it can hardly be considered important.

In 1738, Euler proved this particular claim to be true, [E98] and over the course of his career, Euler also proved (or disproved) almost all of Fermat's other conjectures, including Fermat's Little Theorem and Fermat's Last Theorem for the cases $n = 3$ and $n = 4$. Fermat's Last Theorem is called the "Last" not because it was the last conjecture Fermat ever made, but because after Euler got through with them, it was the last important one that remained to be proved or disproved.

Euler's interest in number theory was first sparked by another Fermat conjecture, that for all values of n, the number $F_n = 2^{2^n} + 1$ is a prime number. Fermat had made his conjecture in several of his letters during the 1630's and 1640's. These numbers are now called *Fermat numbers*, and, indeed, for small values of n, they give us 3, 5, 17, 257 and 65537, all of which are prime numbers. The next Fermat number, taking $n = 5$, is 4,294,967,297.

Euler's mentor in St. Petersburg, Christian Goldbach, alerted Euler to the conjecture in 1729. Euler responded almost immediately that he could make no progress on the problem, but by 1732, close to 100 years after Fermat had originally made the conjecture, Euler had a solution: Fermat was wrong. In Euler's first paper on number theory [E26] Euler announced that 641 divides 4,294,967,297. Later in that same paper, Euler added six of his own conjectures, some equivalent to Fermat's Little Theorem, that if p is prime and if p does not divide a, then p divides $a^{p-1} - 1$, and others that would turn out to be consequences of the Euler-Fermat theorem. See [S1] or [S2] for details.

What Euler did *not* tell us in E26 was how he thought to try to divide 4,294,967,297 by 641. He hadn't simply been dividing by prime numbers until he got to 641. He had a much better way, but he waited about 15 years, until E134, to reveal that secret.

The main purpose of E134, "Theoremata circa divisors numerorum," (Theorems about divisors of numbers) is to give Euler's first proof of Fermat's Little Theorem, but we will be looking at some of the other contents of this paper. After the main result of the paper, he proves a slightly more general version of Fermat's Little Theorem:

Theorem 4. *If neither of the numbers a and b is divisible by the prime number p, then every number of the form $a^{p-1} - b^{p-1}$ will be divisible by p.*

He uses this to prove a theorem about the divisors of numbers that are the sum of two squares:

Theorem 5. *The sum of two squares $aa + bb$ will never be divisible by a prime number of the form $4n - 1$ unless both of the roots a and b are divisible by $4n - 1$.*

Euler is really interested in the contrapositive of Theorem 5, that if a and b do not have a common prime factor of the form $4n - 1$, then all of the odd prime factors of $aa + bb$ must have the form $4n + 1$.

Euler's Theorems 6 and 7 extend the contrapositive of Theorem 5 to sums of two 4th powers (divisible only by odd primes of the form $8n + 1$) and 8th powers (prime divisors of the form $16n + 1$) before giving the general theorem in the form:

Theorem 8. *The sum of two such powers $a^{2^m} + b^{2^m}$, in which the exponents are powers of two, will not admit any divisors except those that are contained in the form $2^{m+1}n + 1$*

When Euler writes "two such powers," he is being just a little bit sloppy, and may be overlooking a condition. He probably means that a and b are relatively prime, but the claim is true if a and b have no common factors except those of the form $2^{m+1}n + 1$.

As Euler then tells us, this gives us a way to try to factor $2^{2^5} + 1$, or 4,294,967,297. Since both 4,294,967,296 and 1 are 32nd powers, and since they are relatively prime, and since $32 = 2^5$, Euler's Theorem 8 tells us that any prime divisors of 4,294,967,297 must have the form $64n + 1$. We can simply try different values of n, until we find a divisor, reach the square root of 4,294,967,297 or give up. Let's try it:

$n = 1$	$64n + 1 = 65$	65 is not prime.
$n = 2$	$64n + 1 = 129$	129 is not prime.
$n = 3$	$64n + 1 = 193$	193 is prime, but does not divide 4,294,967,297.
$n = 4$	$64n + 1 = 257$	257 is prime, but does not divide 4,294,967,297.
$n = 5$	$64n + 1 = 321$	321 is not prime.
$n = 6$	$64n + 1 = 385$	385 is not prime.
$n = 7$	$64n + 1 = 449$	449 is prime, but does not divide 4,294,967,297.
$n = 8$	$64n + 1 = 513$	513 is not prime.
$n = 9$	$64n + 1 = 577$	577 is prime, but does not divide 4,294,967,297.
$n = 10$	$64n + 1 = 641$	641 divides 4,294,967,297 with quotient 6,700,417.

We find the factor 641 after just six divisions.

Euler did not speculate in print on whether the other factor, 6,700,417, is prime. It is prime, but there is no evidence that Euler ever tried to find out.

How much more work would it have been to show that 6,700,417 is prime? Not that much. If 6,700,417 has a prime factor, it has to have one less than its square root, which is a bit over 2588, and, by Theorem 8, any such factors also would have to be of the form $64n + 1$. There are 40 integers less than 2588 of the form $64n + 1$, and we've already checked ten of them. Thirty remain, of which only six, 769, 1153, 1217, 1409, 1601, 2113 are prime, and none of which divide 6,700,417. Euler had already done half the work. Knowing this, we could finish the proof that 6,700,417 is prime with pencil and paper in only a few minutes.

The number 6,700,417 would have been the largest known prime number in Euler's day. When he prepared a table of large prime numbers [E283] in 1762, the largest prime number he mentioned was 2,232,037. Some experts think that Euler knew that 6,700,417 was prime, because it would have been so easy for him prove it. Others think that Euler never thought to use those tools on this particular number, because if he had, he would have told somebody and mentioned it in E283. Experts disagree.

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7

Rational Trigonometry

(March 2008)

Triangles are one of the most basic objects in mathematics. We have been studying them for thousands of years, and the study of triangles, trigonometry, is, to some extent, a part of every mathematical curriculum. Our oldest named theorem, the Pythagorean theorem, is about triangles, though the theorem was known long before Pythagoras. It is probably our most famous and most often proved theorem as well. Hundreds of different proofs are known, [Loomis 1940] and good writers still find interesting things to say about the theorem. [Maor 2007]

The particular branch of trigonometry where we ask that certain parts of the given triangle, sides, angles, medians, area, etc., is called *rational trigonometry*. Though it originally arose from geometry, rational trigonometry is now usually classified as a part of number theory.

For example, for many people, the Pythagorean theorem is particularly interesting when we consider it as a problem in rational trigonometry and ask that the lengths of the sides of the triangle be whole numbers. This is the problem of so-called Pythagorean triples, three whole numbers a, b and c satisfying

$$a^2 + b^2 = c^2.$$

As we all know, the simplest such triple is $(3, 4, 5)$. It is easy to show that there are infinitely many such triples. We can generate all we want by picking two positive integers, m and n, with $m > n$ and letting

$$\begin{aligned} a &= 2mn \\ b &= m^2 - n^2 \\ c &= m^2 + n^2. \end{aligned}$$

It is easy to check that for these values, indeed, $a^2 + b^2 = c^2$. It is slightly less easy to check that if m and n are relatively prime, one odd and the other even, then a, b and c are pairwise relatively prime, so the method is not just generating infinitely many triangles similar to each other. All Pythagorean triples can be generated in this way.

Another way to generate Pythagorean triples is apparently due to Ozanam. He tells us to look at the sequence of rational numbers

$$1\frac{1}{3},\ 2\frac{2}{5}, 3\frac{3}{7}, \ldots, n\frac{n}{2n+1}, \ldots.$$

Each of these numbers, written as an improper fraction, $\frac{a}{b}$, gives two of the three numbers of a Pythagorean triple. We leave it to the reader to find why this is true.

Fibonacci also showed a way to find infinitely many different Pythagorean triples, but neither Fibonacci's nor Ozanam's method gives *all* of them.

It should be no surprise that Euler also worked in rational trigonometry. He wrote about half a dozen papers on the subject, and our purpose in this column is to look at a sequence of four of them, giving better and better solutions to the same problem. The first of those papers [E451] gives the problem right in its title, *Solutio problematis de inveniendo triangulo, in quo rectae ex singulis angulis latera opposita bisecantes sint rationales*, "Solution of the problem of finding a triangle in which the lengths of the straight lines drawn from each angle and bisecting the opposite sides are rational." Euler neglects to mention that he means the sides of the triangle to be rational as well, nor that he means to multiply by the least common denominator and make all these measures integers instead of rational numbers. Euler wrote this paper in 1773.

The other three papers, with their titles in English and the years that Euler wrote them, are

E713 (1778) Investigation of a triangle in which the distance from the angles to its center of gravity is rationally expressed

E732 (1779) An easier solution to the Diophontine problem about triangles, in which the straight lines from the angles to the midpoints of the opposite sides are rationally expressed

E754 (1782) A problem in geometry solved by Diophantine analysis

The last of these was written in French. The others were in Latin, though the second one, E713, has a short summary in French, which we quote below:

> This article, which will give pleasure to the small number of amateurs in indeterminate analysis, contains a very beautiful solution to the problem stated in the title. Here it is in just a few words. Let the sides of the desired triangle be $2a$, $2b$, $2c$, and let the straight lines be drawn from their midpoints to the opposite angles, respectively *f*, *g*, *h*. Take as you please any two numbers q and r and find $M = \frac{5qq-rr}{4qq}$ and $N = \frac{5rr-9qq}{4rr}$. Reduce the fraction $\frac{(M-N)^2-4}{4(M+N)}$ to its lowest terms, and name the numerator x and the denominator y. Then you will have the side $2a = 2qx + (M-N)qy$ and the line $f = rx - \frac{1}{2}(M-N)ry$. Make $p = x + y$ and $s = x - y$, and you will have the sides $2b = pr - qs$ and $2c = pr + qs$ and the lines $g = \frac{3pq+rs}{2}$ and $h = \frac{3pq-rs}{2}$.

The summary shows that the spirit of Euler's solution is like that of the formulas above that give all the Pythagorean triples. We get to choose two numbers, here q and r, with a few restrictions (like we don't want $M + N = 0$, as stated in the text but not the summary.) Then the formulas give the solutions in terms of p and q. As with the solution to the problem of the Pythagorean triples, it is easy to see that all the values, a, b, c, f, g and h, are indeed rational. It is a bit more subtle and a good deal more tedious to check that these values f, g and h are the medians of the triangle with sides $2a$, $2b$ and $2c$. Some of that will be evident from what follows.

Note that Euler mentions that this paper "will give pleasure to the small number of amateurs in indeterminate analysis." To Euler, "indeterminate analysis" is the practice of finding integer or rational solutions to algebraic equations, what we now call and Euler himself would later call Diophantine analysis. He also mentions that he doesn't think that very many people will be interested, that there are only a "small number of amateurs." I think he uses the word "amateurs" a bit differently than we use the same word today. Now it means "people who are not professionals," but to Euler it meant "people who love the subject." I hope we're all "amateurs" in Euler's sense of the word.

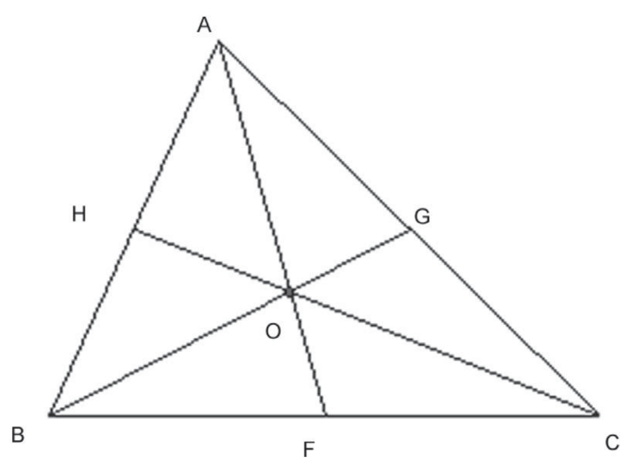
We've seen Euler's beginning, the statement of the problem, and one of his answers. Let's look a bit at his solutions, at some of the things he discovered along the way, and at why he felt the need to return to the problem to improve his solution.

Euler begins the first of his papers, E451 with only the title as preamble and tells us that we should let the sides of the desired triangle be $2a$, $2b$ and $2c$ and the lengths of the medians be f, g and h. Then we want to find rational solutions to the system of equations

$$\begin{aligned} 2bb + 2cc - aa &= ff \\ 2cc + 2aa - bb &= gg \\ 2aa + 2bb - cc &= hh \end{aligned}$$

Euler calls these three equations his "fundamental equations" for this problem.

He doesn't tell us here why these equations have anything to do with the problem, but in the second of the four papers, E713, perhaps he is being a bit more gentle on his "amateurs," for he gives us details and a diagram.



Let ABC be a triangle, with midpoints F, G and H opposite A, B and C, respectively, and medians AF, BG, and CH intersecting at O, the center of gravity. Let $a = BF = CF$, $b = CG = AG$, $c = AH = BH$, $f = AF$, $g = BG$, $h = CH$ and $\omega = \angle AFB$.

Euler claims, without explicitly mentioning the Law of Cosines, that

$$AB^2 = AF^2 + BF^2 - 2AF \cdot BF \cos\omega$$

and

$$AC^2 = AF^2 + CF^2 + 2AF \cdot CF \cos \omega.$$

Add these to get

$$AB^2 + AC^2 = 2AF^2 + 2BF^2$$

or

$$4cc + 4bb = 2ff + 2aa,$$

or

$$ff = 2cc + 2bb - aa, .$$

Similarly,

$$gg = 2aa + 2cc - bb \text{ and } hh = 2aa + 2bb - cc.$$

Thus the problem becomes to find three numbers, a, b, c, for which these three formulae produce squares.

We'll return to E451 and follow Euler on a short tangent. If we use the three fundamental equations, we find that

$$2gg + 2hh - ff = 9aa,$$
$$2hh + 2ff - gg = 9bb,$$
$$2ff + 2gg - hh = 9cc.$$

In the fourth of these papers, E754, Euler describes these equations as "a pleasant property," but that this property "does not contribute in any manner to the solution of the problem." But what is "pleasant" about these equations. They are the same as his three fundamental equations, but with f, g and h substituted for a, b and c, and with $3a$, $3b$ and $3c$ substituted for f, g and h.

This means that if a triangle with sides $2a$, $2b$ and $2c$ has medians of length f, g and h, then a triangle with sides $2f$, $2g$ and $2h$ has medians of length $3a$, $3b$ and $3c$. If the measures in one triangle are all rational, then so are the measures in the other, and so we learn that solutions to this problem in rational trigonometry come in pairs.

But we still don't have any solutions. All of Euler's solutions are rather long, so we will only summarize them

In his first solution, the one given in E451, Euler rewrites his first two fundamental equations as

$$ff = (b-c)^2 + (b-c)^2 - aa = (b-c)^2 + (b+c+a)(b+c-a)$$
$$gg = (a-c)^2 + (a+c)^2 - bb = (a-c)^2 + (a+c+b)(a+c-b).$$

Being a genius at substitution, Euler introduces two new variables, p and q, that enable him to take square roots of these two equations and write them as

$$f = b - c + (b+c+a)p$$
$$g = a - c + (a+c+b)q.$$

After two pages of dense calculations, Euler finds a sixth degree polynomial that gives hh in terms of p and q. Then he finds rational expressions for a, b and $\hat{c}$ in terms of p and q. Hence, if p and q are rational, then so are a, b, c, f and g. That leaves h. So, all Euler has to do is find some rational values of p an q that make his sixth degree polynomial into a perfect square, and at the same time, don't make any of the denominators of his rational expressions equal to zero. It is tedious, but he manages to find several solutions, among which are

$$1. \quad a = 158 \quad b = 127 \quad c = 131 \quad f = 204 \quad g = 261 \quad h = 255$$

and its companion solution, reduced to lowest terms because f, g and h are all multiples of 3,

$$2. \quad a = 68 \quad b = 87 \quad c = 85 \quad f = 158 \quad g = 127 \quad h = 131.$$

Five years later, in 1778, Euler made his second attack on the problem, E713. As we mentiond above, the Summary of this article mentions the "Amateurs of analysis," and he uses the law of cosines to justify his fundamental equations.

During those five years, Euler apparently realized that the "pleasant property" was not just an interesting property of triangles with rational medians, but is a general property of all triangles. He calls it a "most distinguished property" and states it more geometrically than he did before, writing

$$AO^2 + BO^2 + CO^2 = \frac{1}{3}(BC^2 + AC^2 + AB^2).$$

His other calculations are quite similar, but when it comes time to introduce the new variables p and q, he defines them as

$$f = b + c + \frac{p}{q}(b - c + a).$$

This, combined with the first fundamental equation, allows Euler to write a and f, and hence g and h, in terms of b, c, p and q. Things get complicated, but after a while he introduces two more variables, r and s, to make $c + b = pr$ and $c - b = qs$, and then two more, x and y such that $p = x + y$ and $q = x - y$, then t and u so that $\frac{a}{q} = x + ty$ and $\frac{f}{r} = x + uy$, and finally M and N so that $2tx + tty = y + 2Mx$ and $2ux + uuy = y + 2Nx$. In this tower of substitutions, everything ends up depending on q and r, and Euler can find some triangles. We've skipped five pages of details here. The interested reader is encouraged to consult the original sources. The mathematics there is considerably more difficult than the Latin.

In the end, Euler finds that for $q = 1$ and $r = 2$, as well as for $q = 2$, $r = 3$, he gets the same triangle we labeled 1 above, but for $q = 2$, $r = 1$, he finds

$$3. \quad a = 404 \quad b = 377 \quad c = 619 \quad f = 942 \quad g = 975 \quad h = 477.$$

Then for $q = 1$ and $r = 3$, he gets

$$4. \quad a = 3 \quad b = 1 \quad c = 2 \quad f = 1 \quad g = 5 \quad h = 4.$$

Though this is a solution to the Diophantine equations, the sides 3, 1 and 2 do not form a triangle. He gives several other solutions as well.

Hence, this solution lacks two of the properties we admire in the solution to the problem of Pythagorean triples. Two different choices of the variables p and q can give the same solution, and some choices of p and q can give inadmissible solutions. Euler doesn't seem to ask whether or not all rational triangles with rational medians are generated in this way.

Euler's third solution to the problem of rational medinas, E732, followed just a year later, in 1779. For this paper, he called the midpoints of the sides X, Y, and Z instead of F, G and H, and the corresponding lengths of the medians are x, y and z instead of f, g and h. Perhaps this is a symptom of Euler's blindness, as he had been almost entirely blind since unsuccessful cataract surgery in 1773, and he was unable to consult his earlier works on the subject to make his notation consistent.

Using his new notation, Euler transforms two of his three fundamental equations into different forms:

I. $xx - yy = 3(bb - aa)$,

II. $xx + yy = 4cc + aa + bb$, and

III. $zz = 2aa + 2bb - cc$.

These equations are enough different from the others that after Euler makes another sequence of miraculous substitutions, introducing f and g, p and q, m and n, t and u, and finally M, he gets everything in terms of f and g. This takes him just three pages of calculations, and the solution is essentially the same as the one we translated above from E754. A few highlights are

$$m = \frac{5gg - ff}{4gg} \text{ and } n = \frac{5ff - gg}{4ff},$$

exactly as M and N will depend on r and q in E754. Likewise,

$$p = 4(m+n) \quad \text{and} \quad q = (m-n)^2 - 4,$$

almost like his variables x and y are defined in E754, but there he factors out their greatest common divisor.

Now, in terms of f, g, p and q, Euler tells us that the sides of the triangle are $2a$, $2b$ and $2c$, where a, b and c are given by

$$\begin{aligned} a &= (f-g)p + (f+g)q \\ b &= (f+g)p + (f-g)q \\ c &= 2g(m-n)(3m+n) - 8g \\ &= g(m-n)p + 2gq. \end{aligned}$$

He also gives equations for the lengths of the medians, x, y and z.

For his first example, he takes $f = 2, g = 1$ to get his first solution again, then $f = 1, g = 2$ to get the third one. There are no new rational triangles in this paper. Its main improvement over its predecessor, E713, seems to be that its calculations are a bit shorter, and its answer is more concise.

The last of the four papers is much like the third one. Euler wrote it three years later, in 1782, just a year before he died, and for some unknown reason he wrote it in French. The substitutions are slightly different and the resulting algorithm is a bit more streamlined. Moreover, he takes less care to get *integer* results. He is happy to get rational results, then multiply through by a common denominator to make them integer. He gets yet again his examples 1 and 2 above, but this time he gives some new examples, including

5. $a = 159 \quad b = 325 \quad c = 314 \quad x = 309.5 \quad y = 188.5 \quad z = 202$

From this series of papers, we see that even near the end of his life, Euler went back over his earlier results and tried to improve them. His blindness did not impair his amazing powers of calculation or his ability to design ingenious substitutions. Moreover, while his students mostly worked on applied problems, Euler seemed happy to work also on whimsical problems like this, just because they were fun.

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8

Sums (and Differences) that are Squares

(March 2009)

Late in his life, Euler devoted a lot of time and effort to number theory. Indeed, almost 40% of his papers in number theory were published after he had died. Many of these late papers were on Diophantine equations, usually involving square numbers in one way or another.

Usually, Euler does not tell us what motivates the particular Diophantine equations that he chooses to study. Sometimes he does tell us, as in [E754], Problème de géométrie résolu par l'analyse de Diophante, "A problem in geometry solved by Diophantine analysis". In that paper, Euler seeks three integer numbers, x, y and z, such that the three quantities

$$2xx + 2yy - zz,$$
$$2yy + 2zz - xx \text{ and}$$
$$2zz + 2xx - yy$$

are all perfect squares. He explains that x, y and z have these properties exactly when a triangle with sides of length x, y and z has median lines that have rational length. That paper, E754, was one of the subjects of this column in March 2008. [Sandifer 2008] Thus, the Diophantine equations sometimes hide a geometric motivation.

Other times, though, Euler does not tell us what inspired a particular Diophantine problem, as in [E753], Solutio succincta et elegans problematis quo quaeruntur tres numeri tales ut tam summae quam differentiae binorum sint quadrata, "A succinct and elegant solution to the problem of finding three numbers such that the sum or difference of any two of them is a square number." Here again he seeks three integer numbers, x, y and z, where x is the largest and z the smallest, this time with the property that all three sums and all three differences,

$$\begin{array}{lcl} x+y, & & x-y, \\ x+z, & & x-z, \\ y+z & \text{and} & y-z, \end{array}$$

are perfect squares.

We are left to guess *why* Euler thinks such triples are interesting. Euler hints that he has a reason when he opens the paper, "This problem is treated and solved by several authors," but tells us neither who those authors were nor why they were interested. We will return to the question of Euler's motivation at the end of the column.

Before we look at the content of this article, it is probably a good idea to review some of Euler's related works from the 1750s, in particular, [E228], De numeris qui sunt aggregata duorum quadratorum, "On numbers that are the sum of two squares." We will need some lemmas from this article:

Lemma 1. *If* $2n$ *is a sum of two squares, then so also is* n*.*

Proof. Suppose $2n = aa + bb$. Then either a and b are both odd or they are both even. In either case, both $(a + b)/2$ and $(a - b)/2$ are integers, and it is easy to calculate that their squares sum to n.

Lemma 2. *Suppose that* m *is the sum of two squares. Then so also is* $2m$*.*

Proof. Suppose that $m = aa + bb$, with a greater than or equal to b. Then $a + b$ and $a - b$ are the numbers whose squares sum to $2m$.

Lemma 3. *If a number* n *is a sum of two squares in two different ways, then* n *is composite and it can be factored so that each factor is a sum of two squares.*

Proof. The proof is a calculation. If $x = pp + qq = rr + ss$ then $x = (aa + bb)(cc + dd)$ where $p = ac + bd$, $q = ad - bc$, $r = ad + bc$ and $s = ac - bd$.

Returning to the article at hand, we see that Euler proceeds by analysis. He supposes that x, y and z are three numbers that satisfy the conditions of the problem, that is their sums and differences are all squares, and he supposes that x is the largest and z is the smallest. Euler claims without proof that there must be integers p and q such that $x = pp + qq$ and $y = 2pq$. This is not obvious, but it is not hard, either. Suppose that $x + y = a^2$ and $x - y = d^2$. Then $x = \frac{a^2+d^2}{2}$ so, by Lemma 1, $2x$ is the sum of two squares, $p = \frac{a+d}{2}$ and $q = \frac{a-d}{2}$. A little algebra shows that $x = pp + qq$ and $y = 2pq$, as Euler asserted.

Now, because $x = pp + qq$ and $y = 2pq$, we have $x + y = (p + q)^2$ and $x - y = (p - q)^2$, so for that pair, at least, the sum and the difference are squares, as required. Similarly, taking $x = rr + ss$ and $z = 2rs$, we get that $x + z = (r + s)^2$ and $x - z = (r - s)^2$. We are guaranteed from the construction so far that $x + y, x - y, x + z$ and $x - z$ are all squares. For these four conditions to be satisfied (i.e. $x = x$) we get $rr + ss = pp + qq$. For x, y and z to be solutions to the original problem, two more conditions must also be satisfied, that $y + z = 2pq + 2rs$ and that $y - z = 2pq - 2rs$ must also be squares.

Thus Euler has transformed his original problem into a related problem of finding four numbers, p, q, r and s such that

$$x = pp + qq = rr + ss,$$

$$y + z = 2pq + 2rs \text{ is a square and}$$

$$y - z = 2pq \text{ is a square.}$$

Then x is composite, and $x = (aa + bb)(cc + dd)$, where a, b, c and d are as given in Lemma 3. Now we can rewrite y and z in terms of a, b, c and d, and then find $x + z$ and $x - z$. Because both of these are squares, their product, $yy - zz$ is also a square. This works out to be

$$yy - zz = 16abcd(aa - bb)(dd - cc)$$

For this to be a square, the formula

$$ab(aa - bb) \cdot cd(dd - cc)$$

must also give a square.

Now, Euler simplifies his problem by discarding some solutions, supposing that $a = d$ and that there exists an integer n such that

$$c(aa - cc) = nnb(aa - bb).$$

He further supposes that $a = c - b$, perhaps discarding yet more solutions. Then some algebra yields

$$\frac{b}{c} = \frac{nn + 2}{2nn + 1}.$$

Now if we take the obvious possible solution, $b = nn + 2$ and $c = 2nn + 1$, it makes $a = d = nn - 1$, so that the formula $ab(dd - cc)$ reduces to

$$3nn(nn - 1)(nn + 2)^2.$$

For this to be a square, $3(nn - 1)$ must also be a square.

This leaves us to find such n. Euler introduces yet a new pair of variables, f and g, such that

$$3(nn - 1) = \frac{ff}{gg}(n + 1)^2.$$

Solving for n makes

$$n = \frac{ff + 3gg}{3gg - ff},$$

from which it follows that

$$a = d = \frac{12ffgg}{(3gg - ff)^2},$$

$$b = nn + 2 = \frac{3f^4 - 6ffgg + 27g^4}{(3gg - ff)^2} \quad \text{and}$$

$$c = \frac{3f^4 + 6ffgg + 27g^4}{(3gg - ff)^2}.$$

These are certainly rational numbers, but they may not be integers. However, they have a common denominator. Because Euler's problem depends on ratios, we can scale the solution to integers and take

$$a = d = 4ffgg,$$
$$b = f^4 - 2ffgg + 9g^4 \quad \text{and}$$
$$c = f^4 + 2ffgg + 9g^4.$$

From here, we can work backwards to find first p, q, r and s and then x, y and z. The problem is solved, in theory, so Euler gives us some examples.

Example 1. $f = 1$ and $g = 1$.

Then $a = d = 4$, $b = 8$ and $c = 12$. These scale down to $a = d = 1$, $b = 2$ and $c = 3$. This makes $p = 5, q = 5, r = 7$ and $s = 1$, which, in turn, makes $x = 50$, $y = 50$ and $z = 14$. Indeed, the sums and differences are 0, 36, 64 and 100, and they are all squares, as required. Still, with $x = y$, some readers may be dissatisfied and think that Euler threw away all the interesting solutions as he simplified his analysis. So he does another example.

Example 2. $f = 2$ and $g = 1$.

Then $a = d = 16$, $b = 17$ and $c = 33$, so $p = 800, q = 305, r = 817$ and $s = 256$, and we get the solution

$$x = 733025, \quad y = 48800 \quad \text{and} \quad z = 418304.$$

Indeed

$$x + y = 1105^2, \quad x - y = 495^2,$$
$$x + z = 1073^2, \quad x - z = 561^2,$$
$$y + z = 952^2 \quad \text{and} \quad y - z = 264^2.$$

Euler does two more examples. The values $f = 3$ and $g = 1$ give the same solution as Example 1, and the values $f = 1$ and $g = 2$ give nine-digit values of x, y and z, which Euler chooses not to write out explicitly.

Euler seems to be done, but in absolutely typical Euler style, he has more to say. In fact, he has two more things to say. The first he calls a "Note" and the second he calls an "Addition."

In the Note, Euler observes that new solutions can be constructed from old ones. Suppose that x, y and z solve the problem. Then we get another solution by taking

$$X = \frac{yy + zz - xx}{2}, \quad Y\frac{xx + zz - yy}{2} \quad \text{and} \quad Z = \frac{xx + yy - zz}{2}$$

It is immediate to check that the sums are squares, and it is amusing to check that the differences are squares as well.

In the Addition, Euler tells us, "only a few more details are required to solve another problem."

Problem. To find three squares, xx, yy and zz, such that any two differences are squares.

For any values of p and q, if we set $x = pp + qq$ and $y = 2pq$, then $xx - yy = (pp - qq)^2$. Likewise, whatever r and s are, if we set $x = rr + ss$ and $z = 2rs$, then

$xx - zz = (rr - ss)^2$. All that remains is to make sure that the two values of x match up and that $yy - zz$ is a square. This amounts to the two conditions

$$pp + qq = rr + ss \quad \text{and}$$
$$yy - zz = 4(ppqq - rrss).$$

As before, this makes

$$p = ac + bd, \quad q = ad - bc, \quad r = ad + bc \quad \text{and} \quad s = ac - bd.$$

Substitution and factoring shows that for $yy - zz$ to be a square, the product

$$abcd(aa - bb)(dd - cc)$$

must also be a square. We see that this can be done if

$$a = d = nn \pm 1, \quad b = 2mm \mp 1 \quad \text{and} \quad c = nn \mp 2.$$

Indeed, if instead of the integer n we use the rational number $\frac{m}{n}$, we get the following pair of generators:

$$a = d = mm \mp nn, \quad b = 2mm + nn \quad \text{and} \quad c = mm \pm 2nn.$$

Taking simple numbers for m and n, Euler gives us a table of 23 pairs of values of a, b, c and d, each of which gives a solution to the new problem, a total of 46 such solutions.

Then he gives us an example starting from the numbers $m = 2$ and $n = 1$, taking the lower signs. This makes $a = 5$, $b = 7$, $c = 5$ and $d = 2$ (where Euler has reversed c and d, not that it matters). Then $p = 39$, $q = 25$, $r = 45$ and $s = 11$, whence

$$x = 2146, \quad y = 1950, \quad z = 990$$

or, scaling down by the common divisor 2,

$$x = 1073, \quad y = 975, \quad z = 495.$$

A check shows that the differences of the squares of these values are the squares of 448, 952 and 840.

Finally, Euler notes that if x, y and z are a solution to the problem, then another solution can be generated by taking

$$X = 2(yy + zz - xx),$$
$$Y = 2(xx + zz - yy) \quad \text{and}$$
$$Z = 2(xx + yy - zz).$$

Let us turn now to *why* Euler wrote this paper. As we mentioned above, Euler tells us only that others have worked on the problem, not who they were or why they were interested. Rudolf Feuter, the Editor of the volume of the *Opera omnia* in which this article is reprinted, sheds no light on the question either. We are left to figure it out or shrug it off as something we might never know.

On the other hand, we might let the three sums, $x + y$, $x + z$ and $y + z$, be a^2, b^2 and c^2, and the three differences be d^2, e^2 and f^2, respectively. Note that the values a, b, c and

d have a different meaning than they did above. Then a variety of Pythagorean triples arise by calculations like the following:

$$\begin{aligned} a^2 &= x + y \\ &= (x + z) + (y - z) \\ &= b^2 + f^2, \end{aligned}$$

or

$$\begin{aligned} a^2 &= x + y \\ &= (x - y) + (y + z) \\ &= e^2 + c^2. \end{aligned}$$

Such calculations lead to two more Pythagorean triples, $b^2 = c^2 + d^2$ and $e^2 = d^2 + f^2$. So, we have four right triangles and six edges. That makes us think about a tetrahedron. A few minutes of sketches and puzzling reveals that we are looking for a tetrahedron with integer sides and with right triangles for all four faces. The reader is encouraged to draw some pictures. As you try to imagine such a tetrahedron, you should realize that the side of length a is not incident to any right angles and the side of length d has two right angles at each end.

Similarly, we can discover the reason behind the problem in Euler's Addition. Let the differences of squares be $a^2 = x^2 - y^2$, $b^2 = x^2 - z^2$ and $c^2 = y^2 - z^2$. This gives three Pythagorean triples, (y, a, x), (z, b, x) and (z, c, y). Again, we have a tetrahedron, one with three right triangles. The triangle that is *not* a right triangle has sides of length a, b and c. Again, the reader should draw a picture and note that the three right angles do not share a common vertex. Such a tetrahedron would satisfy a different system of Diophantine equations, which we will leave the reader to discover.

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Part III

Combinatorics

9

St. Petersburg Paradox

(July 2007)

If we knew what we were doing, it would not be called research, would it?

—attributed to Albert Einstein (1879–1955)

When I was first learning probability as an undergraduate, I learned about something called "the St. Petersburg Paradox." One version of this paradox is as follows: [J]

> A man is to throw a coin until he throws head. If he throws head at the nth throw, *and not before*, he is to receive £2^n.
>
> What is the value of his expectation?

We learned about this just after we learned about geometric distributions, so we knew that the probability of throwing n heads in a row was $\frac{1}{2^n}$. Our instructor, David Griffeath,[1] asked us to do two things with this problem, to find the average number of times the coin would be tossed in this game, and to find the so-called "value of the game," the amount of money that a player who plays this game should expect to win. We had just learned some formulas for these values, but the infinite series behind those formulas are pretty straightforward. The expected number of tosses is[2]

$$\sum_{n=1}^{\infty} np(n) = \sum_{n=1}^{\infty} \frac{n}{2^n} = 2,$$

and the value of the game is

$$\sum_{n=1}^{\infty} 2^n p(n) = \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{2^n} = \sum_{n=1}^{\infty} 1 = \infty.$$

[1] now at the University of Wisconsin, not to be confused, as I did, with the statistician at the University of Wollongong, David Griffiths. I had assumed they were the same person, one of several over-hasty assumptions I made in writing this column.

[2] A really beautiful proof of this summation is due to an obscure 14th century English mathematician named Richard Swineshead.

At the time, I thought that the "paradox" was that a game with such a short expected duration could still have an infinite value.

Let us move forward several years to an AMS Section meeting in Hoboken, NJ on April 15, 2007, the very day of Euler's 300th birthday. Robert Bradley, President of The Euler Society, gave a talk on Euler's probability and statistics, and he mentioned that one of Euler's posthumously published papers [E811], *Vera aestimatio sortis in ludis*, (On the true valuation of the risk in games) and said that it described the St. Petersburg paradox. I assumed that Euler had posed the St. Petersburg paradox, and resolved to write a column about it.

Euler's article was published in 1862, 79 years after his death in 1783, as part of a two-volume set called the *Opera posthuma*. It is quite short, only four pages in the original, four pages in Pulskamp's English translation, and, because of the extensive footnotes, eight pages in the *Opera omnia.* It was one of the articles that Euler did not bother to see published during his lifetime. Later, we will speculate on why he might have done this.

Moreover, E811 is one of the articles for which Gustav Eneström [En] does not give a date written. Later, we will hazard a guess on this as well.

Turning to the article itself, we see that Euler begins by acknowledging the work of Blaise Pascal, Christiaan Huygens and Jakob Bernoulli. His acknowledgement turns almost immediately to criticism when he writes:[3]

> Following the way of thinking of these men I do not act imprudently if I take up a game where it may happen equally easily that I would either win or lose 100 Rubles. But if all of my wealth is worth only 100 R., I seem to myself to begin this game about to be played most imprudently.

Euler is telling us that to a person with only 100 Rubles, that money is worth more to him than the next 100 Rubles would be. He has discovered one of the properties of what we now call a *utility function*, that the value an individual places on an amount of money depends on how much money that person already has. Usually, the more money you already have, the less value you place on having another Ruble (or dollar), so we would say that utility functions have decreasing first derivatives. This is an important concept in modern economic theory.

He goes on to describe the game behind the St. Petersburg paradox, though he doesn't call it that. Euler does make one trivial change to the game. Rather than tossing a coin, he has us roll a die. For him, the game ends if we roll an odd number, so we win by rolling lots of even numbers.[4]

Now Euler brings a new name into the story, Nicolaus Bernoulli,[5] nephew of the bickering brothers Jakob and Johann. The former of these brothers wrote *Ars conjectandi*,

[3] I follow the Pulskamp translation throughout, unless noted otherwise.

[4] As I write this column, I've been reading Rudolf Taschner's little book [Ta], in which he tells us that ancient Greek and Hebrew numerologists considered even numbers *un*lucky, and odd ones lucky. Euler's description makes the even ones lucky.

[5] There were many Bernoullis. This was the one we call Nicolaus I, (1687–1759). Though both his father (1662–1716) and his grandfather (1623–1708) were named Nicolaus, they were not mathematicians, so sources in the history of mathematics do not give them numbers. It is easy to confuse Nicolaus I with his cousin, Nicolaus II (1695–1726), son of Euler's mentor Johann and a childhood friend of Euler himself.

the first comprehensive study of the theory of probability, published in 1713, and the latter was Euler's mentor in Basel.

According to Euler, Nicolaus believed that, based on his ideas of people's values, given a choice of playing the St. Petersburg game once or receiving a certain payment of 20 Rubles, most people would choose the 20 Rubles. It is interesting that Nicolaus Bernoulli was not among the authors Euler cited earlier in his article, probably because he found reason to criticize Huygens, *et al.*, but he agreed with Nicolaus. It is also interesting that Nicolaus didn't publish much, and apparently didn't publish anything about the St. Petersburg Paradox.

Euler gives some thought to how a man with 20 Rubles would be reluctant to play a game in which he has equal chances of either winning or losing his 20 Rubles, but a very wealthy person with, worth many Rubles (Euler doesn't tell us how many Rubles it took to be "wealthy.") would not be nearly so reluctant. Following a typical Eulerian analysis, he introduces the idea of a *status*, roughly how much the player's wealth would be worth after a particular outcome. If a game or business deal[6] has equal chances of bringing a player to status b or to status c, then Euler proposes that the game be worth $\sqrt{bc}$, that is, the geometric mean of b and c. As a further example, if a game has three equally likely outcomes, leading to statuses b, c or d, then it would be worth $\sqrt[3]{bcd}$.

Probability was a new subject at the time, and analysts hadn't made the step from discrete probability to continuous probability, so Euler's most complicated analysis in this direction has us suppose that there are m ways that the player can be rewarded with status a, n ways to get b and p ways to arrive at status c. We would say that the corresponding probabilities are $\frac{m}{m+n+p}$, $\frac{n}{m+n+p}$ and $\frac{p}{m+n+p}$. In this case, Euler's idea gives the value $\sqrt[m+n+p]{a^m b^n c^p}$. He points out that Huygens would assign the value $\frac{ma+nb+pc}{m+n+p}$ to the same game, while the logarithm of Euler's value would be $\frac{m \ln a + n \ln b + p \ln c}{m+n+p}$.

Next we take our player's current status to be A, and give the player even chances of either winning a or losing b. In this case, where Hugyens' valuation would give the game value of $\frac{a-b}{2}$, Euler's valuations would have us compare $\sqrt{(A+a)(A-b)}$ with A. Euler says that he would be indifferent about playing this game if $\sqrt{(A+a)(A-b)} = A$, which happens if $A = \frac{ab}{a-b}$. On the other hand, if $a = b$, Euler notes that $\sqrt{(A+a)(A-a)} = \sqrt{A^2 - a^2}$ and that is always less than A, "unless my resources be infinite." For finite values of A, he notes that for such a game he expects to lose

$$A - \sqrt{A^2 - a^2} = \frac{1}{2} \cdot \frac{a^2}{A} + \frac{1}{8} \cdot \frac{a^4}{A^3} + \frac{1}{16} \cdot \frac{a^6}{A^5} + \frac{5}{128} \cdot \frac{a^8}{A^6} + \text{etc.}$$

Then, suddenly, without explaining this series or "solving" the St. Petersburg paradox, the paper ends. It looks like Euler didn't publish this paper in his lifetime because he didn't finish it.

Now we must ask *why* Euler abandoned this paper. With the evidence available today, we cannot know for sure, so let us speculate.

Though this series expansion seems to be a bit of a dead-end, it is an interesting exercise (left to the reader) to apply Euler's valuation (not considering the current status parameter A) to the St. Petersburg paradox. It gives a nice, reasonable, finite valuation, so it seems

[6] The introduction of business deals as being subject to the same kind of analysis as games of chance is an exciting innovation.

unlikely that Euler quit because he couldn't think of anything else to write on the subject. Apparently he set it aside for some other reason.

Let's try to guess when Euler wrote E811. Most of his papers on probability date from his Berlin years, especially in the early 1750s, when Frederick the Great asked him to do some work on the national lottery and on life insurance. He wrote all of those papers in French, though, the language of the Berlin Academy, and this paper is in Latin. To me, this makes it seem more likely that Euler wrote it in St. Petersburg, either as a young man between 1728 and 1741, or in his second St. Petersburg years, 1766 to 1783.

But if he had written E811 sometime after 1766, it seems that Euler would have mentioned some of the work of d'Alembert [d'A] or Daniel Bernoulli[7] [B] or the correspondence between Nicolaus Bernoulli and Pierre Rémond de Montmort, [M], all mentioned in Todhunter [To].[8]

A clue! Daniel Bernoulli! Didn't I read a footnote somewhere that said Daniel Bernoulli and Leonhard Euler were friends, and that they lived and worked together in St. Petersburg in the 1730s? So I looked for Bernoulli's article and was delighted to find that it had been translated into English, published in *Econometrica* in 1954 and was available on JSTOR. The article, *Specimen theoriae novae de mensura sortis*, (Exposition of a new theory on the measurement of risk) is brilliant and complete. Many of his ideas are equivalent to Euler's, but expressed quite differently. For example, rather than hypothesizing Euler's geometric mean property valuing games, Bernoulli proposes a principle:

> . . . in the absence of the unusual, the utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed.

From this, he shows that the utility function is a logarithmic curve and then derives Euler's geometric mean property. He derives the same formula for the minimum wealth an individual must have before being willing to undertake a particular risky venture, $A = \frac{ab}{a-b}$, though Bernoulli uses different symbols. Bernoulli, though, in a style that would later be typically Eulerian, does some well-chosen examples involving characters he calls Peter and Paul in one example, then Caius and Sempronius, respectively, in a second and a third example. He uses his utility function to "solve" the St. Petersburg paradox (readers who want to check their work can refer to §19 of Bernoulli's article), both for an initial fortune of zero and for an arbitrary status that Bernoulli denotes by α.

As an addendum to the paper, Daniel tells us that he sent a copy of the paper to his cousin, Nicolaus Bernoulli, who admired the results and directed Daniel to related work done by Cramer[9] in 1728. I've not seen this work of Cramer. Both Daniel and Nicolaus admired it, but it seems that Todhunter did not.

Now, using what we know, let's guess when and why Euler didn't finish E811. Perhaps this is what happened.

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[7] This is Daniel I (1700–1782), another son of Johann, not Daniel's nephew Daniel II (1751–1834). Daniel I was another childhood friend of Euler in Basel and senior colleague in St. Petersburg. When Euler first arrived in St. Petersburg, he rented a room in Daniel's house.

[8] Todhunter, writing in 1865, doesn't mention Euler's E811 at all, probably because it had been published only three years earlier, in 1862. Todhunter does describe several of Euler's other contributions to the subject.

[9] Swiss mathematician Gabriel Cramer (1704–1752) of "Cramer's Law".

Sometime around 1730 or 1731, Daniel Bernoulli learned of the St. Petersburg paradox, perhaps by reading his cousin's correspondence in Montmort's book, or perhaps from Nicolaus himself. He shared the problem with his friend and colleague Leonhard Euler, and they both went to work on the problem. Bernoulli had the advantages of a head start and more access to the thoughts of Montmort and of his cousin Nicolaus. Moreover, Daniel was seven years older and at the time he was probably better than the 23-year old Euler was at these kinds of things. For whatever reason, Bernoulli finished his paper first, and it was better than Euler's. Euler recognized this and set his aside.

The Einstein quote at the beginning of this column was chosen to warn the reader that this story would not proceed in a straight line, from beginning to end. Instead, there are false steps, incorrect assumptions, and things happening out of order. Let us review some of the highlights:

- Euler probably wrote E811 in 1730 or 1731.
- He abandoned it because Daniel Bernoulli's article on the same subject was better.
- The "paradox" in the St. Petersburg problem is not its infinite expected value, but that no reasonable person would pay a large sum to be allowed to play the game.
- David Griffiths, the Wollongong Statistician, is not David Griffeath, probabilist at the University of Wisconsin and chef at the Primordial Soup Kitchen.
- Euler did not solve the St. Petersburg paradox. Euler was not the first person to do everything.

And finally,

- If you know when you start how it's going to end, it doesn't make a very good story, does it?

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10

Life and Death – Part 1

(July 2008)

The history of mortality tables and life insurance is sprinkled with the names of people more famous for other things. American composer Charles Ives, who, like your columnist, worked in Danbury, Connecticut, also invented the insurance agency, so that insurance customers themselves no longer had to negotiate directly with the insurance companies. Edmund Halley, of comet fame, devoted a good deal of energy to calculating one of the earlier mortality tables. [Halley 1693] Henry Briggs, better known for his pioneering work with logarithms, calculated interest tables. Swiss religious reformer John Calvin preached that life insurance was not necessarily immoral usury, as some maintained at the time. Daniel Defoe, author of *Gulliver's Travels*, proposed a national insurance scheme for England in 1697. We find other familiar names, DeMoivre, Fermat, Harriot, Hudde, Huygens, de Witt, van Shooten, Maclaurin, Maseres, Pepys and, of course, Euler.

Euler wrote half a dozen articles that related to mortality and life insurance, and several other articles about lotteries and card games that used many of the same principles of probability. His first excursion into this particular subject was in 1760, when he had been working in Berlin for almost 20 years. Euler was probably inspired to write *Recherches générales sur la mortalité et la multiplication du genre humain*, (General research on mortality and population growth of human kind) [E334] by his colleague at the Berlin Academy, Johann Peter Süssmilch, who published a book on population and mortality tables in 1761.

Süssmilch's interest was in the spirit of the times. Many European cities, including Berlin, had recently taken censuses. In 1747, Berlin took two censuses, the first counting 107,224 residents, and the second differing by less than 200. [Lewin 2003] In 1755, they counted 126,661. In 1748, Euler put some examples in his *Introductio in analysin infinitorum* [E101] to show that observed populations in the hundred thousands and millions were not inconsistent with a world-wide human population of just two at the time of Biblical Creation, assuming only modest growth rates. Many of the names mentioned above as contributing to the history of such tables were contemporaries of Euler, and it is not surprising that he joined in.

Euler was clearly also aware of a mortality table published by Willem Kersseboom (1691–1771) in 1742. Kersseboom's table, reproduced along with the *Opera omnia* edition of E334, gives the number of survivors from among an initial population of 1400 newborns after x years, for x going up to 95 years old. The first few lines of Kersseboom's table are reproduced below.

Age x	Number of survivors l_x	Age x	Number of survivors l_x	Age x	Number of survivors l_x	Age x	Number of survivors l_x
0	1400						
1	1125	11	886	21	808	31	699
2	1075	12	878	22	800	32	687
3	1030	13	870	23	792	33	675
4	993	14	863	24	783	34	665
5	964	15	856	25	772	35	655

Euler uses this same idea, measuring the proportion of survivors among a hypothetical population of N infants born at the same time. He denotes by (1) the proportion of survivors after one year, (2) the proportion after two years, (3) after three years, etc., and he estimates that (125) must be less than one in 100 million. Initially, for the sake of generality Euler assumes only that (1), (2), (3), . . . is a decreasing sequence of fractions between zero and one. Later in the paper Euler assigns particular observed values to these proportions, a few of which are:

$$\begin{array}{llll} (1) = 0.804 & (11) = 0.633 & (21) = 0.577 & (31) = 0.499 \\ (2) = 0.768 & (12) = 0.627 & (22) = 0.571 & (32) = 0.490 \\ (3) = 0.736 & (13) = 0.621 & (23) = 0.565 & (33) = 0.482 \\ (4) = 0.709 & (14) = 0.616 & (24) = 0.559 & (34) = 0.475 \\ (5) = 0.688 & (15) = 0.611 & (25) = 0.552 & (35) = 0.468 \end{array}$$

Except for some round-off errors, Euler's proportions correspond exactly to Kersseboom's data.

Having established his notation, Euler explains how to use it. He starts with a simple example, to calculate how many of these people will die each year. The gives us a small table:

$$\begin{array}{llllll} \text{From} & 0 & \text{years to} & 1 & \text{years there die} & N - (1)N. \\ ” & 1 & ” & 2 & ” & (1)N - (2)N. \\ ” & 2 & ” & 3 & ” & (2)N - (3)N. \\ ” & 3 & ” & 4 & ” & (4)N - (3)N. \\ ” & 4 & ” & 4 & ” & (5)N - (4)N. \end{array}$$

Euler probably gave us this example just to make sure we understand his notation, for he does not count it among the six questions he now sets out to answer:

Question 1. *A certain number of men being given, all of whom are the same age, to find how many will probably still be alive a certain number of years later.*

Euler's analysis is clear and direct. He takes the number of men to be *M* and their age to be *m*. Then the initial number of men is determined by

$$M = (m)N,$$

so

$$N = \frac{M}{(m)}.$$

Thus, *n* years later, these men will be aged $m + n$, so the number of them still alive will be $(m + n)N$, or, in terms of *M*,

$$\frac{(m+n)}{(m)}M.$$

Perhaps to prepare us for questions about mortality and about probability that he will be asking later, Euler adds as a remark that the number of these men that we expect will die over the next *n* years will be

$$\left(1 - \frac{(m+n)}{(m)}\right)M.$$

Question 2. *To find the probability that a man of a certain age will still be alive a certain number of years later.*

Let *m* be the age of the man. From what we learned in answering Question 1, we know that the proportion of men of that age who will be alive *n* years later is

$$\frac{(m+n)}{(m)},$$

so that is the probability that the man will still be alive in *n* years.

Continuing his practice of giving us more of an answer than the question asked for, and because Euler grew up among Bernoullis and he knows the basics of probability very well, Euler adds that the probability that the man will die during the next *n* years is

$$1 - \frac{(m+n)}{(m)}.$$

He also points out that these probabilities become equal when $(m + n) = \frac{1}{2}(m)$. He describes the value of *n* that makes this true as the number of years for which the man's "hope of survival" equals his "dread of death."

Question 3. *We ask the probability that a man of a certain age will die in the course of a given year.*

Using the techniques Euler has just taught us, it is easy to calculate that the probability that a man of age m will die in n years, that is to day that he will die at the age of $m + n$, is just

$$\frac{(n) - (n+1)}{(m)}.$$

Again, Euler answers more than we asked, and tells us that the probability that the man of age m will die between n years and $n + v$ years from now is

$$\frac{(n) - (n+v)}{(m)}$$

and the probability that he will die on a particular date n years from now is

$$\frac{(n) - (n+1)}{365(m)}.$$

Question 4. *To find the term in which a man of a given age can hope to survive, so that it is equally probable that he will die before this term as after.*

This question is curious, for Euler has already answered it in his remarks on Question 2. Here, though, he lets z be the age to which the man can hope to survive, that is the quantity he denoted by $m + n$ in Question 2. The solution is still the value of z that makes $(z) = \frac{1}{2}(m)$. Euler says that we call the interval $z - m$ the *power of life* (*la force de la vie*) of a man aged m years.

Up to this point, all of Euler's questions have been more curiosity than practicality. Now he turns to questions of money and the practical problem of assigning a value to what we call *life annuities*, or in French, *rentes viageres*. A life annuity is a contract to pay a fixed amount of money to a person every year until that person dies. Life annuities tend to be worth more for a younger person (though not always) because the younger person is likely to live longer. In Euler's terms, the younger person usually has a higher power of life. Also, the time value of money (usually) makes money worth more today than the same amount of money will be worth at some future date. Euler has both of these complications in mind when he asks

Question 5. *To determine the life annuity that it is fair to pay each year until his death to a man of any age, in exchange for an amount that he pays in advance.*

Euler tells us the age of the man, m, and the amount he pays in advance, a, and asks us to suppose that there are M such men and that x is the amount that the life annuity will pay each year. He also makes the unstated assumption that there will be no payments during the first year, so after one year the number of men surviving will be $\frac{(m+1)}{(m)}M$, and the total amount of all the life annuities that must be paid in one year will be

$$\frac{(m+1)}{(m)}Mx.$$

Similarly, the total amount to be paid after two years will be

$$\frac{(m+2)}{(m)}Mx,$$

and after three years it will be

$$\frac{(m+1)}{(m)}Mx, \quad \text{etc.}$$

Next, Euler introduces the time value of money. He explains that a sum money S payable in n years at 5 percent interest has a present value of only $\left(\frac{20}{21}\right)^n S$. In the interests of generality, he takes λ to be the annual growth rate, 1.05 in the case of 5% interest, and he notes that the value $\frac{20}{21}$ corresponds to $\frac{1}{\lambda}$.

It is time for another table.

	they ought to pay	which is presently worth
after 1 year	$\frac{(m+1)}{(m)}Mx,$	$\frac{(m+1)}{(m)} \cdot \frac{Mx}{\lambda},$
after 2 years	$\frac{(m+2)}{(m)}Mx,$	$\frac{(m+2)}{(m)} \cdot \frac{Mx}{\lambda^2},$
after 3 years	$\frac{(m+3)}{(m)}Mx,$	$\frac{(m+3)}{(m)} \cdot \frac{Mx}{\lambda^3}$
	etc.	etc.

Fairness requires that the value of what is paid out, that is the sum of the entries in the last column of Euler's table, be equal to the amount paid in. This, and a tiny bit of algebra, makes

$$a = \frac{x}{(m)}\left(\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.}\right),$$

so that

$$x = \frac{(m)a}{\frac{(m+1)}{\lambda} + \frac{(m+2)}{\lambda^2} + \frac{(m+3)}{\lambda^3} + \frac{(m+4)}{\lambda^4} + \text{etc.}}.$$

This answers Question 5, though the calculations involved in actually evaluating the formula for given values of m and λ are quite tedious. In his next paper on the subject [E335] Euler gives some techniques to shorten the calculations, and based on the Kersseboom tables and a 5% interest rate, gives tables of the fair prices for life annuities of 100 *écus* for all ages.

Next, Euler turns to his last and most practical question, one that applies to annuities as they were actually sold in Euler's time:

Question 6. *When the annuities are for newborn babies and the payments do not begin until they have attained a certain age, to determine the amount of these payments.*

This financial instrument roughly corresponds to a modern trust fund, established at the birth of a child but not making payments until that child reaches some age or status, say age 21 or enrolling in college.

The same techniques that answered Question 5 serve us well here, and Euler omits most of the work, jumping straight to the conclusion that

$$a = x\left(\frac{(n)}{\lambda^n} + \frac{(n+1)}{\lambda^{n+1}} + \frac{(n+2)}{\lambda^{n+2}} + \frac{(n+3)}{\lambda^{n+3}} + \text{etc.}\right),$$

where n denotes the year when the first payment is to be made. Solving for x we get

$$x = \frac{a}{\frac{(n)}{\lambda^n} + \frac{(n+1)}{\lambda^{n+1}} + \frac{(n+2)}{\lambda^{n+2}} + \frac{(n+3)}{\lambda^{n+3}} + \text{etc.}},$$

Some readers will have noticed that Euler refers to the lives of *men* throughout. This is not entirely 18th century sexism. Euler knew from reading the works of Kersseboom and Struyck [Kersseboom 1748, Struyck 1740] that men and women had different patterns of mortality, so Euler's results really did apply to *men* and not necessarily to women. They also were aware that mortality was different in the cities than in the countryside and that it varied among countries and climates. Euler doesn't mention it, but other authors also knew that their data was distorted to some degree by migration from the country to the city in a way that made city-dwellers seem longer-lived than their country cousins.

This brings us to the end of the first half of E334, the part of the paper about mortality. It provides a natural stopping point. We plan to return to the second half of this paper next month, when we will see what Euler has to say about "multiplication of human kind," that is to say population growth.

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11

Life and Death – Part 2

(August 2008)

Last month we began this two part series on Euler's work in actuarial science with an account of his study of mortality, the "death" part of "Life and Death." This month we turn to the other half of the equation and ask the mathematical question, "Where do those babies come from?"

To seek his answers, Euler begins with a number of assumptions. Some of them are just to simplify the beginnings of his analysis and will be replaced with more sophisticated assumptions later. Others are simple because he sees no way to gather the data to support more complex ones. Still others are just naïve.

For notation, Euler takes M to be the current population and, taking both births and deaths into account, he takes mM to be the population one year later. He patiently explains that if births and deaths are equal in number, then $m = 1$ and the population will remain the same, that if births exceed deaths then $m > 1$ and the population will increase, etc. This level of detail is unusual, even for Euler. Clearly he expects that some of the people who read this paper do not know higher mathematics.

Euler presents his first assumption on the birthrate, what he call the "multiplication," as follows:

> "Now, having fixed the principle of propagation, which depends on marriages and fertility, it is evident that the number of infants which are born in the course of a year ought to have a certain ratio to the number of living men."

Though Euler himself was the father of 13 children, he was also a man of the 18th century, and like his contemporaries, would have thought it unseemly to mention any role *women* might play in the propagation of the species, other than the oblique reference contained in the word "fertility" (*fécondité*). He also primly and properly assumes that all children are born inside of wedlock, an assumption as untrue then as it is now, at the same time being irrelevant to the mathematics of his model. Perhaps he was just being hopeful, as he had two teenage daughters when he wrote this paper.

Having made this assumption, Euler sets out to avoid using it. He says that he could just take the number of births, N, to be some constant multiple of M, say αM, where α is the measure of fertility, "[b]ut it is difficult to draw from this the consequences about birth rates and other phenomena that depend on it." He doesn't give details.

Instead, paralleling his notation on the population itself, he takes N to be the number of births at present and nN to be number of births in one year. If we take $n = \alpha m$, then this would be consistent with Euler's assumption on birth rates. Instead of calculating n, though, Euler hopes to observe it. He notes that the number of births each year form a geometric progression, increasing or decreasing depending on whether $n > 1$ or $n < 1$.

Now, Euler combines this result with his results from the first half of E334, those that we presented in last month's column. We remind the reader that (m) represents the proportion of a population of infants still alive after m years. This said, he gives us a table:

	Number of births	After 100 years there are still living
at present	N	$(100)N$
after 1 year	nN	$(99)nN$
after 2 years	n^2N	$(98)n^2N$
after 3 years	n^3N	$(97)n^3N$
⋮	⋮	⋮
after 98 years	$n^{98}N$	$(2)n^{98}N$
after 99 years	$n^{99}N$	$(1)n^{99}N$
after 100 years	$n^{100}N$	$n^{100}N$

The numbers in this last column form what we would now call the "age distribution" of the population, a concept that plays a key role in modern demographics and population dynamics. The sum of the numbers in the last column gives the population after 100 years, namely

$$n^{100}N\left(1+\frac{(1)}{n}+\frac{(2)}{n^2}+\frac{(3)}{n^3}+\frac{(4)}{n^4}+\frac{(5)}{n^5}+\text{etc.}\right). \tag{1}$$

Because people are mortal, this is a finite series.

For this and the rest of his analysis to be accurate, it is important that these mortality figures, (1), (2), (3), etc., as well as the value of n be stable, and that they have been stable long enough that the age distribution becomes stable as well. Euler will make this disclaimer at the end of the article.

Now Euler takes the current population to be M, the births per year to be N and claims that

$$\frac{M}{N}=1+\frac{(1)}{n}+\frac{(2)}{n^2}+\frac{(3)}{n^3}+\frac{(4)}{n^4}+\frac{(5)}{n^5}+\text{etc.} \tag{2}$$

He gives only sketchy reasons, but we can fill in some of his steps. If we take M_{100} to be the population in 100 years, then M_{100} is given by formula (1). From the previous table, the number of births in 100 years will be $n^{100}N$. Since the ratio between population and the number of births is taken to be constant, we get that

$$\frac{M}{N} = \frac{M_{100}}{N_{100}} = 1 + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.},$$

as claimed.

For Euler, this is an important result because it allows him to calculate n in terms of M and N. The value of n was difficult to observe directly and was very sensitive to small counting errors, so it was more accurate to calculate it indirectly.

With these theoretical tools in hand, Euler raises, then answers some questions, much like he did in the first half of the paper.

Question 1. *Given the hypotheses of mortality and fertility, and if we know the population, to find how many people there are of each age.*

In modern terms, Euler seeks the age distribution we mentioned above. If we multiply both sides of formula (2) by N, the resulting formula gives the total population M as a sum of the population at age, namely N infants, $\frac{(1)}{n}N$ people of age 1, $\frac{(2)}{n^2}$ people of age 2, ..., and in general $\frac{(a)}{n^a}N$ people of age a.

Question 2. *Given the same things, to find the number of men who die in a year.*

Take M, N and n as before, and note that $\frac{M}{N} = \frac{1}{n}$. Then in a year the population will become nM, so the change in population will be $nM - M$. The number of births will be nN, and the rest of the population change will be accounted for by deaths, so the number of deaths must be $(1 - n)M + nN$.

Question 3. *Knowing how many births and funerals happen during the course of a year, to find the total population and its annual growth, under a given hypothesis of mortality.*

This was a particularly interesting question in Euler's time, to estimate the total population from the numbers of births and deaths, both of which were thought to be easily and accurately available. Note that the "given hypothesis of mortality" of which Euler speaks means the values of those mortality fractions, (1), (2), (3), etc.

Let N be the number of births, as always, and O be the number of funerals. Question 3 asks us to find M given O, N and the "given hypothesis of mortality." From Question 2 we have

$$O = (1 - n)M + nN. \tag{3}$$

so that

$$M = \frac{O - nN}{1 - n}.$$

From this, it is clear that we will have to use the "given hypothesis of mortality" either to find n or to eliminate it from the equation. Towards this end, a bit of algebra gives

$$\frac{M}{N} - 1 = \frac{O - N}{N(1 - n)}.$$

Recall also formula (2):

$$\frac{M}{N} = 1 + \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \frac{(4)}{n^4} + \frac{(5)}{n^5} + \text{etc.} \tag{2}$$

Substituting this into the preceding formula gives

$$\frac{O - N}{N(1 - n)} = \frac{(1)}{n} + \frac{(2)}{n^2} + \frac{(3)}{n^3} + \text{etc.} \tag{4}$$

Now Euler unnecessarily divides the problem into three cases, a stable population, an increasing one, and a decreasing one.

In the first case, the number of births equals the number of deaths, so formula (3) implies that $n = 1$. (Mathematically, we have to admit that perhaps $M = N$, but that cannot happen in reality.) In the case $n = 1$, formula (2) gives

$$M = N\,(1 + (1) + (2) + (3) + (4) + \text{etc.})\,.$$

In the second case, if N, the number of births is greater than O, the number of deaths, then $N - O$ is positive and the population is increasing and that $n > 1$. Likewise, if the number of deaths exceeds the number of births, then the population decreases and $n < 1$.

It seems that Euler has done an incomplete, or at best an evasive job of answering Question 3, for after reading his answer, we still don't know how to find the population M given the numbers of births and deaths, N and O and a "given hypothesis of mortality." In order to provide a number for an answer, we must find a value for n. We're not allowed to use the fact that $n = \frac{N}{M}$ because we don't know M; that's what the question asks us to find. Instead, we have to solve formula (4) for n, but that is a polynomial of degree 100, and it is likely to be difficult to solve. Euler only tells us whether n is greater than or less than one, and not how to find an actual value. We will come back to this in Question 5.

As if he has answered Question 3, Euler asks:

Question 4. *Given the numbers of births and deaths in a year, to find how many of each age there will be among the dead.*

Euler takes M, N, O and n as before, and assumes that we are given N and n. Then he solves his problem with a series of tables. His first table uses the birth rate, n, this year's number of births, N, and the "given hypothesis of mortality" to calculate the age distributions for this year and next year.

Number	this year	next year
of newborns	N	nN
of age one year	$\frac{(1)}{n}N$	$(1)N$
of age two years	$\frac{(2)}{n^2}N$	$\frac{(2)}{n}N$
of age three years	$\frac{(3)}{n^3}N$	$\frac{(3)}{n^2}N$
etc.		etc.

From this we can calculate the number of each age who die each years:

	number of deaths
less than one year	$(1-(1))\,N$
1 to 2 years	$((1)-(2))\,\dfrac{N}{n}$
2 to 3 years	$((2)-(3))\,\dfrac{N}{n^2}$
3 to 4 years	$((3)-(4))\,\dfrac{N}{n^3}$
4 to 5 years	$((4)-(5))\,\dfrac{N}{n^4}$
etc.	etc.

This table answers Question 4, but Euler wants to take it just a little farther. In this table, the sum of the entries in the second column must be the total number of deaths, O. Making that sum and dividing by the common factor N gives

$$\frac{O}{N} = 1 - (1)\left(1 - \frac{1}{n}\right) - \frac{(2)}{n}\left(1 - \frac{1}{n}\right) - \frac{3}{n^2}\left(1 - \frac{1}{n}\right) - \text{etc.}$$

This, he notes, agrees with formulas (2) and (3) above.

Question 5. *Knowing the number of living people as well as the number of births and the number of deaths of each age over the course of a year, to find the law of mortality.*

Solving formula (3) for n gives

$$n = \frac{M-O}{M-N}.$$

This gives a much easier way to find n than the method suggested in Euler's Question 3, though not necessarily cheaper. This way we have to find M. This means spending the time and effort to take a census of all the people living in the city or state. Note though that if M is enough larger than O and N or if O and N are relatively close to each other, small errors in their measurement don't have much effect on n, so perhaps the census doesn't have to be extremely accurate.

Continuing with the problem at hand, let α, β, γ, δ, etc. be the number of deaths at ages less than one year, one to two years, two to three years, etc. This gives values to the entries in the second table from Question 4. Then, from the first line of that table, we get

$$\alpha = (1-(1))\,N,$$

so that

$$(1) = 1 - \frac{\alpha}{N}.$$

Likewise

$$\beta = ((1) - (2))\,\frac{N}{n},$$

so

$$(2) = (1) - \frac{n\beta}{N} = 1 - \frac{\alpha + n\beta}{N}.$$

Continuing in this way,

$$(3) = 1 - \frac{\alpha + n\beta + n^2\gamma}{n},$$

$$(4) = 1 - \frac{\alpha + n\beta + n^2\gamma + n^3\delta}{n},$$

and the pattern is evident.

This is a remarkable result, in my mind the best in the whole article. Euler recognizes clearly, and says as much in his closing paragraphs, that mortality and fertility vary a great deal from one area to another, and that it would be impractical to gather the information that went into Kersseboom's tables for very many locations. However, they routinely kept track of births and deaths, or at least of baptisms and funerals. Euler's Question 5 shows that the information in Kersseboom's tables, though difficult to gather directly, can be recovered from other information that is much easier to collect.

We should note that some of these results can be found using modern tools. We can combine the "law of mortality" with age-specific fertility rates and building a transition matrix to describe how many babies are born to people of various ages and what proportion of people of each age survive to be the next age. Then quantities like Euler's survival rates, (1), (2), etc., the values of n and α and the age distribution can be expressed in terms of eigenvalues, eigenvectors and such. The subject provides an early example in many mathematical modeling courses and a late one for linear algebra courses.

A wise man once said, "You can't always get what you want." You may want mortality tables like Kersseboom's, they are difficult and expensive to prepare. Euler shows us that, with careful analysis and good mathematical modeling, that the wise man was correct when he added, "but if you try sometime, you just might find you can get what you need.

I would like to acknowledge Richard Pulskamp and his translations of many of Euler's work related to probability and statistics. They have been very helpful in these and other columns. They are available on his website and through links from EulerArchive.org.

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Part IV

Analysis

12

e, π and i: Why is "Euler" in the Euler Identity?

(August 2007)

One of the most famous formulas in mathematics, indeed in all of science is commonly written in two different ways:

$$e^{\pi i} = -1 \qquad \text{or} \qquad e^{\pi i} + 1 = 0.$$

Moreover, it is variously known as the Euler identity (the name we will use in this column), the Euler formula or the Euler equation. Whatever its name or form, it consistently appears at or near the top of lists of people's "favorite" results. It finished first in a 1988 survey by David Wells for *Mathematical Intelligencer* of "most beautiful theorems." It finished second in a 2004 survey by the editors of *Physics World* to select the "greatest equations" and it was third in a 2007 survey of participants in an MAA Short Course of "Euler's greatest theorems."[1]

Whether people call it a formula, an equation or an identity, and regardless of which form they use, almost everyone credits the result to Euler. But it is not entirely clear *why* people give him credit for this result, because he never wrote it down in anything remotely like this form, because he wasn't the first one to know the fact behind the formula, and because he himself credited that fact to his mentor, Johann Bernoulli. In this column we will look at the origins of the Euler identity, see what Euler contributed, and consider whether it is correctly named.

Phase 1: 1702 to 1729

There are two formulas that are closely related to the Euler identity. The first we will call the "Euler formula":[2]

$$e^{i\theta} = \cos\theta + i \sin\theta$$

[1] See "Euler's Greatest Hits", pp. 1–5 of *How Euler Did It*, a collection of 40 of these columns published by the MAA.

[2] Throughout this column, we will use i to denote $\sqrt{-1}$, even though Euler did not introduce the more convenient i notation until the 1770's, long after the events in this story.

The Euler identity is an easy consequence of the Euler formula, taking $\theta = \pi$. The second closely related formula is DeMoivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin\theta.$$

This, too, is an easy consequence of the Euler formula, since

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

The relation between DeMoivre's formula and the Euler identity will turn out to be deeper than this.

The English mathematician Roger Cotes (1862–1716) was studying problems in the arc length of spirals. In about 1712, in the course of his investigations, he seems to be the first one to discover a formula equivalent to the Euler formula:

$$\ln(\cos q + i\sin q) = iq.$$

This is easily transformed into the Euler formula by exponentiating both sides, but apparently Cotes never did this. Moreover, Cotes died rather suddenly in 1716 without publishing much of his work on this subject.

I've not found much about Abraham DeMoivre's (1667–1754) discovery of his formula, what he was thinking when he found it, or how rigorous his derivations were. Mactutor [McT] tells us:

> It appears in this form in a paper which de Moivre published in 1722, but a closely related formula had appeared in an earlier paper which de Moivre published in 1707.

Both DeMoivre and Cotes lived in England, though, and in those years of the Newton-Leibniz dispute, Continental mathematicians sometimes made a special effort to ignore English mathematical results.

Meanwhile, on the other side of the English Channel, Johann Bernoulli (1667–1748) was uncovering some of the first geometric properties of complex numbers. In 1702 he gave a formula [Br] for the area of a sector of a circle of radius a, centered at the origin between the x-axis and the radius to the point (x, y) as

$$\frac{aa}{4\sqrt{-1}}\ln\frac{x + y\sqrt{-1}}{x - y\sqrt{-1}}.$$

Twenty-five years later, in 1727, Bernoulli was studying the equation $y = (-1)^x$ with his young student Leonhard Euler. In the course of their discussions, they had to figure out the nature of logarithms of negative numbers. Bernoulli had argued that $\ln(-1) = 0$, since

$$0 = \ln(1) = \ln(-1 \cdot -1) = 2\ln(-1).$$

The same argument applies to any negative number. They were perplexed because they had equally convincing (and flawed) arguments to "prove" that $\ln(-x) = \ln(x)$.

Euler took $x = 0$ in Bernoulli's 1702 formula to find the area of a quarter circle. He reasoned that $\frac{aa}{4\sqrt{-1}}\ln(-1)$ was finite and nonzero. But if Bernoulli were correct that $\ln(-1) = 0$, the area would be zero. Bernoulli was unconvinced, and the issue faded. More details of this episode are given in [Br].

If Euler had taken this argument just one step farther and noticed that the area of a quarter circle is $\frac{\pi a^2}{4}$, he could have solved the equation $\frac{a^2}{4i}\ln(-1) = \frac{\pi a^2}{4}$ and found that $\ln(-1) = \pi i$. From this it follows immediately that $e^{\pi i} = -1$, but Euler did not take this step.

Two years later, Euler was writing [E19] his pioneering work on the gamma function. In one of his examples, he tells us that a particular infinite product turns out to be [S] "$\frac{1}{2}\sqrt{i\ln(-1)}$, which is equal to the side of the square equal to the circle with diameter 1."

Decoding this is a little tricky. On the one hand, "the side of the square equal to the circle with diameter 1" tells us first to find the area of a circle with diameter 1, that is $\frac{\pi}{4}$, then to find a square with the same area, and to find the length of the side of that square. This gives $\frac{\sqrt{\pi}}{2}$. Setting this equal to $\frac{1}{2}\sqrt{i\ln(-1)}$ and applying a bit of algebra, it is easy to conclude that $e^{\pi i} = -1$. Euler doesn't do this, though, nor does he explain his claim that $\frac{1}{2}\sqrt{i\ln(-1)} = \frac{\sqrt{\pi}}{2}$ or why it is equal to the infinite product he had been studying.

By 1729, we have four different people, DeMoivre, Cotes, Bernoulli and Euler (twice), who have found the essential fact behind the Euler identity, but none of them have recognized its importance or written it in anything like the form we recognize today.

Phase 2: The 1740s

Let's jump forward to the 1740s, when Euler was writing his great precalculus textbook, the *Introductio in analysin infinitorum* [E101]. Euler spent most of the 1740s writing this book, then had trouble finding a publisher. Eventually he found a publisher in Switzerland and the book came out in 1748. Today, many people think it is the greatest mathematics book ever written.

Chapter 8 of Euler's *Introductio* is titled "On transcendental quantities which arise from the circle." It is the first time that anyone treats sines, cosines, etc. as functions rather than as ratios, and so it makes an important step towards making functions a fundamental object in mathematics.

Euler spends the first part of the chapter establishing the basic properties of the sine, cosine and tangent functions, very much the way we do them today. Then he begins using complex numbers. He tells us,[3] "Since $(\sin z)^2 + (\cos z)^2 = 1$, we have the factors $(\cos z + i\sin z)(\cos z - i\sin z) = 1$."

He then asks us to

> [c]onsider the following product: $(\cos z + i\sin z)(\cos y + i\sin y)$, which results in
>
> $\cos y\cos z - \sin y\sin z + (\cos y\sin z + \sin y\cos z)i$, which results in ...
>
> $(\cos y + i\sin y)(\cos z + i\sin z) = \cos(y+z) + i\sin(y+z)$

Since multiplication can be regarded as repeated addition, a few lines later he shows that

$$(\cos z \pm i\sin z)^n = \cos nz \pm i\sin nz.$$

This, of course, is DeMoivre's formula. It is not clear whether or not Euler knew of DeMoivre's work, but in the *Introductio* he does not usually cite sources. He also does

[3] We use the Blanton translation, published by Springer in 1988 and 1990.

not seem to consider the possibility that this formula might be true even if n is not an integer.

From DeMoivre's formula he calculates that

$$\cos nz = \frac{(\cos z + i \sin z)^n + (\cos z - i \sin z)^n}{2}, \text{ and}$$

$$\sin nz = \frac{(\cos z + i \sin z)^n - (\cos z - i \sin z)^n}{2i}$$

He boldly takes z to be infinitely small, so that $\sin z = z$ and $\cos z = 1$, and then takes n to be an infinitely large number with $nz = v$, where v is finite, and gets the Taylor series for sine and cosine. Readers who are anxious about the 18th-century use of infinite and infinitesimal numbers may either read the first four chapters of the *Introductio* to become more familiar with the practice, or they can recast Euler's argument into the language of limits. Either way, it is very beautiful mathematics.

A few paragraphs later he uses this version of DeMoivre's formula, taking z infinitely small, j infinitely large and $jz = v$, where again v is finite, to get

$$\cos v = \frac{\left(1 + \frac{iv}{j}\right)^j + \left(1 - \frac{iv}{j}\right)^j}{2} \text{ and}$$

$$\sin v = \frac{\left(1 + \frac{iv}{j}\right)^j - \left(1 - \frac{iv}{j}\right)^j}{2i}.$$

But when j is an infinite number, $e^z = \left(1 + \frac{z}{j}\right)^j$ so these formulas are equivalent to

$$\cos v = \frac{e^{iv} + e^{-iv}}{2} \text{ and } \sin v = \frac{e^{iv} - e^{-iv}}{2i}.$$

Now comes the *coup de grace*. Multiply these equations by 2 and $2i$, respectively, and add them together to get the Euler formula:

$$e^{iv} = \cos v + i \sin v.$$

Euler moves on to apply these results to the practical problems of calculating sines and cosines, without ever considering the special case $v = \pi$ and without explicitly writing down the Euler identity.

The Judgment of History

Early in the 1700s, Cotes, DeMoivre, Johann Bernoulli and Euler himself all had the pieces that could have led them to discover the Euler formula. The problems they were working on did not depend on the Euler formula, though, so none of them had any reason to discover the formula at the time.

In contrast, in the 1740s Euler had good reasons to know the Euler formula, discovering properties of trigonometric functions and finding good ways to approximate them. Moreover, he had a beautiful and convincing demonstration of the Euler formula, satisfying all the standards of rigor of the time and easily translatable into the modern language of limits.

Since Euler's presentation was both complete and well-motivated, it seems like the right thing to do to attach his name to the formula.

The name of the Euler *identity* presents a slightly different problem. Though it is only a special case of the Euler formula, it seems that he never wrote it down. I have made no progress in finding who was the first to do so. The mathematical community seems content and almost unanimous in calling it the Euler identity, and nobody else seems to have a claim that is nearly as good.

And it is one of the most beautiful formulas in all of mathematics.

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13

Multi-zeta Functions

(January 2008)

Two of Euler's best known and most influential discoveries involve what we now call the Riemann zeta function. The first of these discoveries made him famous when he solved the Basel problem. He showed [E41] that the sum of the reciprocals of the square numbers was

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \text{etc.} = \frac{\pi^2}{6}.$$

Euler's second great result [E72] on this topic was what we now call the Euler product formula, and we write it as

$$\sum_{k=1}^{\infty} \frac{1}{k^n} = \prod_{p \text{ prime}} \frac{1}{1 - {}^1/_{p^n}}.$$

For the readers unfamiliar with the zeta function, we'll give a brief introduction.

It has long been known that the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, diverges, but that if we take these terms to some power, as $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$, then the series will converge whenever $n > 1$. The value to which it converges depends on n, and is now denoted $\zeta(n)$.

Euler's first result showed that $\zeta(2) = \frac{\pi^2}{6}$, and his second result showed that $\zeta(n)$ can be written either as an infinite sum or as an infinite product.

Since Euler's time, the zeta function has captured the imaginations of many great mathematicians. In particular, in 1859 Bernhard Riemann, showed that n need not be a real number, and that the zeta function has a natural analytic continuation as a function of a complex variable. Hence the function is traditionally called the *Riemann* zeta function and defined in terms of a complex variable s as

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}.$$

Riemann surmised that his function is zero for infinitely many values of s, and that all its complex roots share the property that their real part is $^1/_2$. For some reason that is still unclear, Riemann's conjecture is known as the "Riemann hypothesis" instead of the "Riemann conjecture." Though it is badly named, it is one of the most important unsolved problems in mathematics today.

Over the years, zeta functions have evolved a number of variations. For example, instead of taking the sum over the ordinary integers, one could take the sum over the integers in some number field. This leads to a topic known as *L*-series. We could also change the numerators in the sum, and look at sums like

$$\sum_{k=1}^{\infty} \frac{\chi(k)}{k^n},$$

where $\chi(k)$ is some function of k. We saw Euler himself do something like this in last month's column, where we describe a series Euler investigated the end of [E190],

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.}$$

This is not exactly an *L*-series, because the denominators form a geometric series, not an arithmetic series, but the nth numerator is given by the number-theoretic function

$$\chi(n) = \text{the number of divisors of } n.$$

This is very much in the spirit of a modern *L*-series.

At a recent section meeting of the MAA, Michael Hoffmann of the US Naval Academy in Annapolis brought to my attention another modern variation of the zeta function, and showed how that variation derived from Euler's work. Most of the remainder of this column is based on what he showed me. [H1992, H2007]

In the modern way, a *multiple zeta value* is defined as

$$\zeta(i_1, i_2, \ldots i_k) = \sum_{n_1 > n_2 \cdots n_k \geq 1} \frac{1}{n_1^{i_1} \cdot n_2^{i_2} \cdots n_k^{i_k}}.$$

Both the motivation and the notation are obscure here. Let's try to untangle both of them at the same time. Let's ask, what would happen if we multiplied together two ordinary zeta functions, say $\zeta(m)$ and $\zeta(n)$? As series, we would get

$$\zeta(m) \cdot \zeta(n) = \left(\sum_{k=1}^{\infty} \frac{1}{k^m}\right) \cdot \left(\sum_{k=1}^{\infty} \frac{1}{k^n}\right)$$

Euler would not have used the Sigma notation, so he might have written this as

$$\left(1 + \frac{1}{2^m} + \frac{1}{3^m} + \frac{1}{4^m} + \text{etc.}\right) \cdot \left(1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \text{etc.}\right)$$

Then he probably would have expanded this to get something like

$$\begin{array}{l} 1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\cdots \\ \frac{1}{2^m}+\frac{1}{2^m}\frac{1}{2^n}+\frac{1}{2^m}\frac{1}{3^n}+\frac{1}{2^m}\frac{1}{4^n}+\frac{1}{2^m}\frac{1}{5^n}+\cdots \\ \frac{1}{3^m}+\frac{1}{3^m}\frac{1}{2^n}+\frac{1}{3^m}\frac{1}{3^n}+\frac{1}{3^m}\frac{1}{4^n}+\frac{1}{3^m}\frac{1}{5^n}+\cdots \\ \frac{1}{4^m}+\frac{1}{4^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{3^n}+\frac{1}{4^m}\frac{1}{4^n}+\frac{1}{4^m}\frac{1}{5^n}+\cdots \\ \cdots \end{array} \tag{1}$$

Now we can take this apart and put it back together a different way. First, note the terms on the "diagonal" of this sum, the first term in the first row, the second in the second row, etc. They sum to form an ordinary zeta function as follows:

$$1+\frac{1}{2^m}\frac{1}{2^n}+\frac{1}{3^m}\frac{1}{3^n}+\frac{1}{4^m}\frac{1}{4^n}+\frac{1}{5^m}\frac{1}{5^n}+\cdots=\zeta\left(m+n\right).$$

On the other hand, the terms below the diagonal sum as

$$\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{3^m}\frac{1}{2^n}+\frac{1}{4^m}+\frac{1}{4^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{3^n}+\frac{1}{5^m}+\text{etc.}$$

This might be a little clearer (or maybe not) if we explicitly include the factors of 1^n in the products. This gives

$$\frac{1}{2^m}\frac{1}{1^n}+\frac{1}{3^m}\frac{1}{1^n}+\frac{1}{3^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{1^n}+\frac{1}{4^m}\frac{1}{2^n}+\frac{1}{4^m}\frac{1}{3^n}+\frac{1}{5^m}\frac{1}{1^n}+\text{etc.}$$

Now we can see clearly that in each denominator, $a^m b^n$ we have $a > b \geq 1$, so we can rewrite the sum in modern Sigma notation as

$$\sum_{a>b\geq 1}\frac{1}{a^m b^n}$$

Glancing back up the page, we see that is exactly what Hoffman defines as the multi-zeta value $\zeta(m, n)$.

Likewise, the terms above the diagonal in our product sum to $\zeta(n, m)$. This gives one of the motivations for multi-zeta values. They arise in multiplying values of the zeta function, and lead to the formula

$$\zeta(m)\zeta(n)=\zeta(m+n)+\zeta(m,n)+\zeta(n,m) \tag{2}$$

A slightly different approach involves defining a different multi-zeta value, using $\geq$ instead of $>$, as

$$\zeta*(m,n)=\sum_{a\geq b\geq 1}\frac{1}{a^m b^n}.$$

This includes the diagonal terms in the big summation, so $\zeta*(m,n)=\zeta(m+n)+\zeta(m,n)$ and it leads to a similar formula about products of zeta values:

$$\zeta(m)\zeta(n)=\zeta*(m,n)+\zeta*(n,m)-\zeta(m+n). \tag{3}$$

All these are modern ideas and modern notations, and they are well documented in the fine bibliography[1] maintained by Michael Hoffman.

I was surprised to learn that these ideas are not of modern origin, but first came from Christian Goldbach in a letter to Euler dated December 24. 1742. [J+W] There, Goldbach uses 18th century notation to find that

$$\zeta*(3,1)=\frac{\pi^2}{72} \text{ and } 2\zeta*(5,1)+\zeta*(4,2)=\frac{19\pi^6}{5670},$$

though he does not claim to know either $\zeta*(5,1)$ or $\zeta*(4,2)$.

Euler responded quickly, though at this time Euler was in Berlin and Goldbach was in Moscow, and the mail was perhaps slower in the middle of the winter. Nonetheless, Euler's letter dated January 19, 1743 contained some additions to Goldbach's results, providing equations for $\zeta*(3,1)$, $\zeta*(5,1)$, $\zeta*(7,1)$, and $\zeta*(9,1)$, in terms of products of the ordinary zeta function. Not all of Euler's claims are correct, though. Hoffman points out that his claim that

$$\zeta*(6,2)=2\zeta(3)\zeta(5)-\frac{3}{2}\zeta(4)^2+\frac{1}{4}\zeta(8)$$

is false, and so were a few others. In fact, this one isn't even very close. According to Maple,™ the left hand side is about 1.6557 and the right hand side is about 0.9868, and Michael Hoffman tells me that nobody has yet found a way to write $\zeta*(6,2)$ as a polynomial function of ordinary zeta values with rational coefficients, and whether or not one exists is an open and active research question.

Euler and Goldbach exchanged a total of five letters on this subject. In the last one, dated February 26, Euler used properties of these multi-zeta functions to give 18-place decimal approximations to $\zeta(n)$ through $n=16$.

As usual, Euler could not leave to a letter what he could expand into a paper, but Euler apparently let nearly 30 years pass before he returned to multiple zeta values. His work became [E477], written in 1771 and published in 1776. There he begins by citing his letters with Goldbach, and describing the series that is now denoted by $\zeta*(m,n)$. He wrote

> In commercio litterario, quod olim com Illustrissimo Goldbachio coluerum, inter alias varii argumenti speculationes circa series in hac forma generalis
>
> $$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}$$

[1] 1 www.usna.edu/User/math/meh/biblio.html

Euler used $\int \frac{1}{z^m}$ to denote what we now write as $\zeta(m)$. He writes P for $\zeta*(m,n)$ and Q for $\zeta*(n,m)$. In Euler's notation, and in the original 1776 publication, he wrote his version of formula 3 like this:

$$1+\frac{1}{2^m}\left(1+\frac{1}{2^n}\right)+\frac{1}{3^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}\right)+\frac{1}{4^m}\left(1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}\right)+\text{etc.}=P$$

$$1+\frac{1}{2^n}\left(1+\frac{1}{2^m}\right)+\frac{1}{3^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}\right)+\frac{1}{4^n}\left(1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}\right)+\text{etc.}=Q$$

ex principio supra stabilito habebimus

$$P+Q=\int\frac{1}{z^m}\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}$$

A few paragraphs later, Euler improves his notation, and denotes $\zeta*(m,n)$ by $\int\frac{1}{z^m}(\frac{1}{y^n})$ instead of by P. Then formula 3 becomes

$$\int\frac{1}{z^m}\left(\frac{1}{y^n}\right)+\int\frac{1}{z^n}\left(\frac{1}{y^n}\right)=\int\frac{1}{z^m}.\int\frac{1}{z^n}+\int\frac{1}{z^{m+n}}$$

Later in the paper, Euler develops some general results about multi-zeta values, especially those that he wrote as $\int\frac{1}{z^n}(\frac{1}{y})$ and we would write as $\zeta*(n,1)$. He shows easily that

$$\int\frac{1}{z^2}\left(\frac{1}{y}\right)=2\int\frac{1}{z^3},\ \text{that is}\ \zeta*(2,1)=2\zeta(3).$$

It takes a bit more work for him to show that

$$\int\frac{1}{z^3}\left(\frac{1}{y}\right)=\frac{3}{2}\int\frac{1}{z^2}.\int\frac{1}{z^2}-\frac{5}{2}\int\frac{1}{z^4},\ \text{that is}\ \zeta*(3,1)=\frac{3}{2}\zeta(2)^2-\frac{5}{2}\zeta(4),$$

and then still more work to find that

$$\int\frac{1}{z^4}\left(\frac{1}{y}\right)=3\int\frac{1}{z^5}-\int\frac{1}{z^2}.\int\frac{1}{z^3},\ \text{that is}\ \zeta*(3,1)=3\zeta(5)-\zeta(2).\zeta(3).$$

Being Euler, he continues for more than 12 pages, stopping with

$$\int\frac{1}{z^9}\left(\frac{1}{y}\right)=3\int\frac{1}{z^2}\cdot\int\frac{1}{z^8}-\int\frac{1}{z^3}\cdot\int\frac{1}{z^7}$$

$$+3\int\frac{1}{z^4}\cdot\int\frac{1}{z^6}-\frac{1}{2}\int\frac{1}{z^5}\cdot\int\frac{1}{z^5}-\frac{11}{2}\int\frac{1}{z^{10}}.$$

We won't translate this into modern notation. In this work, the patterns are not evident, but he applies formula 3 several times and transforms the results into

$$2\int \frac{1}{z^2}\left(\frac{1}{y}\right) = 4\int \frac{1}{z^3}$$
$$2\int \frac{1}{z^3}\left(\frac{1}{y}\right) = 5\int \frac{1}{z^4} - \int \frac{1}{z^2}\cdot \int \frac{1}{z^2}$$
$$2\int \frac{1}{z^4}\left(\frac{1}{y}\right) = 6\int \frac{1}{z^6} - 2\int \frac{1}{z^2}\cdot \int \frac{1}{z^3}$$

and finally

$$2\int \frac{1}{z^9}\left(\frac{1}{y}\right) = 11\int \frac{1}{z^{10}} - 2\int \frac{1}{z^2}\cdot \int \frac{1}{z^8} - 2\int \frac{1}{z^3}\cdot \int \frac{1}{z^7}$$
$$-2\int \frac{1}{z^4}\cdot \int \frac{1}{z^6} - \int \frac{1}{z^5}\cdot \int \frac{1}{z^5}.$$

From this, the pattern is evident. In modern notation, it reads

$$2\zeta * (n, 1) = (n+2)\,\zeta(n+1) - \sum_{i=1}^{n-2} \zeta(n-i)\,\zeta(i+1)$$

This is a form of what is now known as Euler's decomposition formula for the double zeta function, and more than two centuries later it is still an interesting result. Special thanks to Michael E. Hoffman for inspiring and helping with this column.

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14

Sums of Powers

(June 2009)

Many people have been interested in summing powers of integers. Most readers of this column know, for example, that

$$\sum_{i=1}^{k} i = \frac{k(k+1)}{2},$$

and many of us even have our favorite proofs. Those of us who have studied or taught mathematical induction lately are likely to recall that

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6},$$

and it is obvious that

$$\sum_{i=1}^{k} i^0 = k.$$

Indeed, some days it seems like the main reason mathematical induction exists is so that we can make students prove identities like this. Euler knew all of these identities as well. In fact, he knew them at least up to the eighth exponent:

$$\sum_{i=1}^{k} i^8 = \frac{1}{9}x^9 + \frac{1}{2}x^8 + \frac{2}{3}x^7 - \frac{7}{15}x^5 + \frac{2}{9}x^3 - \frac{1}{30}x.$$

Jakob Bernoulli [Bernoulli 1713] had given a comprehensive account of sums of powers in his posthumous epic, *Ars conjectandi*, a masterpiece laying out the complete foundations of the theory of discrete probability. Euler surely knew this book well, as Bernoulli had left it to his son, Nicolaus, to edit and publish the volume, and the younger Bernoulli had been a good friend of Euler.

Bernoulli's solution to the problem involved what is still rather advanced mathematics and a special sequence of numbers known as Bernoulli numbers. His work was complete and correct, but it was by no means easy or elementary. Thus, when Euler came on the scene, the problem was not *how* to sum the powers of integers, but how to do it simply, in a way that might be easy to follow and easy to remember. Euler considered this problem late in his life in an article he wrote in 1776, *De singulari ratione differentiandi et integrandi, quae in summis serierum occurrit*, "On a singular means of differentiating and integrating which occurs in the summing of series." [E642]

To Euler, calculus was easy, and notation was something he made up to fit the circumstances. He was also a genius at spotting patterns and doing calculations. When he approached this problem in 1776, he was also blind and aided by assistants who wrote down his words as Euler dictated articles to them. Euler wrote Σx^n to denote the sum of the nth powers of the first x natural numbers. That is,

$$\Sigma x^n = 1^n + 2^n + 3^n + 4^n + \cdots + x^n.$$

He found Σx^n for n from 0 to 8 and started looking for patterns.

$$\begin{aligned}
\Sigma x^0 &= x \\
\Sigma x^1 &= \frac{1}{2}x^2 + \frac{1}{2}x \\
\Sigma x^2 &= \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x \\
\Sigma x^3 &= \frac{1}{4}x^4 + \frac{1}{2}x^3 + \frac{1}{4}x^2 + * \\
\Sigma x^4 &= \frac{1}{5}x^5 + \frac{1}{2}x^4 + \frac{1}{3}x^3 + * - \frac{1}{30}x \\
\Sigma x^5 &= \frac{1}{6}x^6 + \frac{1}{2}x^5 + \frac{5}{12}x^4 + * - \frac{1}{12}x^2 + * \\
\Sigma x^6 &= \frac{1}{7}x^7 + \frac{1}{2}x^6 + \frac{1}{2}x^5 + * - \frac{1}{6}x^3 + * + \frac{1}{42}x \\
\Sigma x^7 &= \frac{1}{8}x^8 + \frac{1}{2}x^7 + \frac{7}{12}x^6 + * - \frac{7}{24}x^4 + * + \frac{1}{12}x^2 + * \\
\Sigma x^8 &= \frac{1}{9}x^9 + \frac{1}{2}x^8 + \frac{2}{3}x^7 + * - \frac{7}{15}x^5 + * + \frac{2}{9}x^3 + * - \frac{1}{30}x \\
&\text{etc.}
\end{aligned}$$

Here, to help us spot patterns a bit more easily, Euler uses a "*" to indicate that a term of the polynomial is missing. Descartes had also used a "*" as a placeholder 150 years earlier.

Some patterns are quite obvious. For example, the first term of Σx^n is always $\frac{1}{n+1}x^{n+1}$. That looks suspiciously like an integral. The second term is always $\frac{1}{2}x^n$. Also, once a "*" or a negative term is introduced, it stays in that position in all subsequent formulas. It is not so clear from the given formulas, but the terms continue to alternate between zero and nonzero terms, and the signs of the nonzero terms alternate as well. Readers familiar with the Bernoulli numbers will recall that they share similar properties.

There are two other patterns we might not notice, but once we do notice them, their reason becomes obvious. First, there is never a constant term. This is because whenever $x = 0$ we also have $\Sigma x^n = 0$. Likewise, the sum of the coefficients is always equal to 1 because whenever $x = 1$ we also have $\Sigma x^n = 1$.

That was the easy part. Euler sees some more subtle patterns, which he describes as follows. Start with

$$\Sigma x^0 = x,$$

which, for purposes of exposition, we'll write as

$$\Sigma x^0 = x + 0.$$

Multiply the two terms on the right-hand side by

$$\frac{1}{2}x \quad \text{and} \quad \frac{1}{1}x,$$

respectively, and add a linear term of the form αx, where α is chosen to make the sum of the coefficients equal to 1, in accordance with the pattern we saw above. In this case, $\alpha = \frac{1}{2}$ and we get the formula for Σx^1, namely

$$\Sigma x^1 = \frac{1}{2}x^2 + \frac{1}{2}x.$$

Now, multiply the *three* terms (we didn't write "+0" this time, but we count it anyway) by

$$\frac{2}{3}x, \frac{2}{2}x \quad \text{and} \quad \frac{2}{1}x,$$

respectively, and again add a linear term of the form αx, where α is chosen to make the sum of the coefficients equal to 1. This time, $\alpha = \frac{1}{6}$ and we get

$$\Sigma x^2 = \frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x.$$

In general, Euler has us multiply the terms of Σx^n by

$$\frac{n+1}{n+2}x, \frac{n+1}{n+1}x, \frac{n+1}{n}x, \ldots \frac{n+1}{2}x \quad \text{and} \quad \frac{n+1}{1}x,$$

respectively, and add the appropriate linear term and to get the formula for Σx^{n+1}. Euler notes that the values of α, namely 1, $\frac{1}{2}$, $\frac{1}{6}$, 0, $-\frac{1}{30}$, 0, $\frac{1}{42}$, 0, $-\frac{1}{30}$, 0,etc., are exactly the Bernoulli numbers. He has seen them several times before, particularly in his evaluation of the zeta function, $\sum_{k=1}^{\infty} \frac{1}{k^n}$, for even values of n, and in the Euler-Maclaurin summation formula.

Having observed the pattern, Euler now tries to explain it. He introduces calculus, as suggested by the pattern of the first terms of each sum, and claims that

$$\Sigma\ x^1 = 1 \int \partial x\ \Sigma\ x^0,$$

$$\Sigma\ x^2 = 2 \int \partial x\ \Sigma\ x^1,$$

$$\Sigma\ x^3 = 3 \int \partial x\ \Sigma\ x^2,$$

$$\Sigma\ x^4 = 4 \int \partial x\ \Sigma\ x^3,$$

$$\Sigma\ x^5 = 5 \int \partial x\ \Sigma\ x^4,$$

etc.,

and, in general,

$$\Sigma x^{n+1} = (n+1) \int \partial x \Sigma x^n.$$

We can see what Euler is thinking here. He wants to reverse integration and summation and manipulate symbols something like:

$$\begin{aligned}(n+1) \int \partial x \Sigma x^n &= (n+1) \sum \int x^n dx \\ &= (n+1) \sum \frac{1}{n+1} x^{n+1} \\ &= \Sigma x^{n+1}.\end{aligned}$$

This doesn't quite make sense, even though the editors of the *Nova acta academiae scientiarum Petropolitanae* call it *une demonstration rigoreuse*, "a rigorous proof." Let's see how it works.

We know that $\Sigma x^0 = x$. Then making a partial shift to modern notation,

$$\begin{aligned}1 \int \Sigma x^0 dx &= 1 \int x dx \\ &= \frac{1}{2} x^2 + C,\end{aligned}$$

where C isn't really a constant, but instead is a linear term, $\frac{1}{2}x$. That didn't quite work.

Continuing to $n = 2$, Euler claims

$$\begin{aligned}\Sigma x^2 &= 2 \int \partial x \Sigma x^1 \\ &= 2 \int \left(\frac{1}{2}x^2 + \frac{1}{2}x\right) dx\end{aligned}$$

$$= 2\left(\frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 2}x + C\right)$$
$$= \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2C,$$

where again $2C$ isn't really a constant but has to be $\frac{1}{6}x$. We see that the pattern continues; integration gives most of the next formula, but the appropriate linear term has to be added as well.

In a similar vein, Euler gives us a formula involving differentiation,

$$\Sigma x^n = \frac{\partial \cdot \Sigma x^{n+1}}{(n+1)\partial x}.$$

This almost makes sense in the same, quirky symbolic way that the integral formula did, but it doesn't quite work, either. For the case $n = 1$, for example, this would give

$$\begin{aligned}\frac{1}{2}x^2 + \frac{1}{2}x &= \Sigma x^1\\ &= \frac{\partial \cdot \Sigma x^2}{2\partial x}\\ &= \frac{1}{2}\frac{d}{dx}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right)\\ &= \frac{1}{2}\left(x^2 + x + \frac{1}{6}\right),\end{aligned}$$

which isn't true because of the constant term in the last line. To make Euler's differential formula correct, we have to throw away the constant term after taking the derivative. So, Euler's paper isn't as rigorous as his editors thought it was.

Euler closes the paper with a number of examples, mostly summing the values of polynomials, but there is an application to the Euler-Maclaurin formula that takes a bit of insight.

In the end, this is a pleasant paper, but we're left to wonder why Euler wrote this paper? Most of the pattern is in Bernoulli's work from 60 years earlier, though Euler is clearer. I speculate that this wasn't really a research paper, but Euler's effort to teach the material to his assistants, who were also, in many ways, his students. To write the paper, Euler's words had to go through his students, and it is likely that he had the students work out the examples at the end of the paper. They would have learned from the experience.

I've learned from the experience, too. The next time I can't remember $\sum_{i=1}^{n} i^2$, I'm going to remember that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$. I'll integrate that, discard the constant and double it to get $\frac{1}{3}n^3 + \frac{1}{2}n^2$. Then, I'll add the appropriate linear term to get

$$\sum_{i=1}^{n} i^2 = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n.$$

If I can remember.

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15

A Theorem of Newton

(April 2008)

Early in our algebra careers we learn the basic relationship between the coefficients of a monic quadratic polynomial and the roots of that polynomial. If the roots are α and β and if the polynomial is $x^2 - Ax + B$, then $A = \alpha + \beta$ and $B = \alpha\beta$. Not too long afterwards, we learn that this fact generalizes to higher degree polynomials. As Euler said it, if a polynomial

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \cdots \pm N = 0$$

has roots $\alpha, \beta, \gamma, \delta, \ldots, v$, then

$A =$ sum of all the roots	$= \alpha + \beta + \gamma + \delta + \cdots + v,$
$B =$ sum of products taken two at a time	$= \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma +$ etc.
$C =$ sum of products taken three at a time	$= \alpha\beta\gamma +$ etc.
$D =$ sum of products taken four at a time	$= \alpha\beta\gamma\delta +$ etc.
etc., and	
$N =$ product of all roots	$= \alpha\beta\gamma\delta \ldots v.$

These facts are very well known, and Euler has no interest in proving them.

Then there are so many other things to learn that most of us don't learn a closely related system of equations that tells us about the sum of the *powers* of the roots. Indeed, Euler writes the sum of the powers of the roots of the polynomial using the notation

$$\int \alpha = \alpha + \beta + \gamma + \cdots + v,$$

$$\int \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 + \cdots + v^2,$$

$$\int \alpha^3 = \alpha^3 + \beta^3 + \gamma^3 + \cdots + v^3,$$

etc.

This overworks the symbol α, making it both a particular root and at the same time a representative of all the other roots. He also uses the integral sign, $\int$, in one of its 18th century senses, as a summation sign.

With this notation in place, we can state the closely related system of equations we mentioned above. Euler wrote it as

$$\begin{aligned}
&\int \alpha = A,\\
&\int \alpha^2 = A\int \alpha - 2B,\\
&\int \alpha^3 = A\int \alpha^2 - B\int \alpha + 3C,\\
&\int \alpha^4 = A\int \alpha^3 - B\int \alpha^2 + C\int \alpha - 4D,\\
&\int \alpha^5 = A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E,\\
&\int \alpha^6 = A\int \alpha^5 - B\int \alpha^4 + C\int \alpha^3 - D\int \alpha^2 + E\int \alpha - 6F,\\
&\text{etc.}
\end{aligned}$$

Euler attributes these equations to Newton, apparently referring to his *Arithmetica universalis* of 1707. The editors of Euler's *Opera omnia* Series I volume 6, Ferdinand Rudio, Adolf Krazer and Paul Stäckel, cite evidence that the formulas had been known earlier to Girard in 1629 and to Leibniz "certainly not after 1678."

Let's make sure that we know what Euler means by considering an example. The polynomial $x^4 - 10x^3 + 35x^2 - 50x + 24$ has roots 1, 2, 3, and 4. Indeed, it is easy to check two of the coefficients, $A = 1 + 2 + 3 + 4$ and $D = 1 \cdot 2 \cdot 3 \cdot 4$. The other two are a bit more tedious, but

$$\begin{aligned}
&35 = 1\cdot 2 + 1\cdot 3 + 1\cdot 4 + 2\cdot 3 + 2\cdot 4 + 3\cdot 4, \quad \text{and}\\
&50 = 1\cdot 2\cdot 3 + 1\cdot 2\cdot 4 + 1\cdot 3\cdot 4 + 2\cdot 3\cdot 4.
\end{aligned}$$

Also, using Euler's notation for the sums of the powers,

$$\begin{aligned}
&\int \alpha = 1 + 2 + 3 + 4 = 10,\\
&\int \alpha^2 = 1 + 4 + 9 + 16 = 30,\\
&\int \alpha^3 = 1 + 8 + 27 + 64 = 100, \quad \text{and}\\
&\int \alpha^4 = 1 + 16 + 81 + 256 = 354.
\end{aligned}$$

So, Newton's formulas claim that

$$
\begin{aligned}
\int \alpha &= A = 10, \\
\int \alpha^2 &= A \int \alpha - 2B \\
&= 10.10 - 2.35 \\
&= 30, \\
\int a^3 &= A \int \alpha^2 - B \int \alpha + 3C \\
&= 10 \cdot 30 - 35 \cdot 10 + 3 \cdot 50 \\
&= 100, \quad \text{and} \\
\int \alpha^4 &= A \int \alpha^3 - B \int \alpha^2 + C \int \alpha - 4D \\
&= 10 \cdot 100 - 35 \cdot 30 + 50 \cdot 10 - 4 \cdot 24 \\
&= 354,
\end{aligned}
$$

as promised.

Euler used these formulas to great effect throughout his career, notably in his solution to the Basel problem, [E41] and several times in his *Introductio in analysin infinitorum* [E101]. In 1747, he decided to prove them. The result was [E153], a short article, 11 pages, with a long title, Demonstratio gemina theorematis neutoniani quo traditur relatio inter coefficientes cuiusvis aequationis algebraicae et summas potestatum radicum eiusdem, "Proof of the basis of a theorem of Newton, which derives a relation between the coefficients of any algebraic equation and the sums of the powers of the roots of that equation," which was published in 1750, and which contains two very different proofs of the result.

He notes that the first equation requires no proof at all, and the second one is quite easy. He writes

$$A^2 = \alpha^2 + \beta^2 + \gamma^2 + \delta^2 + \varepsilon^2 + \text{etc.} + 2\alpha\beta + 2\alpha\gamma + 2\alpha\delta + 2\beta\gamma + 2\beta\delta + \text{etc.}$$

Thus we get

$$A^2 = \int \alpha^2 + 2B$$

and so

$$\int a^2 = A^2 - 2B = A \int \alpha - 2B.$$

Euler claims that he could prove the other formulas similarly, one at a time, but that it would be a great deal of work. Moreover, he writes that others have found these formulas to be most useful, but nobody seems to have proved them "except by induction." By this he means that they have been observed to be true in a great many cases, and never been seen to be false. Still, he thinks that it is so important that they be proved that he offers to do it twice.

Euler's first proof will be based on calculus. Let

$$x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - Ex^{n-5} + \cdots \pm N = Z$$

Factor Z as

$$Z = (x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\ldots(x-v).$$

Take logarithms and get

$$\ln Z = \ln(x-\alpha) + \ln(x-\beta) + \ln(x-\gamma) + \ln(x-\delta) + \ldots + \ln(x-v).$$

The formal manipulations of Euler's time required that he work with differentials instead of derivatives, so Euler takes the differentials,

$$\frac{dZ}{Z} = \frac{dx}{x-\alpha} + \frac{dx}{x-\beta} + \frac{dx}{x-\gamma} + \frac{dx}{x-\delta} + \cdots + \frac{dx}{x-v},$$

then he divides by dx to get

$$\frac{dZ}{Zdx} = \frac{1}{x-\alpha} + \frac{1}{x-\beta} + \frac{1}{x-\gamma} + \frac{1}{x-\delta} + \cdots + \frac{1}{x-v}.$$

He expands each of the quotients on the right as geometric series to get

$$\frac{1}{x-\alpha} = \frac{1}{x} + \frac{\alpha}{x^2} + \frac{\alpha^2}{x^3} + \frac{\alpha^3}{x^4} + \frac{\alpha^4}{x^5} + \frac{\alpha^5}{x^6} + \text{etc.}$$

$$\frac{1}{x-\beta} = \frac{1}{x} + \frac{\beta}{x^2} + \frac{\beta^2}{x^3} + \frac{\beta^3}{x^4} + \frac{\beta^4}{x^5} + \frac{\beta^5}{x^6} + \text{etc.}$$

$$\frac{1}{x-\gamma} = \frac{1}{x} + \frac{\gamma}{x^2} + \frac{\gamma^2}{x^3} + \frac{\gamma^3}{x^4} + \frac{\gamma^4}{x^5} + \frac{\gamma^5}{x^6} + \text{etc.}$$

etc.

$$\frac{1}{x-\nu} = \frac{1}{x} + \frac{\nu}{x^2} + \frac{\nu^2}{x^3} + \frac{\nu^3}{x^4} + \frac{\nu^4}{x^5} + \frac{\nu^5}{x^6} + \text{etc.}$$

If we add up these series, collect like powers of x, and use the above definitions of the symbols $\int \alpha$, $\int \alpha^2$, $\int \alpha^3$, etc., we get

$$\frac{dZ}{Zdx} = \frac{n}{x} + \frac{1}{x^2}\int \alpha + \frac{1}{x^3}\int \alpha^2 + \frac{1}{x^4}\int a^3 + \frac{1}{x^5}\int \alpha^4 + \text{etc.}$$

But, from the definition of Z as a polynomial, we also have

$$\frac{dZ}{dx} = nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \text{etc.}$$

and so

$$\frac{dZ}{Zdx} = \frac{nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \text{etc.}}{x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + Dx^{n-4} - \text{etc.}}.$$

Now we have two expressions for the same quantity, $\frac{dZ}{Zdx}$. Set them equal to each other, and multiply both sides by Z, the polynomial in the denominator of the second expression. We get

$$\begin{aligned} nx^{n-1} - (n-1)Ax^{n-2} + (n-2)Bx^{n-3} - (n-3)Cx^{n-4} + (n-4)Dx^{n-5} - \text{etc.} \\ = nx^{n-1} + x^{n-2}\int\alpha + x^{n-3}\int\alpha^2 + x^{n-4}\int\alpha^3 + x^{n-5}\int\alpha^4 + \text{etc.} \\ - nAx^{n-2} - Ax^{n-3}\int\alpha - Ax^{n-4}\int\alpha^2 - Ax^{n-5}\int\alpha^3 - \text{etc.} \\ + nBx^{n-3} + Bx^{n-4}\int\alpha + Bx^{n-5}\int\alpha + \text{etc.} \\ - nCx^{n-4} - Cx^{n-5}\int\alpha^2 - \text{etc.} \\ + nDx^{n-5} + \text{etc.} \end{aligned}$$

Now Euler uses one of his favorite tricks and matches the coefficients of powers of x. For the $(n-1)$st power he gets $n = n$. We knew that, but for the other powers, he gets

$$\begin{aligned} -(n-1)A &= \int\alpha - nA, \\ +(n-2)B &= \int\alpha^2 - A\int\alpha + nB, \\ -(n-3)C &= \int\alpha^3 - A\int\alpha^2 + B\int\alpha - nC, \\ +(n-4)D &= \int\alpha^4 - A\int\alpha^3 + B\int\alpha^2 - C\int\alpha + nD \\ &\text{etc.} \end{aligned}$$

From these, just a little bit of algebra gives Newton's result,

$$\begin{aligned} \int\alpha &= A, \\ \int\alpha^2 &= A\int\alpha - 2B, \\ \int\alpha^3 &= A\int\alpha^2 - B\int\alpha + 3C, \\ \int\alpha^4 &= A\int\alpha^3 - B\int\alpha^2 + C\int\alpha - 4D, \\ \int\alpha^5 &= A\int\alpha^4 - B\int\alpha^3 + C\int\alpha^2 - D\int\alpha + 5E, \\ &\text{etc.} \end{aligned}$$

Euler's second proof is almost completely different, and it relies on manipulations that are seldom found in a modern mathematics curriculum. They were better known a century ago when the editors of the *Opera omnia* were planning the contents of each volume of the series. At the time such techniques were grouped under the heading "Theory of equations." Because of the nature of this second proof, the Editors put E-153 in volume 6 of their first series, the volume titled *Algebraic articles pertaining to the theory of equations*.

To illustrate what he has in mind for his second proof, Euler takes the degree $n = 5$. Each of his steps has obvious analogies for higher degrees, and after showing each step in detail for $n = 5$, he tells us what the corresponding result would be for a general value of n. For $n = 5$, the polynomial equation is

$$x^5 - Ax^4 + Bx^3 - Cx^2 + Dx - E = 0$$

and the roots are α, β, γ, δ and ε. If we substitute these roots into the equation, we get the system

$$\begin{aligned}
\alpha^5 - A\alpha^4 + B\alpha^3 - C\alpha^2 + D\alpha - E &= 0,\\
\beta^5 - A\beta^4 + B\beta^3 - C\beta^2 + D\beta - E &= 0,\\
\gamma^5 - A\gamma^4 + B\gamma^3 - C\gamma^2 + D\gamma - E &= 0,\\
\delta^5 - A\delta^4 + B\delta^3 - C\delta^2 + D\delta - E &= 0,\\
\varepsilon^5 - A\varepsilon^4 + B\varepsilon^3 - C\varepsilon^2 + D\varepsilon - E &= 0,
\end{aligned}$$

Sum these, and we get, using the notation above,

$$\int \alpha^5 - A\int \alpha^4 + B\int \alpha^3 - C\int \alpha^2 + D\int \alpha + 5E = 0,$$

so

$$\int \alpha^5 = A\int \alpha^4 - B\int \alpha^3 + C\int \alpha^2 - D\int \alpha + 5E. \qquad (1)$$

We will use the nth degree analog of equation (1) later.

Now we form a sequence of polynomials of lower degrees, based on the coefficients of the original polynomial, namely

I.	$x - 4 = 0$,	and let its root be p;
II.	$x^2 - Ax + B$,	and let one of its roots be q;
III.	$x^3 - Ax^2 + Bx - C$,	and let one of its roots be r; and
IV.	$x^4 - Ax^3 + Bx^2 - Cx + D$,	and let one of its roots be s.

For each of these polynomials, the sum of the roots will be A. For polynomials II, III and IV, the sum of the products of the roots taken two at a time will be B. For III and IV, the sum of the products taken three at a time will be C, and for IV, the product of all four roots will be D.

Now bring the original polynomial equation back into the mix, and we get that

$$\begin{aligned}
\int \alpha &= \int s = \int r = \int q = \int p, \\
\int \alpha^2 &= \int s^2 = \int r^2 = \int q^2, \\
\int \alpha^3 &= \int s^3 = \int r^3, \\
\int \alpha^4 &= \int s^4.
\end{aligned} \tag{2}$$

Now apply equation (1) to polynomial I to get

$$\int p = A.$$

Likewise, applying it to polynomials II, III an IV, we get

$$\begin{aligned}
&\int q^2 = A \int q - 2B, \\
&\int r^3 = A \int r^2 - B \int r + 3C, \quad \text{and} \\
&\int s^3 = A \int s^3 - B \int s^2 + C \int s - 4D.
\end{aligned}$$

Now, into these equations make the substitutions in equations (2) to get Newton's theorem for $n = 5$:

$$\begin{aligned}
&\int \alpha = A, \\
&\int \alpha^2 = A \int \alpha - 2B, \\
&\int \alpha^3 = A \int \alpha^2 - B \int \alpha + 3C, \\
&\int \alpha^4 = A \int \alpha^3 - B \int \alpha^2 + C \int \alpha - 4D, \\
&\int \alpha^5 = A \int \alpha^4 - B \int \alpha^3 + C \int \alpha^2 - D \int \alpha + 5E.
\end{aligned}$$

Euler's comments along the way make it obvious how this can be extended to polynomials of arbitrary degree.

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16

Estimating π

(February 2009)

On Friday, June 7, 1779, Leonhard Euler sent a paper [E705] to the regular twice-weekly meeting of the St.Petersburg Academy. Euler, blind and disillusioned with the corruption of Domaschneff, the President of the Academy, seldom attended the meetings himself, so he sent one of his assistants, Nicolas Fuss, to read the paper to the ten members of the Academy who attended the meeting.

The paper bore the cumbersome title "Investigatio quarundam serierum quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae" (Investigation of certain series which are designed to approximate the true ratio of the circumference of a circle to its diameter very closely."

Up to this point, Euler had shown relatively little interest in the *value* of π, though he had standardized its notation, using the symbol π to denote the ratio of a circumference to a diameter consistently since 1736, and he found π in a great many places outside circles. In a paper he wrote in 1737, [E74] Euler surveyed the history of calculating the value of π. He mentioned Archimedes, Machin, de Lagny, Leibniz and Sharp. The main result in E74 was to discover a number of arctangent identities along the lines of

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{70} + \arctan\frac{1}{99}$$

and to propose using the Taylor series expansion for the arctangent function, which converges fairly rapidly for small values, to approximate π. Euler also spent some time in that paper finding ways to approximate the logarithms of trigonometric functions, important at the time in navigation tables. The paper ends with the intriguing formula

$$\frac{\sin A}{\cos\frac{1}{2}A\cdot\cos\frac{1}{4}A\cdot\cos\frac{1}{8}A\cdot\cos\frac{1}{16}A\cdot\text{etc.}} = \infty\sin\frac{1}{\infty}A = A.$$

Let us resist the temptation to digress too much further and return to the papers Euler wrote in 1779, one of only 24 papers Euler wrote that year. Most of those papers, including E705, were published in the 1790s, more than ten years after Euler's death.

Euler opens E705 with another history of efforts to approximate π, adding the name Ludolph van Ceulen to his list and noting that Sharp had calculated π to 72 digits, Machin to 100 digits and de Lagny to 128 digits, which Euler describes as a "Herculean task."

To begin his own analysis, Euler reminds us of Leibniz's arctangent series, which gives the angle s in terms of its tangent t as

$$s = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \text{etc.}$$

We could take $t = 1$ so that $s = \frac{\pi}{4}$ and get the very-slowly converging approximation

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \text{etc.}$$

This is sometimes called the Lebniz series.

Euler also reminds us that he had previously [E74, §12] shown that if $1 = \frac{a+b}{ab-1}$, that is to say, if $b = \frac{a+1}{a-1}$ then

$$\arctan 1 = \arctan\frac{1}{a} + \arctan\frac{1}{b}$$

This, in turn, is a special case of the formula

$$\arctan\frac{1}{p} = \arctan\frac{1}{p+q} + \arctan\frac{q}{p^2+pq+1}$$

in the case $p = 1$ and $q = a - 1$. [E74, p. 253, §14]

Euler has a whole repertoire of such formulas. Not all of them are mentioned in E74, but they all come easily from the still-more general formula

$$\arctan\alpha = \arctan\beta + \arctan\frac{\alpha-\beta}{1+\alpha\beta}.$$

Without citing any particular formula, Euler proclaims that

$$\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan 1 = \frac{\pi}{4}.$$

This can be found from the first formula above by taking $a = 2$, so that $b = 3$.

This leads to a double series, because from the arctangent series we have

$$\arctan\frac{1}{2} = \frac{1}{2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \text{etc.}$$

and

$$\arctan\frac{1}{3} = \frac{1}{3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \text{etc.}$$

These series decrease "in quadruple ratio", that is to say, each term is less than $1/4$ the size of the previous term, so it converges much more quickly than the series for $t = 1$.

Note that Euler is using something like the ratio test when he describes this convergence as being in "quadruple ratio." Though this is not the ratio of any two consecutive terms, it is the limit of those ratios. Augustin-Louis Cauchy is usually credited with discovering the ratio test in 1821, 42 years after Euler wrote this paper, but only 23 years after it was published.

We can make the series converge more quickly because the denominators are larger if we know that

$$\arctan\frac{1}{2} = \arctan\frac{1}{3} + \arctan\frac{1}{7}.$$

This follows from the second of Euler's arctangent addition formulas, taking $p = 2$ and $q = 1$. Combining this new fact with the formula for arctan 1, it gives

$$\pi = 4\arctan 1 = 8\arctan\frac{1}{3} + 4\arctan\frac{1}{7}.$$

The problem with this is that the second of the arctangent series requires repeated divisions by 49, and though it converges rather quickly, the computations are difficult.
Euler seeks the best of both worlds, rapid convergence and easy calculations. He lets

$$s = t - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \frac{t^9}{9} - \text{etc.}$$

$$stt = t^3 - \frac{t^5}{3} + \frac{t^7}{5} - \frac{t^9}{7} + \text{etc.}$$

so

$$s + sst = t + \frac{2}{3}t^3 - \frac{2}{3\cdot 5}t^5 + \frac{2}{5\cdot 7}t^7 - \text{etc.} = t + s'tt,$$

where this equation defines a new variable s', not to be confused with the derivative of s. Then

$$s' = \frac{2}{3}t - \frac{2}{3\cdot 5}t^3 + \frac{2}{5\cdot 7}t^5 - \frac{2}{7\cdot 9}t^7 + \text{etc.}$$

so that

$$s'tt = \frac{2}{1\cdot 3}t^3 - \frac{2}{3\cdot 5}t^5 + \frac{2}{5\cdot 7}t^7 - \text{etc.}$$

Likewise, by series expansions, he shows that if s'' is defined by the equation

$$s'(1+tt) = \frac{2}{3}t + s''t$$

then

$$s''tt = \frac{2\cdot 4}{1\cdot 3\cdot 5}t^3 - \frac{2\cdot 4}{3\cdot 5\cdot 7}t^5 + \text{etc.}$$

and so on, defining s''', etc.

Solving for s, s', etc., we get

$$\begin{aligned} s &= \frac{t}{1+tt} + \frac{s'tt}{1+tt} \\ s' &= \frac{2t}{3(1+tt)} + \frac{s''tt}{1+tt} \\ s'' &= \frac{2\cdot 4t}{3\cdot 5\,(1+tt)} + \frac{s'''tt}{1+tt} \\ s''' &= \frac{2\cdot 4\cdot 6t}{3\cdot 5\cdot 7(1+tt)} + \frac{s''''tt}{1+tt} \\ &\text{etc.} \end{aligned}$$

Substituting each of these into the one before it gives

$$s = \frac{t}{1+tt} + \frac{2}{3}\cdot\frac{t^3}{(1+tt)^2} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{t^5}{(1+tt)^5} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{t^7}{(1+tt)^7} + \text{etc.},$$

which reduces to

$$s = \frac{t}{1+tt}\left[1 + \frac{2}{3}\left(\frac{tt}{1+tt}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{tt}{1+tt}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{tt}{1+tt}\right)^3 + \text{etc.}\right].$$

This is convenient because each term is the previous term multiplied by $\frac{tt}{1+tt}$ and by a simple fraction of the form $\frac{2n}{2n+1}$.

Euler derives this same formula by a different method that begins by writing the angle s as an integral, $s = \int \frac{dt}{1+tt}$, but we will omit that derivation here.

If we apply Euler's series for s to the identity indicated above, namely

$$\pi = 4\arctan\frac{1}{2} + 4\arctan\frac{1}{3},$$

then for the first part, where $t = \frac{1}{2}$, we get

$$\arctan\frac{1}{2} = \frac{2}{5}\left(1 + \frac{2}{3}\cdot\frac{1}{5} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{5^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{5^3} + \text{etc.}\right)$$

and for the second part, where $t = \frac{1}{3}$, we get

$$\arctan\frac{1}{3} = \frac{3}{10}\left(1 + \frac{2}{3}\cdot\frac{1}{10} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{10^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{10^3} + \text{etc.}\right)$$

Thus, the value for π can be expressed as the sum of two series,

$$\pi = \left\{ \begin{array}{l} \frac{16}{10}\left[1 + \frac{2}{3}\left(\frac{2}{10}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{10}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{10}\right)^3 + \text{etc.}\right] \\ +\frac{12}{10}\left[1 + \frac{2}{3}\left(\frac{1}{10}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{1}{10}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{1}{10}\right)^3 + \text{etc.}\right] \end{array} \right\}.$$

Euler tells us that these two series are obviously much less work because the denominators have factors of 10 and because they are "greatly convergent."

The sum of the given terms, up to the third powers, gives

$$\begin{aligned}\pi &\approx 1.853318094 + 1.286948572 \\ &= 3.140266666\end{aligned}$$

Euler does similar calculations starting with the identity

$$\pi = 8 \arctan \frac{1}{3} + 4 \arctan \frac{1}{7}.$$

He finds

$$\arctan \frac{1}{7} = \frac{7}{50}\left(1 + \frac{2}{3}\cdot\frac{1}{50} + \frac{2\cdot 4}{3\cdot 5}\cdot\frac{1}{50^2} + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\cdot\frac{1}{50^3} + \text{etc.}\right)$$

so

$$\pi = \left\{ \begin{array}{l} \frac{24}{10}\left[1 + \frac{2}{3}\left(\frac{1}{10}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{1}{10}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{1}{10}\right)^3 + \text{etc.}\right] \\ +\frac{28}{50}\left[1 + \frac{2}{3}\left(\frac{2}{100}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{2}{100}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{2}{100}\right)^3 + \text{etc.}\right] \end{array} \right\}.$$

This gives the approximation $\pi \approx 3.141485325$.

In general, each pair of terms in these series adds about one correct decimal place to the approximation.

Euler goes on to lead us through arctangent identities that lead to faster and easier calculations. Because

$$\arctan \frac{1}{3} = \arctan \frac{1}{7} + \arctan \frac{2}{11}.$$

he gets

$$\pi = 12 \arctan \frac{1}{7} + 8 \arctan \frac{2}{11}.$$

Then from the identity

$$\arctan \frac{2}{11} = \arctan \frac{1}{7} + \arctan \frac{3}{79}$$

he gets

$$\pi = 20 \arctan \frac{1}{7} + 8 \arctan \frac{3}{79}.$$

Note how the fraction $\frac{1}{7}$ was problematic when he used the Taylor series for the arctangent because it led to fractions involving 49ths. Here, though, it gives 50ths $\left(\text{disguised as}\,\frac{2}{100}\right)$, and for $t = \frac{3}{79}$, we have

$$\frac{tt}{1+tt} = \frac{9}{6250} = \frac{144}{100\,000}.$$

This gives the convenient series.

$$\arctan\frac{3}{79} = \frac{237}{6250}\left[1 + \frac{2}{3}\left(\frac{144}{100\,000}\right) + \frac{2\cdot 4}{3\cdot 5}\left(\frac{144}{100\,000}\right)^2 + \frac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\frac{144}{100\,000}\right)^3 + \text{etc.}\right]$$

These two arctangent approximations lead to Euler's best explicit approximating series of the paper,

$$\pi = \left\{\begin{array}{l} \dfrac{28}{10}\left[1 + \dfrac{2}{3}\left(\dfrac{2}{100}\right) + \dfrac{2\cdot 4}{3\cdot 5}\left(\dfrac{2}{100}\right)^2 + \dfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\left(\dfrac{2}{100}\right)^3 + \text{etc.}\right] \\ \quad + \dfrac{30336}{100\,000}\left[1 + \dfrac{2}{3}\left(\dfrac{144}{100\,000}\right) + \dfrac{2\cdot 4}{3\cdot 5}\left(\dfrac{144}{100\,000}\right)^2 + \text{etc.}\right] \end{array}\right\}.$$

Euler calculates the first of these series accurately to 12 decimal places and the second to 17 places, but for some reason he doesn't add them together to give an approximation of π. We will speculate on this mystery at the end of the column.

Eighteen pages into a twenty-page paper, Euler suddenly changes gears and goes back to the original arctangent series,

$$\arctan t = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \frac{1}{7}t^7 + \frac{1}{9}t^9 - \text{etc.}$$

He denotes by $\sum$ the partial sum of the first n terms of this series, so

$$\sum = t - \frac{1}{3}t^3 + \frac{1}{5}t^5 - \cdots \pm \frac{t^{2n-1}}{2n-1}.$$

Now, changing the meaning of the symbol s from what it had meant earlier in the article, he lets s denote the remainder term, so that

$$s = \frac{t^{2n+1}}{2n+1} - \frac{t^{2n+3}}{2n+3} + \frac{t^{2n+5}}{2n+5} - \text{etc.},$$

where we notice he's being a little sloppy about the ambiguity of the signs.
All this makes

$$\arctan t = \sum \pm s$$

Euler parallels his series manipulations from sections 4 and 5 find series for s, for stt, then $s(1+tt)$ to define s' with the relation

$$s(1+tt) = \frac{t^{2n+1}}{2n+1}s'tt.$$

Similarly, he defines s" with

$$s'(1+tt) = \frac{2t^{2n+1}}{(2n+1)(2n+3)} + s''tt, \textit{etc}.$$

Substituting, then factoring, gives

$$\begin{aligned} s &= \frac{2^{2n+1}}{(2n+1)(1+tt)} + \frac{2^{2n+3}}{(2n+1)(2n+3)(1+tt)^2} \\ &\quad + \frac{2^{2n+5}}{(2n+1)(2n+3)(2n+5)(1+tt)^3} + \text{etc.} \\ &= \frac{2^{2n+1}}{(2n+1)(1+tt)}\left(1 + \frac{2tt}{(2n+1)(2n+3)} + \frac{2\cdot 4t^4}{(2n+1)(2n+3)(1+tt)^2} + \text{etc.}\right) \end{aligned}$$

Euler does no examples with this series, but he tells us that it is even more convergent than the preceding one because the denominators, with those factors $(2n+1)$, etc. make the denominators much larger than the numerators.

Just ten days later, Euler sent Nicholas Fuss to the Academy meeting with another paper [E706] on approximating π. This one was titled "De novo genere serierum rationalium et valde convergentium quibus ratio peripheriae ad diametrum exprimi posttest" (On a new kind of strongly convergent rational series which is able to express the ratio of the circumference to the diameter). This paper is much shorter, only five pages in the original, seven pages in the *Opera omnia*,

Euler notes that

$$4 + x^4 = (2+2x+xx)(2-2x+xx)$$

$$\int \frac{(2+2x+xx)dx}{4+x^4} = \int \frac{dx}{2-2x+xx}.$$

He denotes this last integral by $\odot$, the astrological symbol for the Sun, so that

$$\odot = \int \frac{dx}{2-2x+xx}.$$

But this last integrates to give an arctangent, so that

$$\odot = \arctan\frac{x}{2-x}$$

where, as so often happens, Euler means us to take the particular antiderivative that is zero when $x = 0$.

Now, Euler returns to the form $\int \frac{(2+2x+xx)dx}{4+x^4}$, which he rewrites as

$$\begin{aligned}\int \frac{(2+2x+xx)dx}{4+x^4} &= 2\int \frac{dx}{4+x^4} + 2\int \frac{xdx}{4+x^4} + \int \frac{xxdx}{4+x^4} \\ &= 2A + 2B + C \\ &= \arctan \frac{x}{2-x}\end{aligned}$$

where Euler uses, instead of A, B and C, the astrological symbols, ♄, ♃ and ♂ which are the symbols for Saturn, Jupiter and Mars, respectively.

Euler expands each of these integrands as series, then integrates to get

$$A = \frac{x}{4}\left[1 - \frac{1}{5}\cdot\frac{x^4}{4} + \frac{1}{9}\left(\frac{x^4}{4}\right)^2 - \frac{1}{13}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right]$$

$$B = \frac{xx}{8}\left[1 - \frac{1}{3}\cdot\frac{x^4}{4} + \frac{1}{5}\left(\frac{x^4}{4}\right)^2 - \frac{1}{7}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right]$$

$$C = \frac{x^3}{4}\left[\frac{1}{3} - \frac{1}{7}\cdot\frac{x^4}{4} + \frac{1}{11}\left(\frac{x^4}{4}\right)^2 - \frac{1}{15}\left(\frac{x^4}{4}\right)^3 + \text{etc.}\right]$$

Now, using the facts that

$$\pi = 8\arctan\frac{1}{3} + 4\arctan\frac{1}{7}$$

and that

$$\arctan\frac{x}{2-x} = 2A + 2B + C$$

to get his grand result,

$$\pi = \left\{\begin{array}{l} 2\left(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ +\frac{1}{2}\left(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ +\frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{etc.}\right) \\ \frac{1}{2}\left(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ +\frac{1}{16}\left(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{etc.}\right) \\ +\frac{1}{64}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{etc.}\right) \end{array}\right\}$$

Euler claims that this converges very rapidly and that it is very easy to use because the series only use powers of two. Euler makes no effort to perform the calculations, nor does he try it on any of his other arctangent identities like

$$\pi = 20 \arctan \frac{1}{7} + 8 \arctan \frac{3}{79}$$

or

$$\pi = 4 \arctan 1 = 8 \arctan \frac{1}{3} + 4 \arctan \frac{1}{7}$$

Thus, over the course of these two papers, Euler has given us instructions on how to calculate π to many decimal places, though he has not undertaken the calculations himself. Indeed, these ideas were "in the air" of the era. The Slovenian mathematician and artilleryman Baron Jurij Vega adapted the ideas in [E74] and calculated π to 140 decimal places, in a book he first published in 1784, [Vega 1835, vol. 2] breaking de Lagny's record of 128 digits. In an effort to be admitted as a foreign member of the St. Petersburg Academy, he sent a synopsis [Vega 1798] to the Academy, which they published in the same issue of their *Nova acta* as Euler's two articles E705 and E706.

We close by speculating why Euler did not finish his calculation in E705, and at the same time why he used such strange symbols in his calculations in E706. In 1779, Euler was blind and in ill health. He relied on his team of assistants, Fuss, Gmelin, his son J. A. Euler and others, to write up the details and to check the calculations. Now, suppose that Euler and his assistants were just about to finish writing E705 as an 18-page paper when suddenly Euler had another idea. They might have been too distracted by finishing their calculations to write up the new ideas. That would explain both the unfinished calculations and the abrupt change of gear at the end of the paper.

Then Euler may have continued to think about the problem and came across the idea that he used in E706. Euler didn't send another paper to the Academy until August 12, so perhaps he was ill, or maybe he left early for the July vacation days, and it would be like his students to "play" a bit when the Master was away and use those funny symbols as they wrote up E706. This is, of course, speculation. Though it is consistent with the times and the personalities involved, we can't know if it is true.

Note: The photograph at the beginning of this column is of a 20-foot tall sculpture of the Greek letter π, planned and built by artist Barbara Grygutis in 2008 and standing outside Henry Abbot Technical High School in Danbury, CT, less than three minutes from Exit 6 on I-84 westbound, five minutes from Exit 5 eastbound. At night, it is illuminated with programmable LED lights.

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17

Nearly a Cosine Series

(May 2009)

To look at familiar things in new ways is one of the most fruitful techniques in the creative process. When we look at the Pythagorean theorem, for example, and we ask, "what if it's *not* a right triangle?" it leads us to the Law of Cosines. When we ask, "what if -1 *did* have a square root?" we discover complex variables. To find variations on a familiar theme isn't the only tool in our creative repertoire, but it is one of the most reliable.

But it doesn't always work. This month we'll look at a variation on the Taylor series for $\cos x$. It seems like a pretty good idea, and Euler does his best to make something of it, but after a few promising results, it just doesn't go anywhere.

The Taylor series for $\cos x$ is very familiar:

$$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} - \frac{x^6}{1 \cdot 2 \cdot 3 \cdots 6} + \frac{x^8}{1 \cdot 2 \cdot 3 \cdots 8} - \text{etc.}$$

Indeed, combining it with the corresponding Taylor series for $\sin x$ and for e^x, then taking x to be an imaginary number, $i\theta$, leads to one of the most popular ways to prove the Euler identity, $e^{i\theta} = \cos\theta + \sqrt{-1}\sin\theta$, and its famous special case, $e^{\pi i} = -1$. [Sandifer August 2007]

Sometime late in 1776 or early in 1777, as he neared the age of 70, Euler looked at this series and asked what he would get if he changed the denominators. Euler had asked such questions before. Note that the denominator on the term of degree m is what we now write $m!$ and call "m-factorial." Early in his career, when he was only 22 years old, he had discovered what we now call the Gamma function by asking what would happen if m were a fraction instead of a whole number, and this had been a very productive line of inquiry, as we have seen in previous columns. [Sandifer September 2007, Sandifer October 2007, Sandifer November 2007]

Instead of changing the number of factors in the products, Euler wondered what would happen if he changed where the products begin, say starting at a number n instead of starting with 1. That is to say, he asked about the series

$$1 - \frac{x^2}{n(n+1)} + \frac{x^4}{n(n+1)(n+2)(n+3)} - \frac{x^6}{n\cdots(n+5)} + \text{etc.}$$

Euler doesn't give this series a name. Usually, he just specifies a series by stating a value of n. We will usually do that as well, but when we want to compare the series for two different values of n, we'll distinguish them by using the notation $\cos_n x$.

Obviously, if $n = 1$, this is the same as the Taylor series for $\cos x$, that is $\cos_1 x = \cos x$. After a moment's reflection we notice that the series is undefined if $x = 0$ or if x is any negative integer, but what about other values of n, larger integers or fractions?

Euler knows that there are no bounds and few rules about how to *be* creative, but that once you have an idea, especially in mathematics, you should pursue it methodically. Euler begins by looking at the cases $n = 1$, $n = 2$ and $n = 3$.

As we saw, $\cos_1 x = \cos x$, but Euler takes note of its important properties in case they become part of a pattern that persists for larger values of n. For him, the most important property is that the roots form an arithmetic sequence, extending in the positive and the negative direction. The smallest positive root is $\frac{\pi}{2}$ and the difference between consecutive roots is π.

In the case $n = 2$, the series is

$$1 - \frac{x^2}{2 \cdot 3} + \frac{x^4}{2 \cdot 3 \cdot 4 \cdot 5} - \frac{x^6}{2 \cdots 7} + \frac{x^8}{2 \cdots 9} - \text{etc.}$$

This is the Taylor series for $\frac{\sin x}{x}$, one of Euler's favorites because of the crucial role it played in his solution to the Basel problem some four decades earlier. Except for a gap at $x = 0$, the roots of this formula are all multiples of π, positive and negative. They too form an arithmetic sequence.

Moving on to $\cos_3 x$, where $n = 3$, we get the series

$$1 - \frac{x^2}{3 \cdot 4} + \frac{x^4}{3 \cdot 4 \cdot 5 \cdot 6} - \frac{x^6}{3 \cdots 8} + \text{etc.}$$

Euler tells us that *evidens est seriei propositae summam esse*, "it is obvious that the given series sums to be"

$$\frac{2(1 - \cos x)}{x^2}.$$

It wasn't that obvious to me, but it's true. It follows from multiplying the given series by $\frac{x^4}{1 \cdot 2}$. I'll leave out the details and hope that the reader is as amused by filling them in as I was.

Again with the exception of $x = 0$, this has roots every time $\cos x = 1$, that is whenever x is a nonzero multiple of 2π. And again, the roots form an arithmetic sequence. There seems to be a pattern!

So we move with confidence to the case $n = 4$. The series is

$$1 - \frac{x^2}{4 \cdot 5} + \frac{x^4}{4 \cdot 5 \cdot 6 \cdot 7} - \frac{x^6}{4 \cdots 9} + \text{etc.}$$

Readers who enjoyed checking Euler's claim for the case $n = 3$ might also enjoy discovering that this series sums to

$$\frac{6(x - \sin x)}{x^3}.$$

We see that when $x = 0$, this equals 1, either the hard way (a triple application of l'Hôpital's rule) or the easy way (substitute $x = 0$ into the series itself). Because $x = 0$ is the only root of the numerator, we see that $\cos_4 x$ has no roots at all. The pattern is broken!

What happened? Euler tries to explain it by noting that in the case $n = 3$, all the roots were double roots, and they were twice as far apart as they had been in the case $n = 2$. He says that, as in the case of polynomials, a pair of real roots will "coalesce" into a double root as they are about to become imaginary roots. He also claims a small consolation because the function goes to zero as x goes to infinity.

Disappointed, but undeterred, Euler prepares to study the cases where n is a fraction. For any value of n, he defines s by the equation

$$\frac{s}{x^{n+1}} = \cos_n x.$$

This makes

$$s = x^{n-1} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+3}}{n(n+1)(n+2)(n+3)} - \text{etc.}$$

Now Euler is going to use his calculus tricks on the series. Introduce a new function z by taking

$$s = x^{n-1} - z$$

so that

$$z = \frac{x^{n+1}}{n(n+1)} - \frac{x^{n+3}}{n\cdots(n+3)} + \frac{x^{n+5}}{n\cdots(n+5)} - \text{etc.}$$

Now take the second derivative of this with respect to x and get

$$\frac{\partial\partial z}{\partial x^2} = x^{n-1} - \frac{x^{n+1}}{n(n+1)} + \frac{x^{n+3}}{n\cdots(n+3)} - \text{etc.} = s,$$

from which we have

$$\frac{\partial\partial z}{\partial x^2} + z = x^{n-1}.$$

Now, Euler tells us, "the whole business (*totum negotium*) reduces to the solution of this second degree equation."

Alas, Euler can't solve this equation either, so this approach turns out to be a dead end. According to to Maple$^{\text{TM}}$, this differential equation has a solution in terms of Lommel functions, which are supposedly related to Bessel functions. But Euler couldn't know that because Lommel wrote in the 1880s, a hundred years after Euler died.

Euler tries something else. In the case $n = 1/2$, we get

$$\cos_{1/2} x = 1 - \frac{4x^2}{1\cdot 3} + \frac{16x^4}{1\cdot 3\cdot 5\cdot 7} - \frac{64x^6}{1\cdot 3\cdots 11} + \text{etc.}$$

To find the roots, Euler sets this equal to zero and makes the substitution $z = 4xx$, where we should note that this z is a new variable not the same as the function z that appeared in the differential equations above. So, Euler wants to solve

$$0 = 1 - \frac{z}{1\cdot 3} + \frac{zz}{1\cdots 7} - \frac{z^3}{1\cdots 11} + \frac{z^4}{1\cdots 15} - \text{etc.}$$

Thwarted in his analytical efforts to solve it, he resorts to numerical methods. He observes that this is an alternating series with decreasing terms. A number of years earlier, [E212, E247, Sandifer June 2006] Euler had developed a technique for accelerating the convergence of such a series. In general, if we write a general alternating series with decreasing terms as

$$1 - a + b - c + d - e + \text{etc.}$$

and if we take

$$\begin{aligned} \alpha &= 1 - a \\ \beta &= 1 - 2a + b \\ \gamma &= 1 - 3a + 3b - c \\ &\text{etc.} \end{aligned}$$

then the sum is given by

$$\frac{1}{2} + \frac{\alpha}{4} + \frac{\beta}{8} + \frac{\gamma}{16} + \text{etc.}$$

Euler uses this to estimate that $z = 4.20$, hence $x = 1.025$.

Euler checks his work with a similar but more accurate method he says is due to Daniel Bernoulli and estimates that $z = 3.31$ so $x = .909$, significantly smaller than $\frac{\pi}{2} \approx 1.571$, which is the smallest positive root of $\cos_1 x$. Euler studies at the ratio between .909 and 1.571. He does a continued fraction expansion and decides that the ratio is likely to be $1 : \sqrt{3}$, and suspects that the root of $\cos_{\frac{1}{2}} x$. that he has been pursuing is probably $x = \frac{\pi}{2\sqrt{3}}$, or 0.90695. This is an astonishing bit of guesswork on Euler's part.

Euler can carry this no farther and he does not try to find the second or third roots of $\cos_{\frac{1}{2}} x$. Instead, he moves on to $n = 1/4$. This time he goes straight to Bernoulli's method and estimates that the smallest root of $\cos_{\frac{1}{4}} x$ is $x = 0.5717$, which he guesses is $\frac{\pi}{\sqrt{30}}$, or about 0.57356.

The case $n = 1/3$ leads to an estimate that $x = .6875$, a value that Euler says is not easily related to π, as the other two values had been.

At this point, Euler must be a bit discouraged. He gives up on chasing roots for fractional powers of n without even noting that as n gets smaller, the value of the smallest root seems to get smaller as well. He spends a few more pages trying to identify some of the complex roots that arise when $n > 3$, and is unable to find relations that describe them. Despite the paucity of his results, Euler ends his article with the remark, "Still, I believe that by extending this argument, in which several excellent ingenuities occur, Geometry would not be displeased."

Euler was satisfied with this paper, and hoped that someone could add to it. That took a hundred years, after the development of the idea of special functions, an idea for which Euler planted the seeds, but did not live to seem them sprout and flourish.

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18

A Series of Trigonometric Powers

(June 2008)

The story of Euler and complex numbers is a complicated one. Earlier in his career, Euler was a champion of equal rights for complex numbers, treating them just like real numbers whenever he could. For example, he showed how to integrate $\int \frac{1}{x^2+1}dx$ without using inverse trigonometric functions. He factored $x^2 + 1 = (x + \sqrt{-1})(x - \sqrt{1})$, then used partial fractions to rewrite

$$\frac{1}{x^2+1} = \frac{\frac{1}{2}\sqrt{-1}}{x+\sqrt{-1}} - \frac{\frac{1}{2}\sqrt{-1}}{x-\sqrt{-1}},$$

then integrated this difference to get

$$\int \frac{1}{x^2+1}dx = \frac{1}{2}\sqrt{-1}\ln\frac{x+\sqrt{-1}}{x-\sqrt{-1}}.$$

Euler typically omitted constants of integration until he needed them, and also seldom used i in place of $\sqrt{-1}$. His role in that particular notational innovation is exaggerated.

Euler struck a second, and better-known blow for justice for complex numbers when he took the variable in the exponential function e^x to be an imaginary number, say $x = \theta\sqrt{-1}$, and showed that

$$e^{\theta\sqrt{-1}} = \cos\theta + \sqrt{-1}\sin\theta.$$

Euler continued to use complex numbers late in his life, but his applications seem to me to be less sweeping and more technical, showing how they solved a variety of specific problems. This month we look at one such problem from 1773.

The title of E447 is “Summatio progressionum $\sin\varphi^\lambda + \sin 2\varphi^\lambda + \sin 3\varphi^\lambda + \mathrm{K} + \sin n\varphi^\lambda$, $\cos\varphi^\lambda + \cos 2\varphi^\lambda + \cos 3\varphi^\lambda + \mathrm{K} + \cos n\varphi^\lambda$.” Right off, this is confusing to the modern reader, because Euler writes $\sin\varphi^\lambda$ where we would write $\sin^\lambda\varphi$ and mean $(\sin\varphi)^\lambda$. For this, we will use the modern notation.

Euler begins by asking us to let

$$\cos\varphi + \sqrt{-1}\sin\varphi = p \quad \text{and}$$
$$\cos\varphi - \sqrt{-1}\sin\varphi = q.$$

Then, from de Moivre's formula, we have

$$\cos n\varphi = \frac{p^n + q^n}{2} \quad \text{and}$$
$$\sin n\varphi = \frac{p^n - q^n}{2\sqrt{-1}},$$

and because $\sin^2\varphi + \cos^2\varphi = 1$, we have

$$pq = 1.$$

Properties of geometric series tell us

$$p^\alpha + p^{2\alpha} + p^{3^\alpha} + \text{L} + p^{n\alpha} = \frac{p^\alpha(1 - p^{n\alpha})}{1 - p^\alpha}$$

and

$$q^\alpha + q^{2\alpha} + q^{3^\alpha} + \text{L} + q^{n\alpha} = \frac{q^\alpha(1 - q^{n\alpha})}{1 - q^\alpha}.$$

If we add these together and repeatedly apply the identities $pq = 1$ and $p^{k\alpha} + q^{k\alpha} = 2\cos k\alpha\varphi$ (a consequence of de Moivre's formula), we get

$$-1 + \frac{\cos n\alpha\varphi - \cos(n+1)\alpha\varphi}{1 - \cos\alpha\varphi}.$$

Likewise, if we subtract the q-series from the p-series we get

$$\frac{\sin\alpha\varphi - \sin(n+1)\alpha\varphi + \sin n\alpha\varphi}{1 - \cos\alpha\varphi}\sqrt{-1}.$$

Euler uses an integral sign, $\int$, where we would use a summation sign, $\sum$, so he writes these results as

$$\int (p^{n\alpha} + q^{n\alpha}) = -1 + \frac{\cos n\alpha\varphi - \cos(n+1)\alpha\varphi}{1 - \cos\alpha\varphi} \quad \text{and}$$
$$\int (p^{n\alpha} - q^{n\alpha}) = \frac{\sin\alpha\varphi + \sin n\alpha\varphi - \sin(n+1)\alpha\varphi}{1 - \cos\alpha\varphi}\sqrt{-1}. \qquad (1)$$

Now Euler is ready to work on the sums in the title of the article. He takes $\lambda = 1$, and his two series become

$$s = \sin\varphi + \sin 2\varphi + \sin 3\varphi + \text{L} + \sin n\varphi$$
$$= \int \sin n\varphi \quad \text{and}$$
$$t = \cos\varphi + \cos 2\varphi + \cos 3\varphi + \text{L} + \cos n\varphi$$
$$= \int \cos n\varphi.$$

Because of de Moivre's identities,

$$\sin n\varphi = \frac{p^n - q^n}{2\sqrt{-1}} \quad \text{and}$$

$$\cos n\varphi = \frac{p^n + q^n}{2},$$

these two series can be rewritten as

$$2s\sqrt{-1} = \int (p^n - q^n) \quad \text{and}$$

$$2t = \int (p^n + q^n)$$

But from formula (1) above, and taking $\alpha = 1$, this gives

$$s = \frac{\sin\varphi + \sin n\varphi - \sin(n+1)\varphi}{2(1-\cos\varphi)} \quad \text{and}$$

$$t = -\frac{1}{2} + \frac{\cos n\varphi - \cos(n+1)\varphi}{2(1-\cos\varphi)}.$$

Note how unexpectedly simple these formulas are. They are each the sum of n terms using only the terms at the beginning and the terms at the end, without using any of the terms in between.

Now take $\lambda = 2$ so that

$$\begin{aligned} s &= \sin^2\varphi + \sin^2 2\varphi + \mathrm{L} + \sin^2 n\varphi \\ &= \int \sin^2 n\varphi \quad \text{and} \\ t &= \cos^2\varphi + \cos^2 2\varphi + \mathrm{L} + \cos^2 n\varphi \\ &= \int \cos^2 n\varphi. \end{aligned}$$

Recalling that $pq = 1$ we get

$$\begin{aligned} \sin^2 n\varphi &= (\sin n\varphi)^2 \\ &= \left(\frac{p^n - q^n}{2\sqrt{-1}}\right)^2 \\ &= \frac{p^{2n} - 2p^n q^n + q^{2n}}{-4} \\ &= \frac{1}{2} - \frac{p^{2n} + q^n}{4}. \end{aligned}$$

Similarly,

$$\cos^2 n\varphi = \frac{1}{2} + \frac{p^{2n} + q^{2n}}{4}.$$

Summing these, we get

$$4s = 2\int 1 - \int (p^{2n} + q^{2n}) \quad \text{and}$$
$$4t = 2\int 1 + \int (p^{2n} + q^{2n})$$

Obviously, $\int 1 = n$, so, using formula (1) we get

$$s = \frac{n}{2} + \frac{1}{4} - \frac{\cos n\varphi - \cos 2(n+1)\varphi}{4(1-\cos 2\varphi)} \quad \text{and}$$
$$t = \frac{n}{2} - \frac{1}{4} + \frac{\cos n\varphi - \cos 2(n+1)\varphi}{4(1-\cos 2\varphi)}.$$

It is reassuring to note that $s + t = n$, as it should be, because s is a sum of n squared sines and t is a sum of the corresponding squared cosines.

Euler does $\lambda = 3$ and $\lambda = 4$, and his expressions for s and t grow first to three, then to four terms, though the terms grow no more complicated, except for involving higher powers of 2. Moreover, his expressions have the same general form.

Let's look a bit more closely at this expression for s in the case $\lambda = 2$, the sum of the squares of a sequence of sines. Note how the last term does not increase as the number n increases. Also, if $\cos 2\varphi$ is not very close to 1, then the denominator in the last term is not very small. Moreover, the two cosines in the numerator are always between -1 and 1, so their difference is between -2 and 2. Consequently, the last term is bounded between two values, $+M$ and $-M$, that do not depend on n. Hence, s is always between $\frac{n}{2} + \frac{1}{4} + M$ and $\frac{n}{2} + \frac{1}{4} - M$. Thus, as n goes to infinity, so also does s. The same reasoning applies to t.

Note that this was not the case for the series corresponding to $\lambda = 1$. The last terms in the expressions for s and t are both bounded, by the same argument we gave above, but neither expression contains a term that goes to infinity as n increases.

Indeed, these remarks about $\lambda = 1$ are true for all odd exponents. That is to say, if λ is odd, then neither s nor t increase without bound as n increases, but for λ even, the series behaves like $\lambda = 2$, and both s and t grow without bounds.

Euler notices this, too, and wants to examine it a bit. In Euler's time, there were several notions of the *value* of a series. One of them, proposed by Jakob Bernoulli, was that the value was the limit of the average value of the partial sums. Using this notion, the series

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - \text{etc.}$$

would have value equal to $^1/_2$, because half the time the partial sums are 1 and half the time they are zero. Hence, the weighted average value of the partial sums is $^1/_2$.

This is apparently the notion that justifies Euler's next steps. Taking $\lambda = 1$, he has shown that

$$s = \sin\varphi + \sin 2\varphi + \sin 3\varphi + \mathsf{L} + \sin n\varphi$$
$$= \frac{\sin\varphi + \sin n\varphi - \sin(n+1)\varphi}{2(1-\cos\varphi)}.$$

Euler argues that the average value of $\sin n\varphi - \sin(n+1)\varphi$ is zero, so, if we let n go to infinity the value of the now-infinite series s can be considered to be

$$s = \frac{\sin\varphi}{2(1-\cos\varphi)}.$$

The same analysis makes the infinite series

$$t = -\frac{1}{2}.$$

Modern analysts throughout the world cringe at this, because Euler has given an exact, finite sum to two series for which the terms do not converge to zero. The analysts don't let us do that anymore.

Perhaps Euler realizes we may have doubts about this particular result, for he reassures us that it is easy to show that this makes sense. He rewrites t as

$$t = \frac{\cos\varphi - 1}{2(1-\cos\varphi)}.$$

Multiplying both sides by $2 - 2\cos\varphi$ and writing t as the series it represents, Euler gets

$$\begin{aligned}\cos\varphi - 1 &= (2-2\cos\varphi)t \\ &= (2-2\cos\varphi)(\cos\varphi + \cos 2\varphi + \cos 3\varphi + \mathrm{L}) \\ &= 2\cos\varphi + 2\cos 2\varphi + 2\cos 3\varphi + 2\cos 4\varphi + \text{etc.} \\ &- 2\cos^2\varphi - 2\cos\varphi\cos 2\varphi - 2\cos\varphi\cos 3\varphi - \text{etc.}\end{aligned}$$

Now, from the angle addition formula for cosines we know that in general

$$2\cos a\cos b = \cos(a-b)\cos(a+b).$$

Applied to the negative terms of the preceding series, this makes

$$\begin{aligned}2\cos^2\varphi &= 1 + \cos 2\varphi, \\ 2\cos\varphi\cos 2\varphi &= \cos\varphi + \cos 3\varphi, \\ 2\cos\varphi\cos 3\varphi &= \cos 2\varphi + \cos 4\varphi, \\ 2\cos\varphi\cos 4\varphi &= \cos 3\varphi + \cos 5\varphi, \\ 2\cos\varphi\cos 5\varphi &= \cos 4\varphi + \cos 6\varphi, \\ 2\cos\varphi\cos 6\varphi &= \cos 5\varphi\cos 7\varphi, \\ &\text{etc.}\end{aligned}$$

Now, substituting these for those negative terms, and, at the same time rearranging the terms a bit, we get

$$\begin{aligned}\cos\varphi - 1 &= 2\cos\varphi + 2\cos 2\varphi + 2\cos 3\varphi + 2\cos 4\varphi + \text{etc.} \\ &- 1 - \cos\varphi - \cos 2\varphi - \cos 3\varphi - \cos 4\varphi - \text{etc.} \\ &- \cos 2\varphi - \cos 3\varphi - \cos 4\varphi - \text{etc.}\end{aligned}$$

Note how, when we substituted $1 + \cos 2\varphi$ for $2\cos^2\varphi$, we put the 1 in the second row of the new expression, and the $\cos 2\varphi$ in the third row. Likewise for all the other substitutions. As modern mathematicians, we benefit from the work of Cauchy and we know that such rearrangements of terms may not be valid unless the series involved are absolutely convergent, and that the series in question here are not absolutely convergent. Today, Euler would have to find another way to do this.

Getting back to our formulas, let's rewrite the preceding formula, aligned a bit differently, so that things that cancel can be seen more clearly. We get

$$\begin{aligned}\cos\varphi - 1 = {} & 2\cos\varphi + 2\cos 2\varphi + 2\cos 3\varphi + 2\cos 4\varphi + \text{etc.}\\ & - 1 - \cos\varphi - \cos 2\varphi - \cos 3\varphi - \cos 4\varphi - \text{etc.}\\ & - \cos 2\varphi - \cos 3\varphi - \cos 4\varphi - \text{etc.}\end{aligned}$$

which is clearly true.

This justifies Euler's claim that, for infinite values of n,

$$t = -\frac{1}{2}.$$

Euler thought he was finished, but the Editor's summary at the beginning of the volume of the *Novi commentarii* mentions that he later added an appendix "Summatio generalis infinitarum aliarum progressionum ad hoc genus referendarum" (Summation of infinitely many general progressions related to this kind). It contains a theorem and two examples.

Theorem. *If we know the sum of a progression*

$$Az + Bz^2 + Cz^3 + Dz^4 + \mathsf{L} + Nz^n,$$

then it always permits us to sum the two progressions

$$S = Ax\sin\varphi + Bx^2\sin 2\varphi + Cx^3\sin 3\varphi + \mathsf{L} + Nx^n\sin n\varphi$$

and

$$T = Ax\cos\varphi + Bx^2\cos 2\varphi + Cx^3\cos 3\varphi + \mathsf{L} + Nx^n\cos n\varphi.$$

The proof is straightforward, but in the course of the proof, Euler introduces the function notation as he generally uses it in the 1760s. When he writes

$$\Delta : z,$$

he means us to substitute the function defined by the progression

$$Az + Bz^2 + Cz^3 + Dz^4 + \mathsf{L} + Nz^n.$$

Though he had used the modern $f(x)$ function notation briefly in the 1730s, Euler did not stick with that notation, and from the 1760s until his death in 1783, he and his assistants used this notation with a symbol, usually an upper-case Greek letter, followed by a colon, and then the variable.

Then he notes that, with p and q as before, that is,

$$p = \cos\varphi + \sqrt{-1}\sin\varphi \quad \text{and}$$
$$q = \cos\varphi - \sqrt{-1}\sin\varphi,$$

his function notation gives

$$2S\sqrt{-1} = \Delta : px - \Delta qx \quad \text{and}$$
$$2T = \Delta : px + \Delta : qx.$$

Then he gives examples.

Example 1. If all the coefficients in $\Delta : z$ are equal to 1 and if the series is taken to be an infinite series, then

$$\Delta : z = \frac{z}{1-z}.$$

Then, from the equations in the proof of his theorem as well as the identities $p - q = 2\sqrt{-1}\sin\varphi$, $p + q = 2\cos\varphi$ and $pq = 1$, Euler gets that

$$S = \frac{x\sin\varphi}{1 - 2x\cos\varphi + x^2} \quad \text{and}$$
$$T = \frac{x\cos\varphi - x^2}{1 - 2x\cos\varphi + x^2}.$$

Typically, Euler checks that his result agrees with what he already knows. In a corollary, he finds that for the special case $x = 1$, this gives back the formulas from earlier in the paper, that

$$S = \frac{\sin\varphi}{2(1-\cos\varphi)} \quad \text{and}$$
$$T = -\frac{1}{2}.$$

As a second example, Euler takes

$$\Delta : z = \ln\frac{1}{1-z}$$
$$= z + \frac{1}{2}z^2 + \frac{1}{3}z^3 + \frac{1}{4}z^4 + \mathrm{L},$$

and he finds that

$$S = \arctan\frac{x\sin\varphi}{1 - x\cos\varphi} \text{ and}$$
$$T = -\frac{1}{2}\ln(1 - 2x\cos\varphi + x^2).$$

We leave the details to the reader.

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Gamma the Function

(September 2007)

Euler gave us two mathematical objects now known as "gamma." One is a function and the other is a constant. The function, $\Gamma(x)$, generalizes the sequence of factorial numbers, and is the subject of this month's column. A nice history of the gamma function is found in a 1959 article by Philip Davis, [D] and a shorter one is online at [Anon.]. The second gamma, denoted γ, is a constant, approximately equal to 0.577, and, if things go as planned, it will be the subject of next month's column. In 2003, Julian Havel wrote a book about gamma the constant. [H]

When Euler arrived in St.Petersburg in 1728, Daniel Bernoulli and Christian Goldbach were already working on problems in the "interpolation of sequences." Their problem was to find a formula that "naturally expressed" a sequence of numbers. For example, the formula n^2 "naturally expresses" the sequence of square numbers, 1, 4, 9, 16, ..., and $\frac{n(n+1)}{2}$ expresses the sequence of triangular numbers, 1, 3, 6, 10, 15, ... Both of these are well defined for fractional values of n, so they were said to *interpolate* the sequences.

Earlier mathematicians including Thomas Harriot and Isaac Newton had developed an extensive calculus of finite differences to help find formulas that matched various sequences of values, and their work helped lead to the invention of calculus. In fact, one way to understand the discovery of logarithms is that they resulted from the interpolation of geometric series.

Bernoulli and Goldbach were stumped trying to interpolate two particular sequences. The first was the sequence we now call the factorial numbers, 1, 2, 6, 24, 120, 720, etc. They called it the "hypergeometric progression." The second was the sequence of partial sums of the harmonic series, $1, 1+\frac{1}{2}, 1+\frac{1}{2}+\frac{1}{3}$, etc.

Shortly after he learned of the problems, Euler solved both of them. This month we are interested in his solution of the first one, in which he showed us how to give meaning to expressions like $\left(2\frac{1}{2}\right)!$, as a natural interpolation between $2! = 2$ and $3! = 6$.

Euler announced his solution in a letter to Christian Goldbach dated October 13, 1729. He began his letter, "Most Celebrated Sir: I have been thinking about the laws by which a series may be interpolated.... The most Celebrated [Daniel] Bernoulli suggested that I write to you." He goes on to proclaim that the general term of the "series" 1, 2, 6, 24, 120, etc. (at the time, people used the words series, sequence and progression interchangeably) is given by

$$\frac{1 \cdot 2^n}{1+n} \cdot \frac{2^{1-n} \cdot 3^n}{2+n} \cdot \frac{3^{1-n} \cdot 4^n}{3+n} \cdot \frac{4^{1-n} \cdot 5^n}{4+n} \text{etc.} \tag{1}$$

There is a subtle reason that Euler used the form given in (1) rather than the more "obvious" form

$$\frac{1}{1+n} \cdot \frac{2}{2+n} \cdot \frac{3}{3+n} \cdot \frac{4}{4+n} \text{etc.} \tag{2}$$

The reason involves absolute convergence of infinite products, and something called "Gauss's criterion." In fact, (1) converges as it is written, as the limit of its partial products, but to make (2) converge, we must interpret it as

$$\lim_{k \to \infty} \frac{1}{1+n} \cdot \frac{2}{2+n} \cdot \frac{3}{3+n} \cdots \frac{k}{k+n} \cdot k^n, \tag{3}$$

or else the limit will be zero. We won't go into details here, but instead refer the reader to Walker [W]. Euler clearly knew that something like Gauss's criterion was necessary when he made his definition, but then he doesn't use the criterion much in his exposition.

Euler's exposition in the letter of October 1729 is very brief, but he gave more details and consequences in an article, De progressionibus transcendentibus, seu quarum termini generals algebraice dari nequeunt, "On transcendental progressions, or those for which the general term is not given algebraically." [E19] In the article he tells us, without doing the calculations, that if n is 0 or 1, then the product is 1. For n equal to 2 and 3, he gives us a little more, telling us that

$$\frac{2 \cdot 2}{1 \cdot 3} \cdot \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{4 \cdot 4}{3 \cdot 5} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \text{etc., which he says equals 2,}$$

and

$$\frac{2 \cdot 2 \cdot 2}{1 \cdot 1 \cdot 4} \cdot \frac{3 \cdot 3 \cdot 3}{2 \cdot 2 \cdot 5} \cdot \frac{4 \cdot 4 \cdot 4}{3 \cdot 3 \cdot 6} \cdot \frac{5 \cdot 5 \cdot 5}{4 \cdot 4 \cdot 7} \cdot \text{etc., which equals 6.}$$

As a more complicated example, Euler takes $m = {}^1\!/_2$ and gets an infinite product:

$$\frac{1}{1+{}^1\!/_2} \cdot \frac{2}{2+{}^1\!/_2} \cdot \frac{3}{3+{}^1\!/_2} \cdot \frac{4}{4+{}^1\!/_2} \cdot \text{etc.}$$

This, in turn, equals

$$\frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \frac{8}{9} \cdot \text{etc.} \tag{4}$$

It seems hopeless to try to evaluate (4), but even at age 22, Euler had read a great deal of the mathematical literature. In particular, he knew that in 1665, John Wallis had found that

$$\frac{\pi}{4} = \frac{2 \cdot 4}{3 \cdot 3} \cdot \frac{4 \cdot 6}{5 \cdot 5} \cdot \frac{6 \cdot 8}{7 \cdot 7} \cdot \frac{8 \cdot 10}{9 \cdot 9} \cdot \text{etc.}$$

From this, it is not hard to find that the value given in (4) is $\frac{\sqrt{\pi}}{2}$. Philip Davis [D] speculates that Euler recognized a connection between π and areas and integration. Then, when he saw that the value of his infinite product involved π, he thought to try to rewrite the infinite product as an integral. After a good deal of work, described in detail both in [D] and in [S], he finds that his infinite product equals

$$\int_0^1 (\ln x)^n dx$$

and that this, too, is well defined for fractional values of n as well as for negative values of n that are not integers.

Near the end of the paper, Euler proposes an "application," though he admits that the example might not be very useful. He writes:

> "To round off this discussion, let me add something which certainly is more curious than useful. It is known that $d^n x$ denotes the differential of x of order n and if p denotes any function of x and dx is taken to be constant then ... the ratio of $d^n p$ to dx^n can be expressed algebraically. ... We now ask, if n is a fractional number, what the value of that ratio should be."

Euler is proposing that we use his new function to find what we now call "fractional derivatives," and he gives us some examples. For this, we will use modern notation, and use what we now call the gamma function, denoted $\Gamma(x)$. We note that if x is a nonnegative integer, then $\Gamma(x+1) = x!$. Let's have a look at some of the elementary properties of k-th derivatives of x^n and watch for a pattern:

first derivative	nx^{n-1},
second derivative	$n(n-1)x^{n-2}$,
third derivative	$n(n-1)(n-2)x^{n-3}$,
...	
kth derivative	$n(n-1)(n-2)\cdots(n-k+1)x^{n-k} = \dfrac{n!}{(n-k)!}x^{n-k}$,

where we have to be a little careful that $k \le n$ so that $n - k \ge 0$ and so $(n-k)!$ will be defined. This problem goes away if k is not an integer.

Armed with the gamma function, now we can define the kth derivative even if k is a fraction, as follows:

$$k\text{th derivative of } x^n \text{ is } \frac{\Gamma(n+1)}{\Gamma(n-k+1)}x^{n-k},$$

or, more like the way Euler wrote it,

$$\frac{d^k(z^e)}{dz^n} = z^{e-n}\frac{\int_0^1 (-\ln x)^e dx}{\int_0^1 (-\ln x)^{e-n} dx},$$

where, to Euler, e is just an exponent, and does not yet have today's connotations as a special constant.

Using this idea, Euler takes $n = 1$ and $k = {}^1\!/_2$, to find that the ${}^1\!/_2$th derivative of x is

$$\frac{\Gamma(2)}{\Gamma(3/2)}x^{1/2} = \frac{1}{\sqrt{\pi}/2}\sqrt{x} = \frac{2\sqrt{x}}{\sqrt{\pi}}$$

Euler does not take this any farther, but it is easy for us to see at least one way to do it. If we would like to take fractional derivatives of some more complicated function $f(x)$, then we can try to take the Taylor series for f, and apply Euler's fractional derivative formula to each term. Readers with the skills and the software are encouraged to experiment with this with their favorite mathematical software like Maple™ or Mathematica™. One good place to start might be to take a polynomial like $f(x) = (x-1)(x-2)(x-3)(x-4)(x-5)$ and build an animation of the graph of its kth derivatives, say for x between 0 and 6, as k increases from 0 to 5. We know that f itself has 5 roots, its first derivative has 4, its second has 3, etc. It is interesting to watch what happens to the roots as k increases, until, after five derivatives, all the roots disappear.

Trigonometric functions, like $f(x) = \sin x$ are also interesting.

Readers who know about the properties of Fourier series and Laplace transforms may know that they, too, can be used to define fractional derivatives.

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20

Gamma the Constant

(October 2007)

Sam Kutler, now retired from St. John's College in Annapolis, once pointed out that there are three great constants in mathematics, π, e and γ, and that Euler had a role in all three of them. Euler did not discover e or π, but he gave both of them their names. In contrast, Euler discovered, but did not name γ, the third and least known of these constants.

This γ is usually known as the Euler-Mascheroni constant, acknowledging both the work Euler did in discovering the constant in about 1734, (more on this later) and the work of Lorenzo Mascheroni (1750–1800). Mascheroni was a priest and a professor of mathematics at the University of Pavia in Italy. As Mascheroni studied Euler's books on integral calculus, he took careful notes and extended several of Euler's results, especially those involving the constant that now bears his name. Mascheroni published his notes in 1790 under the title *Adnotationes ad calculum integrale Euleri*. The editors of Euler's *Opera Omnia* have republished Mascheroni's *Adnotationes* as an appendix to the second volume of Euler's integral calculus in Series 1 volume 12 of the *Opera Omnia*. Mascheroni's *Adnotationes* are a model of a wonderful way to learn mathematics: find an excellent book on the subject and work through it, theorem by theorem, working examples, checking proofs, and extending results when you can. It doesn't make very exciting reading, though. It's a bit like reading someone else's homework assignments, watching that person struggle, but eventually master difficult concepts.

Euler made his first steps towards discovering gamma the constant in the same letter to Christian Goldbach dated October 13, 1729 in which he also first mentioned gamma the function. Goldbach and Daniel Bernoulli had been working on "interpolating a sequence" by finding a function that "naturally expresses" the sequence, and that is also defined for fractional values. We learned in last month's column how the gamma function interpolates the series of factorial numbers.

Bernoulli and Goldbach were also working, without much success, to interpolate the partial sums of the harmonic series. In modern notation, they were looking for a function $f(x)$ such that, if n is a positive integer, then $f(n) = \sum_{k=1}^{n} \frac{1}{k}$. In the letter, Euler hinted at his solution, claiming that he had found such a function, and that $f(\frac{1}{2}) = 2 - 2\ln 2$, but Euler did not give details. Instead, he published the details a few years later as *De summatione innumerabilium progressionum*, "On the summation of innumerably many progressions." [E20]

In his article, Euler asks us to look at the integral $\int_0^1 \frac{1-x^n}{1-x}dx$. If n is a positive integer, we can expand the integrand as a geometric series and get

$$\frac{1-x^n}{1-x} = 1 + x + x^2 + \cdots + x^{n-1}.$$

Integrating this gives

$$\begin{aligned}\int_0^1 \frac{1-x^n}{1-x}dx &= \int_0^1 (1 + x + x^2 + \cdots + x^{n-1})dx \\ &= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^n}{n}\right)\Big|_{x=0}^{x=1} \\ &= 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\end{aligned}$$

Taking $f(n) = \int_0^1 \frac{1-x^n}{1-x}dx$ gives the nth partial sum of the harmonic series. Since the integral is well-defined even if n is not an integer, we see that f is a function that interpolates the partial sums of the harmonic series. It is the function that Bernoulli and Goldbach had been unable to discover.

If we take $n = {}^1\!/_2$, we get, by a rather tricky bit of integration, the details of which we will omit, just like Euler did,

$$\begin{aligned}f(\frac{1}{2}) &= \int_0^1 \frac{1-\sqrt{x}}{1-x}dx \\ &= \int_0^1 \frac{1}{1-\sqrt{x}}dx \\ &= \left(2\sqrt{x} - 2\ln(1+\sqrt{x})\right)\Big|_{x=0}^{x=1} \\ &= 2 - 2\ln 2\end{aligned}$$

as Euler had claimed in his letter to Goldbach.

Euler tries a similar trick on the sum of the reciprocals of perfect, taking a stab at the Basel problem, but he gets an integral that he's unable to evaluate. He does manage to approximate it, though, and thus makes a major step towards his solution to the Basel problem, which he discovered just two years later. Details are in my book. [S]

In both of these projects, partial sums of the harmonic series, and the Basel problem, Euler notices how sums like $\sum_{k=1}^n f(k)$ are closely related to integrals $\int_0^n f(x)dx$, and that as n gets large, the difference between the two seems to converge to a constant.

In E20, Euler didn't follow up on this idea, but he soon returned to it with E25, *Methodus generalis summandi progressions*, "General methods of summing progressions." There he gives his first account of what we now call Euler-Maclaurin summation. Using Euler's notation, if s is the sum of a sequence the terms of which are "naturally expressed" by a function $t(n)$, then Euler found that

$$s = \int t dn + \alpha t + \frac{\beta dt}{dn} + \frac{\gamma d^2 t}{dn^2} + \frac{\delta d^3 t}{dn^3} + \text{etc.}$$

where the Greek letters are constants that will eventually turn out to be related to the Bernoulli numbers. It will take Euler 20 years to discover that relationship, though.

This formula tells us that the difference between a sum, s, and an integral, $\int t dn$, is equal to $\alpha t + \frac{\beta dt}{dn} + \frac{\gamma d^2 t}{dn^2} + \frac{\delta d^3 t}{dn^3} + \text{etc.}$ For some functions t, this error term converges quickly and is easy to estimate. Maclaurin was interested in using finite sums to approximate definite integrals, but Euler used improper integrals to approximate infinite series and definite integrals to approximate finite sums. Since the Euler's and Maclaurin's approaches were so different, the summation formula bears both their names, even though Euler found his version at least eight years before Maclaurin's work.

For Euler and Maclaurin, each series or function had its own error term, given by the formula above. They were able to calculate the error terms for particular cases. The error term for $1/x$ is approximately 0.577, the value we now call γ, while for $1/x^2$ it is approximately 0.645. Mascheroni put his own name next to Euler's in 1790 by calculating the error terms corresponding to many other series and functions, and showing how all the error terms involved the value γ.

Finally we turn to the question, who first called it γ? Havil [H, p. 90]. Dunham [D] and Glaisher [G] all tell us that it was Mascheroni. Twice I have checked the *Opera Omnia* edition of Mascheroni's *Adnotationes*, and Mascheroni consistently uses the symbol A. Jeff Miller [Mi] cites a source that says it was Euler in 1781 who first used γ. I have checked all of Euler's 1781 works, and I find him using A and C, but not γ.

260 15. **Bretschneider,** *theoriae logarithmi integralis lineamenta nova.*

7. $\mathrm{li}(1\pm x) = c + \int \frac{[l(1\pm x)]^0}{1!} \cdot \frac{\partial x}{x} + \int \frac{[l(1\pm x)]^1}{2!} \cdot \frac{\partial x}{x} + \int \frac{[l(1\pm x)]^2}{3!} \cdot \frac{\partial x}{x} + \dots$

Jam vero a cl. *Eulero* est demonstratum, esse

8. $(-1)^n \int_0^1 \frac{[l(1-x)]^n}{n!} \cdot \frac{\partial x}{x} = \frac{{}^n\mathfrak{F}_0}{n.n!} + \frac{{}^{n+1}\mathfrak{F}_1}{(n+1)(n+1)!} + \frac{{}^{n+2}\mathfrak{F}_2}{(n+2)(n+2)!} + \dots$

$= \sigma_{n+1} = \frac{1}{1^{n+1}} + \frac{1}{2^{n+1}} + \frac{1}{3^{n+1}} + \frac{1}{4^{n+1}} + \dots$

Adhibitis igitur in aequatione (7.) signis inferioribus et posito $x = 1$, invenitur

9. $c = \frac{1}{2}\sigma_2 - \frac{1}{3}\sigma_3 + \frac{1}{4}\sigma_4 - \frac{1}{5}\sigma_5 + - \dots,$

qui est valor quantitatis illius constantis, quam vir cl. *Kramp* in summatione seriei harmonicae investigavit et in numeros sequentes $\gamma = c =$ 0,577215 664901 532860 618112 090082 3.. computavit.

Miller and Glaisher also cite an 1835 article by Carl Anton Bretschneider [B], and, indeed, in the paragraph shown above, from page 260 of that article, we find Bretschneider using γ to denote the constant. We also see him citing Euler and, elsewhere on the page, Mascheroni, and the work of "vir cl. *Kramp*," who calculated γ to 31 decimal places. This is curious, since the value cited is the same as the one calculated by Mascheroni, given to 31 decimal places but correct only to 19 places. The value to 31 places generated by Maple™ is 0.577215 664901 532860 6.

So, who *really* first called it γ? As is the case for many questions of the form "who was the first to . . .?" I'm not sure, but I don't think it was Euler or Mascheroni. Though the history of the constant γ is confusing and riddled with errors, and the secondary sources disagree, [D, G, H, Mi] I think it was probably Bretschneider. He's not very famous, and perhaps he deserves to be known for this, if for nothing else.

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21

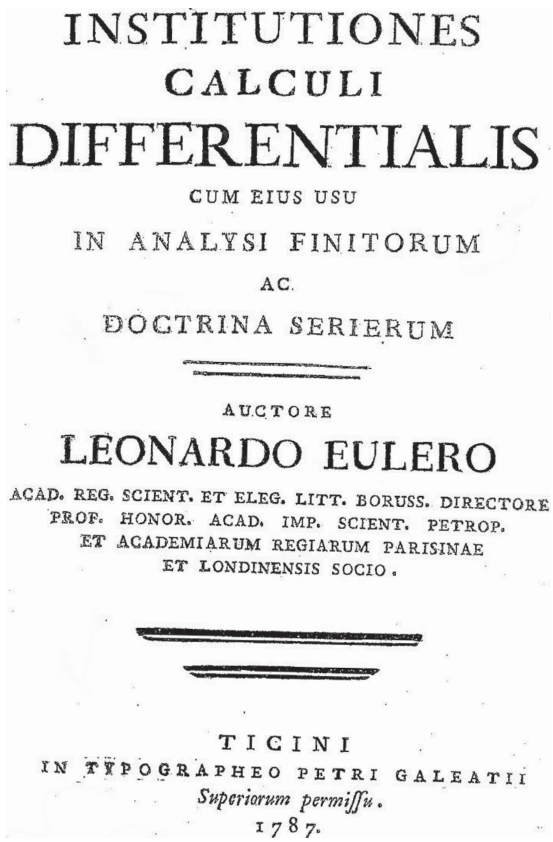
Partial Fractions

(June 2007)

Sometimes Euler has a nice sense of showmanship and a flair for a "big finish." At the end of a long, sometimes difficult work, he'll put a beautiful or particularly interesting result to reward his reader for making it clear to the end. He doesn't always do this, but when he does, it seems like a real treat.

For example, at the end of one of his papers on number theory [E228] in which Euler is studying numbers that are sums of two squares, he shows that 1,000,009, a number that had appeared on several lists of primes among the smallest seven-digit prime numbers, was in fact not a prime because it could be written as a sum of two squares two different ways, $1000^2 + 3^2$ and $235^2 + 972^2$, and he did this without giving the prime factorization of 1,000,009.

In another example of a big finish, Euler ends his first paper on the gamma function [E19] with an optimistic speculation that his "interpolation of the hypergeometric series," as he called it, could be used to define fractional derivatives, though he could not imagine what use they could possibly have.


INSTITUTIONES
CALCULI
DIFFERENTIALIS
CUM EIUS USU
IN ANALYSI FINITORUM
AC
DOCTRINA SERIERUM

AUCTORE
LEONARDO EULERO
ACAD. REG. SCIENT. ET ELEG. LITT. BORUSS. DIRECTORE
PROF. HONOR. ACAD. IMP. SCIENT. PETROP.
ET ACADEMIARUM REGIARUM PARISINAE
ET LONDINENSIS SOCIO.

TICINI
IN TYPOGRAPHEO PETRI GALEATII
Superiorum permissu.
1787.

This month's column is about a treat at the end of Euler's differential calculus book, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, "Lessons in differential calculus with its use in finite analysis and the study of series," *Calculi differentialis* for short. Euler wrote the book about 1750 and it was published in 1755. It is number 212 in Eneström's index. The title page of a 1787 edition published in Italy is shown on page 141.

Euler divides his book into two parts. The first part "contains a complete explanation of this calculus" and has been translated into English by John Blanton. It has nine chapters and was 278 pages in the 1755 edition, 213 pages in the *Opera omnia.* The second part "contains the use of this calculus in finite analysis and the study of series." It is another 18 chapters, 600 pages in the original and 459 pages in the *Opera omnia*. This part has not been translated into English. The whole volume in its original is almost 900 pages long, has no illustrations, exercises or applications to "real life" problems. And it only covers differential calculus. Euler published another *three* volumes on integral calculus about 15 years later. The whole set is more than 2500 pages long. And some people complain that modern calculus textbooks are too long.

In part 2 chapters 10 and 11 of the *Calculi differentialis*, Euler tells us about maxima and minima. Chapter 15 is titled *De valoribus functionum, qui certis casibus videntur indeterminate*, "On the values of functions, which in certain cases are seen to be indeterminant." By this he means L'Hôpital's rule, though he never mentions L'Hôpital himself.

Chapter 16 has the intriguing title *De differentiatione functionum inexplicabilium*, "On the differentiation of inexplicable functions." Euler doesn't tell us what an "inexplicable function" is, only what it is not. It is not rational or algebraic or given by any of the usual transcendental functions. He gives us two examples:

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x},$$

the partial sum of the harmonic series, interpolated so that x may not be a positive integer, and

$$1 \cdot 2 \cdot 3 \cdot 4 \cdots x,$$

the factorial numbers, again interpolated so that the function is defined even if x is not a positive integer. Euler began his studies of these two functions in 1729 in E19 and E20.

There is a beautiful surprise in chapter 16 while Euler is studying the function $S = 1 \cdot 2 \cdot 3 \cdot 4 \cdots x$. We would call this the gamma function today, with the small transformation that $\Gamma(x+1) = S(x)$. In one of his examples he finds an expression for $\frac{dS}{S}$ and notes that when $x = 0$ his expression gives

$$\frac{dS}{S} = -dx \cdot 0.577215664901325.$$

Just a page earlier he had identified this constant that begins 0.577 as the one that arises from comparing the ln n with the nth partial sum of the harmonic function, what we now call the Euler-Mascheroni constant and denote by the symbol γ. Since $S(0) = 1$, we can rewrite this using the modern Γ-notation as

$$\Gamma'(1) = -\gamma.$$

This result is the highlight of Emil Artin's classic little book *The Gamma Function* [A], and gives a beautiful and unexpected link between two different objects that share the name *gamma*. Euler knew the theorem in 1755.

This would have made a fine ending for the book, but Euler had a better one. Euler used a trick that Beethoven would use 50 years later in his *Fifth Symphony*: If you have two really great endings, use them both. Use the better one second.

So we get to chapter 18, the last chapter of the second part of *Calculi differentialis*, titled *De usu calculi differentialis in resolutione fractionum*, "On the use of differential calculus in the resolution of fractions." By this, Euler means what we now call "partial fractions." Euler had introduced partial fractions in his 1748 masterpiece, *Introductio in analysin infinitorum*, "Introduction to the analysis of the infinites," [E101, E102] which he regarded as a prerequisite for calculus, so he felt confident his readers would know about them. Today we usually see partial fractions as an integration technique, though they are gradually drifting out of the curriculum. Euler will use them as an integration technique too in his integral calculus textbooks, but here he is considering them as an application of differential calculus.

He reminds us that any rational function $\frac{P}{Q}$ can be rewritten as a sum of "simple fractions" for which the denominators are either irreducible factors of the denominator Q, or powers of those factors. Euler assumes without mentioning it that the degree of P is less than the degree of Q because he thought he had made that clear in the *Introductio*. He notes that the simplest case occurs when Q is a product of distinct linear factors because then $\frac{P}{Q}$ can be rewritten as a sum where the numerators are constants and the denominators are just those same linear factors. He further reminds us of the forms that the partial fractions will have if Q has some repeated factors, some factors that are irreducible quadratics, or even repeated irreducible quadratic factors.

Rather than telling us his technique right away, Euler leads us through its derivation. He takes $\frac{P}{Q}$ to be his rational function and assumes that Q has a simple (not repeated) factor $f+gx$. To make sure we understand what he means, he tells us that this implies that there is a polynomial S such that $Q=(f+gx)S$ and such that $f+gx$ is not a factor of S. He writes the simple fraction that arises from the factor $f+gx$ as $\frac{A}{f+gx}$, (though he uses the Fraktur alphabet where we use a common "A") and he lumps the rest of the partial fraction expansion into the rational function $\frac{V}{S}$. This makes

$$\frac{P}{Q}=\frac{A}{f+gx}+\frac{V}{S}.$$

Euler solves this for V to get

$$V=\frac{P-AS}{f+gx}.$$

Now for V to be a polynomial, the denominator on the right, $f+gx$, must divide the numerator, $P-AS$, which implies that if we substitute $x=\frac{-f}{g}$ into the numerator, it must vanish. But when $P-AS=0$ it implies that $A=\frac{P}{S}$. This gives us a way to find A in the numerator of $\frac{A}{f+gx}$, which is part of the partial fraction expansion of $\frac{P}{Q}$.

So, when $x = \frac{-f}{g}$

$$A = \frac{P}{S} = \frac{(f + gx)P}{(f + gx)S} = \frac{(f + gx)P}{Q}.$$

To make this meaningful in modern notation, we would have to take appropriate limits, but since the meaning is clear, we will stubbornly persist in using Euler's 18th century notation. But this last expression is indeterminate of the form $\frac{0}{0}$, so what Euler taught us in chapter 15 about L'Hôpital's rule applies. We can take differentials (as they always did in the 18th century. Derivatives came later.) We get

$$A = \frac{(f + gx)dP + Pgdx}{dQ}.$$

The first term in the numerator disappears because $f + gx = 0$, leaving us with

$$A = \frac{gPdx}{dQ}.$$

To summarize in modern notation, when $f + gx$ is a simple factor of the denominator Q of a rational function $\frac{P}{Q}$, then we can find the coefficient A in the term $\frac{A}{f+gx}$ of the partial fraction expansion of $\frac{P}{Q}$ by taking

$$\lim_{x \to \frac{-f}{g}} \frac{gP}{dQ/dx}.$$

Note that the factor S is used in the derivation, but it does not appear in the result itself, so we don't have to divide Q by $f + gx$.

This is Euler, so of course there are examples. Example 1 is to find the coefficient corresponding to the factor $1 + x$ in the rational function $\frac{x^9}{1+x^{17}}$. Here $P = x^9$, $Q = 1 + x^{17}$, and it isn't hard to find S if we wanted to. Also $f = g = 1$. Using modern notation we get

$$\begin{aligned} A &= \lim_{x \to -1} \frac{1.(x^9)}{d(1 + x^{17})/dx} \\ &= \lim_{x \to -1} \frac{x^9}{17x^{16}} \\ &= \frac{-1}{17} \end{aligned}$$

In Euler's second example, he finds that the coefficient corresponding to the factor $1 - x$ in $\frac{x^m}{1-x^{2n}}$ is $A = \frac{1}{2n}$.

There are other examples, and there are still a few details about repeated factors and irreducible factors in the denominator, but the clever reader can probably figure out what to do. The repeated root case, for example, begins by rewriting

$$\frac{P}{Q} = \frac{V}{S} + \frac{A}{(f + gx)^n} + \frac{B}{(f + gx)^{n-1}} + \frac{C}{(f + gx)^{n-2}} + \text{etc.}$$

and applying L'Hôpital's rule several times.

What a nice way to end a differential calculus book, with partial fractions as an application of derivatives, rather than as an integration technique. I conducted a brief survey of 20th and 21st century English language calculus textbooks and asked a number of colleagues. Only one colleague, educated in a highly competitive university in China, had seen this in his four-semester mathematical analysis course. For the rest of us, it has been forgotten. It might be nice to remember it.

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22

Inexplicable Functions

(November 2007)

Imagine my surprise when I was looking at Euler's *Calculi differentialis*. [E212] There, deep into part 2 (the part that John Blanton hasn't translated yet), I saw the odd title of chapter 16, *De differentiatione functionum inexplicabilium*, "On the differentiation of inexplicable functions." That made me curious. What was an "inexplicable function?"

In the nine chapters of part 1, Euler had taught us how to take derivatives of polynomials, of algebraic functions, of transcendental functions, to take higher derivatives, and to solve certain kinds of differential equations. Part 2 is about twice as long as part 1, both in number of pages (278 vs 602) and number of chapters (18 vs 9). The first nine chapters of part 2 mostly involve series. Chapters 10 to 13 are about applications of calculus to finding maxima and minima and to finding roots of equations, and the last five chapters seem to be a kind of grab bag of applications like interpolation and partial fractions. Inexplicable functions are in that grab bag.

Though he defines inexplicable functions as those that are neither algebraic nor transcendental, he doesn't have a very complete idea of what a transcendental function is, and he has only two kinds of examples of inexplicable functions. He might as well have defined inexplicable functions as being functions that are like his examples.

Both of his examples come out of the work on the interpolation of sequences that he presented in his letter to Goldbach of October 13, 1729. That work led to the results on the gamma function and the constant gamma that we have described in the last two columns.

Euler's first example generalizes the partial sums of the harmonic series. In this paper he calls all of his inexplicable functions S, but we'll call this particular one $H(x)$ and write it as

$$H(x) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x},$$

where x is not necessarily a whole number. Even though Euler had shown in 1729 [E20] that this could be defined as a definite integral

$$H(x) = \int_0^1 \frac{1 - y^x}{1 - y} dx,$$

he tells us here that this function "can not be explained in any way." This is probably because the *Calculi differentialis* was designed as a textbook, and its only prerequisite was his *Introductio in analysin infinitorum*. He did not expect that readers would know a result from a 25 year old research paper.

Euler also considers related sums like $1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2x-1}$ and $1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{x^n}$, where, again, x is not necessarily an integer.

It should be no surprise that Euler's second class of examples generalizes the factorial function. It isn't quite the gamma function, because the gamma function is a shift of the factorial function $\Gamma(n+1) = n!$. Euler denoted this by S as well, but we'll call his version $F(x)$. (F is for *factorial*, a word that was not used in Euler's time.) Euler's definition amounts to

$$F(x) = 1 \cdot 2 \cdot 3 \cdot 4 \cdots x,$$

where, again, x need not be a whole number. He also considers products like $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{7}{8} \cdots \frac{2x-1}{2x}$, and a couple of other examples built out of his two basic kinds of inexplicable functions. His fifth example is to differentiate our $F(x)$. It follows three examples based on sums and the one based on products.

Euler had to work without the benefits of subscript notations, which would not become popular for another 50 years, so much of his notation here will seem quite awkward. He took

$$S = A + B + C + D + \cdots + X,$$

where X is the value of the x-th term of the summation. Here, x is allowed to be a fraction, and for the generalized harmonic function, $X = 1/x$. He denotes by X', X'', X''', etc., the values of the $(x+1)$-st, $(x+2)$-nd, $(x+3)$-rd, etc., and he denotes the "term at infinity" by $X^{|\infty|}$. Then he writes successive sums as

$$\begin{aligned} S' &= S + X' \\ S'' &= S + X' + X'' \\ S'' &= S + X' + X'' + X''', \quad \text{and finally} \\ S^{|\infty|} &= S + X' + X'' + X''' + \cdots + X^{|\infty|}. \end{aligned}$$

We take ω to be an infinitely small number. Then, to take the derivative of the sum given by S, Euler wants to compare the sum of the first x terms to $\sum$, the sum of the first $x + \omega$ terms. Much like he did with the Ss and the Xs, he takes Z to be the term corresponding to $x + \omega$, and calls successive terms Z', Z'', Z''', and the term at infinity $Z^{|\infty|}$. Continuing his analogy between $\sum$ and S, he writes

$$\begin{aligned} \sum{}' &= \sum + Z' \\ \sum{}'' &= \sum + Z' + \sum{}'' \\ \sum{}''' &= \sum + Z' + Z'' + Z''', \quad \text{and finally} \\ \sum{}^{|\infty|} &= \sum + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}. \end{aligned}$$

Euler is not explicitly concerned with whether the series that give $S^{|\infty|}$ and $\sum^{|\infty|}$ converge. In fact, he knows that they diverge, but he is concerned with *how* they diverge. He tells us, "Now, the nature of the series S, S', S'', S''', etc., when it is continued to infinity, will be like an arithmetic progression if the sequence of terms X, X', X'', X''', etc. converges when it is continued to infinity."

That's not very clear, but it means that if $\lim_{n\to\infty} X^{(n)} = b$, then the sequence $X^{|\infty|}, X^{|\infty+1|}, X^{|\infty+2|}$, etc. is like an arithmetic sequence with difference b.

Euler denotes the value to which the Xs converge by $X^{|\infty+1|}$, and claims that $\sum^{|\infty|} = S^{|\infty+\omega|}$. Further, $S^{|\infty+\omega|}$ ought to lie naturally between $S^{|\infty|}$ and $S^{|\infty+1|}$. Since, at infinity, the Xs form an arithmetic sequence, Euler is comfortable interpolating $S^{|\infty+\omega|}$, giving it the value

$$\sum{}^{|\infty|} = S^{|\infty+\omega|} = S^{|\infty|} + \omega X^{|\infty+1|}.$$

Euler gathers his tools to get first that

$$\sum{}^{|\infty|} = S^{|\infty+\omega|} = S + X' + X'' + X''' + \cdots X^{|\infty|} + \omega X^{|\infty+1|},$$

and then that

$$\sum{}^{|\infty|} = \sum + Z' + Z'' + Z''' + \cdots + Z^{|\infty|}.$$

Together, these last two equations relate $\sum$ and S as

$$\sum = S + \omega X^{|\infty+1|} + X' + X'' + X''' + \cdots - Z' - Z'' - Z''' - \text{etc.} \tag{1}$$

Moreover, if the terms X, X', X'', X''' go to zero, that is to say if $X^{|\infty+1|} = 0$, then we get to ignore the term $\omega X^{|\infty+1|}$ in this expression.

We can do this calculation in the special case of Euler's generalized harmonic sum. Then we get

$$S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{x}.$$

Euler rearranges the terms in his formula for $\sum$ to get

$$\sum = S + (X' - Z') + (X'' - Z'') + (X''' - Z''') + \text{etc.}$$

Here, because $X^{|\infty+1|} = 0$, we can leave out the term $\omega X^{|\infty+1|}$.

Furthermore,

$$X' = \frac{1}{x+1} \qquad Z' = \frac{1}{x+1+\omega} \qquad X' - Z' = \frac{\omega}{(x+1)(x+1+\omega)}$$

$$X'' = \frac{1}{x+2} \qquad Z'' = \frac{1}{x+2+\omega} \qquad X'' - Z'' = \frac{\omega}{(x+2)(x+2+\omega)}$$

$$X''' = \frac{1}{x+3} \qquad Z''' = \frac{1}{x+3+\omega} \qquad X''' - Z''' = \frac{\omega}{(x+3)(x+3+\omega)}$$

etc. etc. etc.

In a transformation that seems to hold absolutely no hope of progress, Euler expands each of the factors $\frac{1}{x+k+\omega}$ as a Taylor series to get

$$\frac{1}{x+1+\omega} = \frac{1}{x+1} - \frac{\omega}{(x+1)^2} + \frac{\omega^2}{(x+1)^3} - \frac{\omega^3}{(x+1)^4} + \text{etc.},$$

$$\frac{1}{x+2+\omega} = \frac{1}{x+2} - \frac{\omega}{(x+2)^2} + \frac{\omega^2}{(x+2)^3} - \frac{\omega^3}{(x+2)^4} + \text{etc., and so forth.}$$

Finally, Euler substitutes these expansions in for $X' - Z'$, $X'' - Z''$, etc. in his expression for Σ, subtracts S, substitutes dS for $\Sigma - S$, and substitutes dx for ω to get his final, and rather disappointing answer:

$$\begin{aligned} dS = dx & \left(\frac{1}{(x+1)^2} + \frac{1}{(x+2)^2} + \frac{1}{(x+3)^2} + \frac{1}{(x+4)^2} + \text{etc.}\right) \\ & - dx^2 \left(\frac{1}{(x+1)^3} + \frac{1}{(x+2)^3} + \frac{1}{(x+3)^3} + \frac{1}{(x+4)^3} + \text{etc.}\right) \\ & + dx^3 \left(\frac{1}{(x+1)^4} + \frac{1}{(x+2)^4} + \frac{1}{(x+3)^4} + \frac{1}{(x+4)^4} + \text{etc.}\right) \\ & - dx^4 \left(\frac{1}{(x+1)^5} + \frac{1}{(x+2)^5} + \frac{1}{(x+3)^5} + \frac{1}{(x+4)^5} + \text{etc.}\right) + \text{etc.} \end{aligned}$$

The basic idea here was to make the step from S to $\sum$ by using the terms X', X'', X''', etc. to count up to infinity, to make the step from $X^{|\infty|}$ to $Z^{|\infty|} = X^{|\infty+\omega|}$ using the properties of the limit of the Xs, and then to count back from infinity using the terms Z', Z'', Z''', etc. It is clever, but outrageous.

As we mentioned earlier, Euler does a few more examples interpolating sums of sequences, including the special series closely related to the Riemann zeta function:

$$S = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots + \frac{1}{x^n}.$$

Before he leads us on to inexplicable functions built from products rather than sums, Euler does a bit more analysis to help us understand what happens if the terms of S neither vanish nor converge to some nonzero value. Formula (1) above describes the difference between S and $\sum$ if the terms converge, and if they converge to zero, then we get to ignore the term $\omega X^{|\infty+1|}$.

If the terms don't converge, then things are considerably more complicated. Let us look at the special case where the terms don't converge, but the difference between the terms does converge, as happens, for example, with the sequence whose general term is $\ln x$.

Consider three consecutive partial sums "at infinity" that Euler would denote as $S^{|\infty|}$, $S^{|\infty+1|}$ and $S^{|\infty+2|}$. Their first differences will be $S^{|\infty+1|} - S^{|\infty|} = X^{|\infty+1|}$ and $S^{|\infty+2|} - S^{|\infty+1|} = X^{|\infty+2|}$. Now we see that the second differences will be $X^{|\infty+2|} - X^{|\infty+1|}$, and we are assuming that this difference between the terms does converge. Note that the difference between the terms is the *second* difference between the partial sums.

Now we compare $S^{|\infty|}$to $\sum^{|\infty|}$ and get

$$\sum{}^{|\infty|} = S^{|\infty+\omega|} = S^{|\infty|} + \omega X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}(X^{|\infty+2|} - X^{|\infty+1|})$$

From this, we get a kind of second-difference analogue to Formula 1:

$$\begin{aligned}\sum &= S + X' + X'' + X''' + X'''' + \text{etc.}\\ &+ \omega X^{|\infty+1|} + \frac{\omega(\omega-1)}{1\cdot 2}(X^{|\infty+2|} - X^{|\infty+1|})\\ &- Z' - Z'' - Z''' - Z'''' - \text{etc.}\end{aligned}$$

For $X^{|\infty+1|}$, Euler substitutes $X' + (X'' - X') + (X''' - X'') + (X'''' - X''') + (X''''' - X'''') + \text{etc.}$, and for $X^{|\infty+2|} - X^{|\infty+1|}$ he writes $X'' - X' + ((X''' - 2X'' + X') + (X'''' - 2X''' + X'') + (X''''' - 2X'''' + X''') + \text{etc.})$. This gives him the astonishing but awkward formula,

$$\begin{aligned}\sum &= S + X' + X'' + X''' + X'''' + \text{etc.}\\ &+ \omega X' + \omega((X'' - X') + (X''' - X'') + (X'''' - X''') + (X''''' - X'''') + \text{etc.})\\ &+ \frac{\omega(\omega-1)}{1\cdot 2}X'' - \frac{\omega(\omega-1)}{1\cdot 2}X' \qquad (2)\\ &+ \frac{\omega(\omega-1)}{1\cdot 2}((X''' - 2X'' + X') + (X'''' - 2X''' + X'')\\ &+ (X''''' - 2X'''' + X''') + \text{etc.})\\ &- Z' - Z'' - Z''' - Z'''' - \text{etc.}\end{aligned}$$

Note that this applies to series for which the general terms need not converge, but the difference between consecutive terms do converge, and that $\ln 1 + \ln 2 + \ln 3 + \cdots + \ln x$ is one such series.

Similar, but even more complicated formulas are possible for sums for which the second differences or third differences of the general terms converge.

Now we are ready to sketch how Euler took the derivative of his version of the gamma function, the function we are writing as

$$F(x) = 1\cdot 2\cdot 3\cdot 4\cdots x.$$

First he takes logarithms of both sides and gets

$$\ln F(x) = \ln 1 + \ln 2 + \ln 3 + \ln 4 + \cdots + \ln x$$

In this series the terms do not vanish, but the differences between the terms do vanish. Euler demonstrates this with the following calculation:

$$\ln(\infty + 1) - \ln(\infty) = \ln\left(1 + \frac{1}{\infty}\right) = \frac{1}{\infty} = 0.$$

This gives Euler license to use his "astonishing but awkward" Formula (2). It seems to lead to mayhem as he substitutes dx for ω and then replaces terms like $X' - Z'$ with Taylor series approximations. Eventually it simplifies a bit and he gets

$$\begin{aligned}\ln F(x) = {} & x\left(\ln\frac{2}{1} + \ln\frac{3}{2} + \ln\frac{4}{3} + \ln\frac{5}{4} + \text{etc.}\right) \\ & - x\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \text{etc.}\right) \\ & + \frac{1}{2}x^2\left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \text{etc.}\right) \\ & - \frac{1}{3}x^3\left(1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.}\right) \\ & + \frac{1}{4}x^4\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \text{etc.}\right) + \text{etc.}\end{aligned} \tag{3}$$

Note the first two series on the right. The first series expands as

$$\ln 2 - \ln 1 + \ln 3 - \ln 2 + \ln 4 - \ln 3 + \ln 5 - \ln 4 + \text{etc.}$$

This telescopes to give

$$\ln 1 - \ln(n+1) = -\ln(n+1)$$

Meanwhile, the second series in Formula (3) is just the harmonic series. We know that as n goes to infinity, the difference between the logarithm and the nth partial sum of the harmonic series approaches the Euler-Mascheroni constant, now denoted γ. Euler knew this value to be approximately 0.5772156649015325, and he wrote it out like that rather than denoting it by a symbol. We will use γ.

As an additional concession to modern notation, we'll also note that the other series that appear in Euler's expression for $\ln F(x)$ are values of the Riemann zeta function, and we'll do what Euler couldn't do and write them as $\zeta(2)$, $\zeta(3)$, $\zeta(4)$, etc. With these notations, Euler's expression for $\ln F(x)$ can be written as

$$\ln F(x) = -x\gamma + \frac{1}{2}x^2\zeta(2) - \frac{1}{3}x^3\zeta(3) + \frac{1}{4}x^4\zeta(4) - \text{etc.}$$

Now for the climax. This differentiates to give

$$\frac{dF(x)}{F(x)} = -\gamma dx + x\zeta(2)dx - x^2\zeta(3)dx + x^3\zeta(4) - \text{etc.}$$

Taking $x = 0$ so that $F(0) = 0! = 1$, and this last formula gives the derivative of the factorial function in terms of the constant gamma:

$$\left.\frac{dF(x)}{dx}\right|_{x=0} = -\gamma,$$

or, in terms of the gamma function,

$$\Gamma'(1) = -\gamma.$$

This ends the story that has extended over our last three columns. Euler discovered both objects, gamma the function and gamma the constant, early in his career while working on problems in the "interpolation of functions," that is, giving meaningful values to functions that are initially defined only on the integers. Later, Euler showed that the derivative of the function at $x = 1$, is the negative of the constant. Though this result is widely known, it does not seem so well known that the result is due to Euler. Finally, well after Euler's death, the two different objects happed to be given the same name.

What a remarkable coincidence.

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23

A False Logarithm Series

(December 2007)

Solving a good research question should open more doors than it closes. One of Euler's lesser papers, *Methodus generalis summandi progressiones* ("General methods of summing progressions") [E25] is more noteworthy for the things it started than the things it finished. The principal role of the paper is as one of a sequence of papers that led to Euler's development of the Euler-Maclaurin summation formula. That sequence began in 1729 with a letter to Goldbach containing results that Euler later published in 1738 in [E20], and continued through [E25], [E43], [E46], [E47] up to [E55], *Methodus universalis series summandi ulterius promota*, written in 1736 and published in 1741.

This sequence of papers has a wonderful plot. First Euler examines the relations between the partial sums of the harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}$ and the logarithm function, $\ln n$. Then he sharpens these relations by using the fact that the function $1/x$ "naturally expresses" the terms of the harmonic series and that $\int_1^n \frac{1}{x} dx = \ln n$. He extends his results first to other more general series of reciprocals like $1 + \frac{1}{2^k} + \frac{1}{3^k} + \frac{1}{4^k} + \cdots + \frac{1}{n^k}$, then between partial sums of other series of reciprocals and their corresponding integrals $\int_1^n \frac{1}{x^k} dx$, and then to functions in general, $\sum_{i=1}^n f(i)$ and $\int_1^n f(x)dx$. It is delightful to watch the young Euler sharpen his tools and his insights from one year to the next between the years 1729 and 1736.

At the end of E25, though, Euler casts a glance in another direction, and poses two series for which the methods he had used to evaluate so many other series do not seem to work:

$$1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots + \frac{1}{2^n - 1} + \text{etc. and } \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{27} + \text{etc.}$$

The pattern for the second of these series is not obvious, but Euler explains that "the general term is $\frac{1}{a^\alpha - 1}$ where a and α denote integers greater than one." Goldbach had shown that this series sums to 1, and Euler expanded on Goldbach's technique in [E72], one of Euler's greatest papers, to discover the Euler product formula. See [S, BPV, D].

This month's topic, though is the first of these series. It comes up again almost 15 years later in [E190], *Consideratio quarumdam serierum quae singularibus proprietatibus sunt praeditae* ("Consideration of some series which are distinguished by special properties"). In E190, Euler studies the series

$$s = \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^3} + \frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} + \frac{(1-x)(a-x)(a^2-x)(a^3-x)}{a^6-a^{10}} \text{ etc.}$$

Here, the numerators add a factor of $a^n - x$ at each term, and the denominators involve exponents that are triangular numbers. In modern notation, one might write

$$s = \sum_{n=0}^{\infty} \frac{\prod_{k=0}^{n} (a^k - x)}{a^{T(n)} - a^{T(n+1)}}$$

where $T(n)$ is the nth triangular number, given by $T(n) = n \cdot (n+1)/2$.

It takes some thought to recognize what "special properties" might distinguish this series, even if we follow 18th century style and ignore questions of convergence. Adventurous readers may want to play with the series for a few minutes before reading on.

Note that if $x = a$, then the first term equals 1, and all the rest of the terms are zero, since they have a factor of $a - x$ in their numerators. Hence $s(a) = 1$.

Further, if $x = a^2$ then the first term reduces to $1 + a$ and the second term reduces to $1 - a$, while all the other terms vanish, so $s(a^2) = 2$. Similarly, Euler observes that $s(a^n) = n$ for positive whole numbers n, though Euler only shows us the calculations up to $n = 3$ and claims to have done them himself up to $n = 5$. Later in the article, Euler gives a proof that $s(a^n) = n$ when n is a positive integer, but at this point he gives only evidence. This evidence, though, naturally leads to the conjecture that $s(x) = \log_a x$.

But it isn't. Euler demonstrates this by showing that, for $a = 10$, $s(9) = 0.89705\ 05852\ 10673\ 21224$ but $\log 9 = 0.95424\ 2509$ (though the editors of the *Opera omnia* note that Euler made an error in his calculation of $s(9)$. It should be $0.89777\ 85865\ 88$. It still isn't log 9.) It is entertaining to check this using your favorite computer algebra system.

Euler apparently picked $a = 10$ and $x = 9$ rather charitably. In fact your computer algebra system can show that most of the time, $s(x)$ and $\log_a x$ are not very close together at all. Euler did not have a computer algebra system. He does have an easier way, not involving approximations, to show that $s(x)$ is not a logarithm function. He takes $x = 0$, and he "knows" that $\log 0 = -\infty$. On the other hand, for $x = 0$, his series gives

$$\frac{1}{1-a} + \frac{1}{1-aa} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \frac{1}{1-a^5} + \text{etc.}$$

This series has a finite sum, so $s(x)$ cannot be the logarithm function.

Note, though, that this series is the negative of one of the series that Euler proposed at the end of E25, and here again he tells us, "this series cannot be summed."

It must have both disappointed and excited Euler that $s(x)$ was not the logarithm function. On the one hand, if the series *were* the logarithm function, then it would have provided an unusually fast converging means of calculating logarithms. On the other hand, since $s(x)$ is *not* the logarithm function, it challenged one of his basic assumptions. He had two functions, $s(x)$ and $\log_a x$ that "naturally expressed" the same sequence of numbers; that is, they agreed at infinitely many values of x, yet they were not the same function.

This is in section 4 of this 32-section article. Euler spends most of the rest of this article studying properties of his series, showing how much it really does differ from the logarithm functions, and also showing rigorously that it does agree with the logarithm function at integer powers of a. We'll omit that, and refer the interested reader to the Mattmueller translation available on The Euler Archive. [E190]

Instead, we will leap forward to section 28, where Euler returns to the series from E25,

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.}$$

Of this, Euler writes (in the Mattmueller translation), " . . . for $a > 1$, even though it is finite and can easily be determined by approximations, [it] cannot be expressed neither in rational nor in irrational numbers. It appears therefore especially worth the effort that mathematicians investigate the nature of that transcendental quantity by which its sum is expressed." Here, Euler calls "irrational" what we would call "algebraic," though he uses the word "transcendental" in its modern sense.

Unable to express the series exactly, he sets out to give good approximations. He defines a new series s, not the same one he also denoted by s earlier in this article, as

$$s = \frac{1}{a-z} + \frac{1}{a^2-z} + \frac{1}{a^3-z} + \frac{1}{a^4-z} + \frac{1}{a^5-z} + \text{etc.}$$

Then the series Euler wants to approximate is the value of this series s when $z = 1$. He sets to work on this new series. Euler skips a few steps here that we will put in. The first term of this series can be expanded into a geometric series as

$$\frac{1}{a-z} = \frac{1/a}{1-z/a} = \frac{1}{a} + \frac{z}{a^2} + \frac{z^2}{a^3} + \frac{z^3}{a^4} + \text{etc.}$$

Likewise, the second term expands as

$$\frac{1}{a^2-z} = \frac{1/a^2}{1-z/a^2} = \frac{1}{a^2} + \frac{z}{a^4} + \frac{z^2}{a^6} + \frac{z^3}{a^8} + \text{etc.}$$

The other terms expand similarly. Euler "reverses the order of summation" to gather together like powers of z, and gives us s in a different form as

$$\begin{aligned} s = &\frac{1}{a} + \frac{1}{a^2} + \frac{1}{a^3} + \frac{1}{a^4} + \frac{1}{a^5} + \text{etc.} \\ &+ z\left(\frac{1}{a^2} + \frac{1}{a^4} + \frac{1}{a^6} + \frac{1}{a^8} + \frac{1}{a^{10}} + \text{etc.}\right) \\ &+ z^2\left(\frac{1}{a^3} + \frac{1}{a^6} + \frac{1}{a^9} + \frac{1}{a^{12}} + \frac{1}{a^{15}} + \text{etc.}\right) + \text{etc.} \end{aligned}$$

This done, Euler uses a trick that he had also used in E25. He knows that most of the error in the approximation of the sum of a series occurs in the first few terms of the series. To reduce this effect, he takes A to be the sum of the first n terms of s; that is

$$A = \frac{1}{a - z} + \frac{1}{a^2 - z} + \frac{1}{a^3 - z} + \frac{1}{a^4 - z} + \cdots + \frac{1}{a^n - z}.$$

This leaves

$$s = A + \frac{1}{a^{n+1} - z} + \frac{1}{a^{n+2} - z} + \frac{1}{a^{n+3} - z} + \frac{1}{a^{n+4} - z} + \text{etc.}$$

Euler again expands this into geometric series and collects like terms to get

$$\begin{aligned} s = A &+ \frac{1}{a^{n+1}} + \frac{1}{a^{n+2}} + \frac{1}{a^{n+3}} + \frac{1}{a^{n+4}} + \text{etc.} \\ &+ z\left(\frac{1}{a^{2n+2}} + \frac{1}{a^{2n+4}} + \frac{1}{a^{2n+6}} + \frac{1}{a^{2n+8}} + \text{etc.}\right) \\ &+ z^2\left(\frac{1}{a^{3n+3}} + \frac{1}{a^{3n+6}} + \frac{1}{a^{3n+9}} + \frac{1}{a^{3n+12}} + \text{etc.}\right) + \text{etc.} \end{aligned}$$

Each line in this expression is a geometric series, so he can sum those to get

$$s = A + \frac{1}{a^n(a-1)} + \frac{z}{a^{2n}(aa-1)} + \frac{z^2}{a^{3n}(a^3-1)} + \frac{z^3}{a^{4n}(a^4-1)} + \text{etc.}$$

For the series Euler proposed in E25, the case $a = 2$ and $z = 1$, this gives

$$s = A + \frac{1}{1 \cdot 2^n} + \frac{1}{3 \cdot 2^{2n}} + \frac{1}{7 \cdot 2^{3n}} + \frac{1}{15 \cdot 2^{4n}} + \frac{1}{31 \cdot 2^{5n}} + \text{etc.}$$

He takes $n = 4$ and so sums the first four terms of the series to get

$$A = 1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} = 1.542857142857141.$$

This is incorrect in the last decimal place, which should be a 3. That makes

$$s = A + \frac{1}{16 \cdot 1} + \frac{1}{16^2 \cdot 3} + \frac{1}{16^3 \cdot 7} + \frac{1}{16^4 \cdot 15} + \text{etc.}$$

Euler sums the first 15 terms of this series to get

$$s = A + 0.063638009558149$$
$$= 1.606695152415291$$

This agrees exactly with my computer algebra system for the infinite series.

This is the best Euler can do with that series from E25. He has one last remark, though. If we go back to the series from the beginning of section 28,

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \frac{1}{a^5-1} + \text{etc.}$$

and if we expand each of the terms as geometric series, then collect like powers of a, we get the form

$$s = \frac{1}{a} + \frac{2}{a^2} + \frac{2}{a^3} + \frac{3}{a^4} + \frac{2}{a^5} + \frac{4}{a^6} + \frac{2}{a^7} + \frac{4}{a^8} + \frac{3}{a^9} + \text{etc.},$$

where the nth numerator counts the number of divisors of n. Euler does not try to explain why this is true (though it is), but he does tell us that the numerator in the term $\frac{4}{a^6}$ is a 4 because the exponent 6 has four divisors, namely 1, 2, 3 and 6. For prime exponents, the numerator will always be 2, and for composite exponents it will always be greater than 2. These numerators are easy to calculate, and for the special case $a = 10$, it is easy for us to sum the series.

$$s = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999} + \text{etc.}$$

Euler gives the sum to 30 decimal places:

$$s = 0.122324243426244526264428344628.$$

It is clear that the series has number theoretic properties, but Euler did not pursue them any farther. Those properties are related to what we now call q-series. They were extensively studied by such great mathematicians as Gauss, Cauchy, Jacobi, Sylvester and Ramanujan and are still of great interest today.

I'd like to thank Warren Johnson, now at Connecticut College, for bringing this article to my attention, for helping me understand its connections with q-series, and for helpful comments on the text itself.

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24

Introduction to Complex Variables

(May 2007)

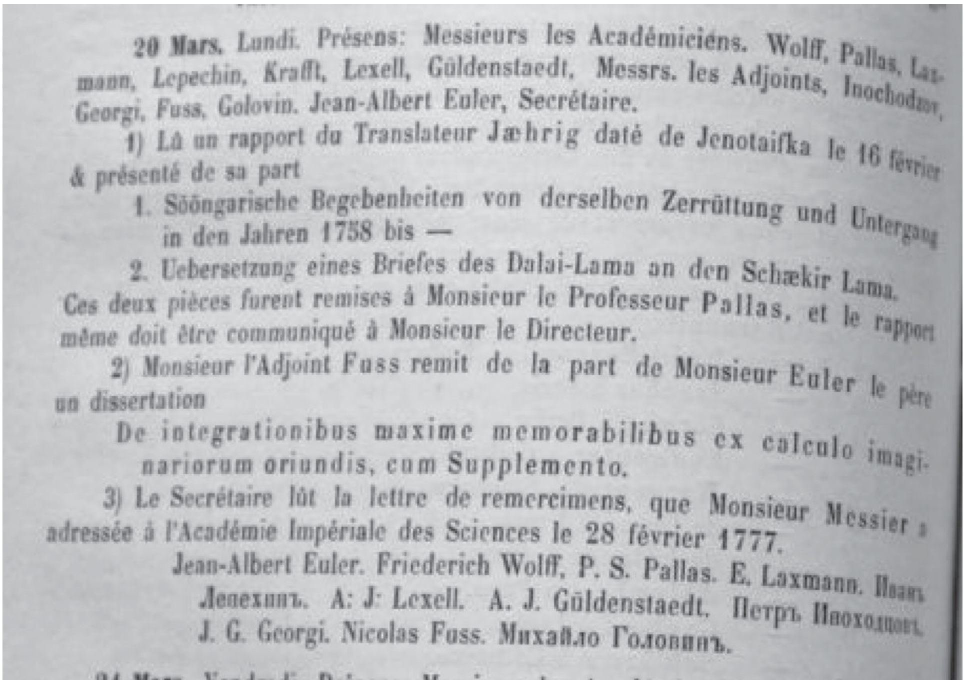
On Monday, March 20, 1777 the Imperial Academy of Sciences of St. Petersburg had one of its regular meetings. Except for holidays and occasional special meetings, they met twice a week on Mondays and Fridays, a total of 70 or 80 meetings per year.

This particular meeting wasn't much different from the other meetings they had that year, though it was a little shorter than most. The minutes from that meeting are shown in the photograph below. [SPA] It opened with a report from the Academy's translator, a Mr. Jaehrig, including his account of some letters between the Dalai Lama and the Sakya Trizin Lama, and it closed with the reading of a letter of thanks from someone named Monsieur Messier. In between, one of the Adjoint Members of the Academy, Nicolas Fuss, submitted on behalf of Leonhard Euler two articles, De integrationibus maxime memorabilibus ex calculo imaginariorum oriundis, (E656, "On some most memorable integrations arising from the calculus of the imaginaries") and its sequel, E657, Supplementum ad dissertationem praecedentem circa integrationem formulae $\int \frac{z^{m-1}dz}{1-z^n}$ casu, quo ponitur $z = v(\cos\varphi + \sqrt{-1}\sin\varphi)$, "Supplement to the preceding article about the integration of the formula $\int \frac{z^{m-1}dz}{1-z^n}$ by setting $z = v(\cos\varphi + \sqrt{-1}\sin\varphi)$"). Eleven days later, on March 31, Fuss brought five more articles, including one related to these two, Ulterior disquisitio de formulis integralibus imaginaries, (E694, "Later article on imaginary integral formulas").

The Academy's most famous member, Leonhard Euler, blind for more than five years, would turn 70 years old in just a month and seldom attended the regular meetings any more. Instead, he stayed at his home a few blocks from the Neva, the river through St. Petersburg, and his assistants went there to work with him. His assistants would do the actual writing, and they would sometimes work out the details of the calculations, but the articles were almost always published under Euler's name.

In 1777, Euler and his assistants sent more than 50 articles to the Academy. They would be "presented" to the Academy, that is, the manuscript was handed over to the Secretary of the Academy. Euler and company wrote articles too fast for the Academy's publishers, but those articles that were to be published without delay also had to be read aloud at a meeting of the Academy. These two articles weren't published until the 1789 issue of the Academy's journal, and that issue wasn't actually printed until 1793, 16 years after they were written, and these articles apparently escaped being read before the Academy.

So, what are these "most memorable integrations" of which Euler writes? The main point of the articles is to show that calculus with complex numbers is possible, and that it works a lot like calculus with real numbers.



20 Mars. Lundi. Présens: Messieurs les Académiciéns. Wolff, Pallas, Laxmann, Lepechin, Krafft, Lexell, Güldenstaedt. Messrs. les Adjoints, Inochodzow, Georgi, Fuss, Golovin. Jean-Albert Euler, Secrétaire.

1) Lû un rapport du Translateur Jæhrig daté de Jenotaifka le 16 février & présenté de sa part

1. Söongarische Begebenheiten von derselben Zerrüttung und Untergang in den Jahren 1758 bis —

2. Uebersetzung eines Briefes des Dalai-Lama an den Schækir Lama.

Ces deux pièces furent remises à Monsieur le Professeur Pallas, et le rapport même doit être communiqué à Monsieur le Directeur.

2) Monsieur l'Adjoint Fuss remit de la part de Monsieur Euler le père un dissertation

De integrationibus maxime memorabilibus ex calculo imaginariorum oriundis, cum Supplemento.

3) Le Secrétaire lût la lettre de remercimens, que Monsieur Messier a adressée à l'Académie Impériale des Sciences le 28 février 1777.

Jean-Albert Euler. Friederich Wolff. P. S. Pallas. E. Laxmann. Иванъ Лепехинъ. A: J: Lexell. A. J. Güldenstaedt. Петръ Иноходцовъ. J. G. Georgi. Nicolas Fuss. Михайло Головинъ.

Minutes of the Petersburg Academy from March 20, 1777

Euler begins asking us to consider a differential Zdz, where Z is a function of what he calls an "imaginary" quantity z. He writes its integral as $\int Zdz = \Delta : z$, where $\Delta : z$ is Euler's function notation. We would write $f(z)$ or $\Delta(z)$.

Now we separate everything in sight into real and imaginary parts. We take $z = x + y\sqrt{-1}$. (Euler and his students have not yet adopted the symbol i to denote $\sqrt{-1}$. They do that later in 1777.) Also, $Z = M + N\sqrt{-1}$ and $\Delta : z = P + Q\sqrt{-1}$.

Euler is very patient with us here, and explains that

$$Zdz = (dx + dy\sqrt{-1})(M + N\sqrt{-1})$$

and that the real and the imaginary parts (he uses those words) are $Mdx - Ndy$ and $(Ndx + Mdy)\sqrt{-1}$ respectively, and that $P = \int (Mdx - Ndy)$ and $Q = \int (Ndx + Mdy)$.

Euler's patience lapses for a moment here when he just tells us, without giving details, that "because of the integrability criteria" it follows that

$$\frac{\partial M}{\partial y} = -\frac{\partial N}{\partial x} \text{ and } \frac{\partial N}{\partial y} = \frac{\partial M}{\partial x}.$$

In fact, it is easy to derive these formulas. We need only know that the mixed partial derivatives of Z have to be equal, but it isn't as easy as the last few steps have been. He does this step in more detail 11 days later in his third article, E694.

These are, of course, the Cauchy-Rieman equations, used to such great effect two or three mathematical generations later by Augustin-Louis Cauchy (1789–1857) and Bernhard Riemann (1826–1866).

Now, Euler wants to show that certain calculus facts familiar for ordinary functions of real numbers are also true for complex numbers. He begins with $\int z^n dz = \frac{z^{n+1}}{n+1}$. As was customary at the time, Euler neglects the constant of integration unless he needs it.

In z^n, Euler substitutes $z = x + y\sqrt{-1}$, then expands the resulting binomial as

$$\left(x + y\sqrt{-1}\right)^n = x^n + \left(\frac{n}{1}\right) x^{n-1} y\sqrt{-1} - \left(\frac{n}{2}\right) x^{n-2} yy - \left(\frac{n}{3}\right) x^{n-3} y^3 \sqrt{-1} + \text{etc.}$$

Euler and his students had only recently started writing the binomial coefficients as $\left(\frac{n}{k}\right)$. Sometime over the next few decades, people started to omit the fraction bar, leaving us with the modern notation, $\binom{n}{k}$. We will use Euler's notation. Separating his expanded binomial into M, its real part, and N, the imaginary part, he gets

$$M = x^n - \left(\frac{n}{2}\right) x^{n-2} yy + \left(\frac{n}{4}\right) x^{n-4} y^4 - \left(\frac{n}{6}\right) x^{n-6} y^6 + \text{etc.}$$

and

$$N = \left(\frac{n}{1}\right) x^{n-1} y - \left(\frac{n}{3}\right) x^{n-3} y^3 + \left(\frac{n}{5}\right) x^{n-5} y^5 - \text{etc.}$$

He does a similar substitution and expansion with the right hand side, $\frac{z^{n+1}}{n+1}$, and separates it into its real and imaginary parts, P and Q, then multiplies by $n + 1$, to get what he thinks P and Q *should* be, if his integral formula is correct:

$$(n+1)P = x^{n+1} - \left(\frac{n+1}{2}\right) x^{n-1} yy + \left(\frac{n+1}{4}\right) x^{n-3} y^4 - \left(\frac{n+1}{6}\right) x^{n-5} y^6 + \text{etc.}$$

and

$$(n+1)Q = \left(\frac{n+1}{1}\right) x^n y - \left(\frac{n+1}{3}\right) x^{n-2} y^3 + \left(\frac{n+1}{5}\right) x^{n-4} y^5 - \text{etc.}$$

On the other hand, he knows that P *has to be* given by $P = \int (Mdx - Ndy)$. Substituting his expressions for M and N gives the rather formidable expression:

$$P = \int \left\{ \begin{array}{l} dx \left(x^n - \left(\frac{n}{2}\right) x^{n-2} y^2 + \left(\frac{n}{4}\right) x^{n-4} y^4 - \left(\frac{n}{6}\right) x^{n-6} y^6 + \text{etc.}\right) \\ -dy \left(\left(\frac{n}{1}\right) x^{n-1} y - \left(\frac{n}{3}\right) x^{n-3} y^3 + \left(\frac{n}{5}\right) x^{n-5} y^5 - \text{etc.}\right) \end{array} \right\}$$

Euler sets out to show that these two expressions for P are equal. He says that he will do this by integrating "by parts," but he really means that he will integrate parts of the actual value of P given in the integral, and then show that they equal the corresponding parts of the expression $(n + 1)P$.

For the term that has x^{n+1} in its integral, he gets

$$\int x^n dx = \frac{x^{n+1}}{n+1},$$

and that agrees with the corresponding part of $(n+1)P$.

For the terms that have x^{n-1} in their integral, he gets

$$-\int \binom{n}{2} x^{n-2}y^2 dx - \int \binom{n}{1} x^{n-1} y dy$$

and shows that these integrate to give their corresponding part of $(n+1)P$ as well. He continues, pairing $\int \binom{n}{4} x^{n-4}y^4 dx$ with $\int \binom{n}{3} x^{n-3}y^3 dy$, and then stops, saying that the pattern is clear. He also omits the details in showing that the expression that follows from $Q = \int (N dx + M dy)$ agrees with the expression given as $(n+1)Q$.

Euler agrees that that was hard work, and offers us an easier way. It is slightly less general, since the binomial series expansions can be made to work even if n is not an integer and his easier way only works when n is a positive integer. But it is considerably shorter and a good deal more elegant. From x and y, Euler creates two new variables, $v = \sqrt{xx + yy}$ and an angle φ chosen so that, as Euler writes it, "tang.$\varphi = \frac{y}{x}$." This makes

$$x = v\cos\varphi \text{ and } y = v\sin\varphi.$$

Their differentials are

$$dx = dv\cos\varphi - vd\varphi\sin\varphi \text{ and } dy = dv\sin\varphi + vd\varphi\cos\varphi.$$

Now Euler can use deMoivre's formula to get

$$(x + y\sqrt{-1})^n = v^n(\cos n\varphi + \sqrt{-1}\sin n\varphi).$$

This has real and imaginary parts M and N respectively equal to

$$M = v^n \cos n\varphi$$

and

$$N = v^n \sin n\varphi.$$

A similar calculation shows that, for $\int z^n dz = \frac{z^{n+1}}{n+1}$, we would have to have

$$P = \frac{v^{n+1}\cos(n+1)\varphi}{n+1} \text{ and } Q = \frac{v^{n+1}\sin(n+1)\varphi}{n+1}.$$

On the other hand, we know that $P = \int (Mdx - Ndy)$ and $Q = \int (Ndx + Mdy)$. Euler skips a few steps in his substitution. We'll skip some, too, but not as many as Euler did. For P, we get

$$\begin{aligned} P &= \int (Mdx - Ndy) \\ &= \int ((v^n \cos n\varphi)(dv\cos\varphi - vd\varphi\sin\varphi) - (v^n \sin n\varphi)(dv\sin\varphi + vd\varphi\cos\varphi)) \end{aligned}$$

Now, after a careful expansion and application of the trigonometric identities

$$\begin{aligned}\cos(n+1)\varphi &= \cos(\varphi + n\varphi)\\ &= \cos\varphi\cos n\varphi - \sin\varphi\sin n\varphi\end{aligned}$$

and

$$\begin{aligned}\sin(n+1)\varphi &= \sin(\varphi + n\varphi)\\ &= \sin\varphi\cos n\varphi + \cos\varphi\sin n\varphi\end{aligned}$$

we get that

$$P = \int \frac{v^{n+1}}{n+1}\cos(n+1)\varphi \text{ and } Q = \frac{v^{n+1}\sin(n+1)\varphi}{n+1},$$

as promised.

Indeed, Euler's second solution took him only a page of calculations to find both P and Q, whereas his first method had taken two and a half pages to find only the first three terms of P. Things aren't quite as rosy as he would have us believe, though, because he does skip a good number of easy but paper-consuming calculations.

We have described only the first five of the 44 pages of Euler's first paper. The second paper adds 17 pages, and the third another 18. Euler goes on (and on and on) to apply the same methods to integrate $\frac{dz}{1+zz}$ and get, as we would expect, $\tan^{-1}z$, and $\frac{dz}{1+z}$. He goes on to integrate the more general form $\frac{z^{m-1}dz}{1+z^n}$, which contains the last two as special cases.

Gradually, he comes to appreciate the power and convenience of the substitution $z = v(\cos\varphi + \sqrt{-1}\sin\varphi)$, and devotes the second paper to that substitution. The third paper gives Euler's derivation of the Cauchy-Riemann formulas in more detail, and then attacks some more general integrals like $\int \frac{z^{m-1}dz}{(a\pm bz^n)^\lambda}$.

Over the next few months, Euler expands his use of complex numbers in calculus, and in a paper [E671] presented to the Academy on May 5, 1777, as he is studying the integral of $\frac{d\varphi\cos\varphi}{\sqrt[n]{\cos n\varphi}}$, he writes that he will be using imaginary numbers and that "I will use i to denote $\sqrt{-1}$." His notation caught on.

On a technical level, we've seen exciting developments here. We see Euler discovering the Cauchy-Riemann formulas more than a decade before Cauchy was even born, and almost 50 years before Riemann, and we've done calculus with complex numbers.

Something has happened on a philosophical level as well. For most of his life, Euler was content to use a principle that Leibniz had called the Principle of Continuation. This said, roughly, that similar things ought to behave similarly. This gave Euler reason to use the same rules of calculation with infinite and infinitesimal numbers that he used for finite numbers and to treat solid bodies as if they were point masses. The Principle of Continuation *should* have allowed Euler to integrate complex functions just like he integrated real ones.

We can only speculate why Euler chose to try to be analytically rigorous when writing about complex variables. Perhaps he wrote this paper to explain the use of complex numbers to his students, especially to Nikolas Fuss. That could also explain why Euler was sometimes very careful about including details, so that his students would understand, but at other times he skipped them, to leave gaps for his students to fill in.

On the other hand, perhaps he or the people he was writing for did not agree that complex numbers, or, as he called them, *imaginary* numbers, are similar to real numbers, so he was not comfortable applying the Principle of Continuation.

We won't even worry about why Cauchy and Riemann got their names on those equations instead of Euler.

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The Moon and the Differential

(October 2009—A Guest Column by Rob Bradley)

Euler's output was split fairly evenly between pure and applied mathematics, the latter including many topics that we would today classify as physics. Most of his papers fall decisively into one category or the other, but it wasn't at all rare for one of his works of applied mathematics to include new techniques or results in analysis. This frequently happened in the Paris Prize competition, for example, where the questions were generally of a practical nature. This month, we'll look at an astronomical paper [E401] that proposes numerical techniques for approximating a body's velocity and acceleration. Remarkably, one of the results in E401 was probably the first step in the development of the calculus of operations and seems to have influenced Lagrange's foundational program for the calculus.

Euler read E401, "A New Method for Comparing Observations of the Moon to the Theory," to the Berlin Academy on February 6, 1766, just a few months before his return to St. Petersburg. Because he quoted astronomical data from the summer of 1765, it's nearly certain that his results date that year. Nevertheless, the paper appeared in the Berlin Academy's volume for 1763, which was only published in 1770. Euler probably had it included in this volume, despite its later composition date, because it's a follow-up to E398, "A New Method for Determining the Perturbations in the Motion of Heavenly Bodies Caused by their Mutual Attraction," which he read to the Academy on July 8, 1762. Because of the long publication delay in the 1763 volume, Euler was able to arrange matters so that E398 was immediately followed by three papers that build upon it results: E399, read on December 18, 1763, which applies the methods E398 to the moon, E400, read on December 4, 1765, which considers the general three body problem, and E401.

Euler begins E401 by summarizing what he had done in E398. Supposing

> ". . . first that both the position of the body in question, as well as its motion, i.e. its speed and direction, are exactly known for a given epoch; and further, knowing at the same time the accelerations produced by the forces exerted on the body, I showed how we may assign the position and motion of the body from this, not just an instant later, but for a considerable enough time after the first instant."

Euler observes that it would be quite useful to apply this method to the determination of the motion of the moon and the construction of lunar tables. However, he says that he initially despaired of ever being able to measure the moon's velocity with sufficient accuracy, having "neither the fortitude nor the patience to undertake a task of this kind; but... I found a method, by using various observations of the moon, made on several consecutive days, to ascertain for each one the true speed and direction of the moon." That is, he had figured out a numerical method for determining the moon's velocity (and for that matter its acceleration) from a sequence of observations of the moon's position. He describes this in the following proposition.

Lemma. *"For the abscissas $\zeta = 0, \zeta = 1, \zeta = 2, \zeta = 3, \zeta = 4,$ etc., knowing their ordinates p, q, r, s, t, etc. on a curve, it is required to find the differential values both of the first degree $\frac{\mathrm{d}p}{\mathrm{d}\zeta}, \frac{\mathrm{d}q}{\mathrm{d}\zeta}, \frac{\mathrm{d}r}{\mathrm{d}\zeta}, \frac{\mathrm{d}s}{\mathrm{d}\zeta}$, etc., as well as the second degree $\frac{\mathrm{dd}p}{\mathrm{d}\zeta^2}, \frac{\mathrm{dd}q}{\mathrm{d}\zeta^2}, \frac{\mathrm{dd}r}{\mathrm{d}\zeta^2}, \frac{\mathrm{dd}s}{\mathrm{d}\zeta^2}$, etc., taking the differential $\mathrm{d}\zeta$ to be constant."*

Euler's notation may seem a little strange to modern readers. The letters p, q, r, etc., don't represent different functions, only different values of a given function, which we might call $z = f(\zeta)$. Therefore we would write something more like

$$\left.\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=0}, \quad \left.\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=1}, \quad \left.\frac{\mathrm{d}z}{\mathrm{d}\zeta}\right|_{\zeta=2}, \ldots$$

where Euler has written $\frac{\mathrm{d}p}{\mathrm{d}\zeta}, \frac{\mathrm{d}q}{\mathrm{d}\zeta}, \frac{\mathrm{d}r}{\mathrm{d}\zeta}$, etc. It's even clearer in Lagrange's derivative notation, not yet invented in 1765: Euler is simply trying to find $f'(0), f'(1), f'(2), f'(3), \ldots$ as well as $f''(0), f''(1), f''(2), f''(3), \ldots$.

Euler's solution to the problem posed in this lemma begins with some notation. He defines z as we have done, then he sets $q - p = \Delta p, r - 2q + p = \Delta^2 p, s - 3r + 3q - p = \Delta^3 p$, etc. If we let $z_n = f(n)$, then these are the forward differences $\Delta z_0, \Delta^2 z_0, \Delta^3 z_0, \ldots$. "Given this", says Euler, "we know that we have:

$$\begin{aligned} z &= p + \Delta p \cdot \frac{\zeta}{1} + \Delta^2 p \cdot \frac{\zeta(\zeta-1)}{1.2} + \Delta^3 p \cdot \frac{\zeta(\zeta-1)(\zeta-2)}{1.2.3} \text{ etc., or} \qquad (1) \\ &= p + \Delta p\zeta + \frac{1}{2}\Delta^2(\zeta\zeta - \zeta) + \frac{1}{6}\Delta^3 p(\zeta^3 - 3\zeta^2 + 2\zeta) \\ &+ \frac{1}{24}\Delta^4 p(\zeta^4 - 6\zeta^3 + 11\zeta^2 - 6\zeta) \\ &+ \frac{1}{120}\Delta^5 p(\zeta^5 - 10\zeta^4 + 35\zeta^3 - 50\zeta^2 + 24\zeta) \\ &\quad \text{etc.''} \end{aligned}$$

Equation (1) is sometimes called Newton's Forward Difference Formula. Many of us have come across it in a numerical methods course, see e.g., [Burden 2001, p. 127], where the name is usually applied to an interpolating polynomial, rather than an infinite series. If $\{x_n\}$ is a sequence with constant differences $h = \Delta_{x_i}$, then

$$p_n(t) = \sum_{k=0}^{n} \tbinom{t}{k} \Delta^k f(x_0) \qquad (2)$$

is the unique polynomial of degree $\leq n$ with the property that $p_n(k) = f(x_k)$ for $k = 0, 1, 2, \ldots, n$. When $x \in [x_0, x_n]$ and $x = x_0 + th$, then $f(x) \approx p_n(t)$. In Euler's application, $x_0 = 0$ and $h = 1$, so that t in equation (2) is just his variable ζ. Furthermore, if $f(x)$ is a well-behaved function, then under certain conditions we will have $p_n(t) \to f(x)$ as $n \to \infty$, which more or less justifies Euler's claim in equation (1).

Next, Euler differentiates equation (1) to get

$$\begin{aligned}\frac{\mathrm{d}z}{\mathrm{d}\zeta} &= \Delta p + \frac{1}{2}\Delta^2 p(2\zeta - 1) + \frac{1}{6}\Delta^3 p(3\zeta\zeta - 6\zeta + 2) \\ &\quad + \frac{1}{24}\Delta^4 p(4\zeta^3 - 18\zeta^2 + 22\zeta - 6) \\ &\quad + \frac{1}{120}\Delta^5 p(5\zeta^4 - 40\zeta^3 + 105\zeta^2 - 100\zeta + 24) \\ &\quad \text{etc.,}\end{aligned}$$

an expression that is sometimes called Markoff's Formula [MathWorld]. Euler then differentiates a second time to find a similar expression for $\frac{\mathrm{dd}z}{\mathrm{d}\zeta^2}$.

Finally, Euler substitutes $\zeta = 0, \zeta = 1, \zeta = 2$, etc., to find the first and second order differential quantities he set out in the statement of the lemma. The first of these is

$$\frac{\mathrm{d}p}{\mathrm{d}\zeta} = \Delta p - \frac{1}{2}\Delta^2 p + \frac{1}{3}\Delta^3 p - \frac{1}{4}\Delta^4 p + \frac{1}{5}\Delta^5 p - \text{etc.} \tag{3}$$

For good measure, he also derives formulas for the third and fourth derivatives.

Now let's skip ahead about half a century, to 1814. In that year, François-Joseph Servois (1768–1847) published a paper called "Essay on a new method of exposition for the differential calculus" [Servois 1814a] in *Annales de mathématiques pures et appliqués*. Often called "Gergonne's *Annales*" after its editor, this was the first journal ever to be devoted entirely to mathematics. Servois' paper contained the following remarkable *definition*:

$$\Delta z - \frac{1}{2}\Delta^2 z + \frac{1}{3}\Delta^3 z - \cdots = dz, \tag{4}$$

for an arbitrary function z. "This is the complete definition of a new function of z," says Servois, "polynomial or even *infinitinomial*,[1] in general, which I call the *differential*."

In this paper, Servois was grappling with the foundational problem of the calculus. At the beginning of the 19th century, there were three competing foundational schools on the European Continent: those who thought that differentials were acceptable or at least could be made suitably rigorous, those who wanted to base calculus on the limit—still an informal notion at that time—and a third group who, following Lagrange (1736–1813), defined derivatives not via limits, but rather through the coefficients of a function's power series expansion. Servois was a disciple of Lagrange and his paper was full of formal series manipulations, including a derivation of the expansion (1). Although he was reasonably

[1] Servois coined this term (*infinitinôme*) here. Although this word never caught on, he also introduced the mathematical terms "distributive" and "commutative" in this paper.

sympathetic to the limit approach, as he demonstrated in a philosophical essay that followed immediately in the pages of Gergonne's *Annales* [Servois 1814b], he wanted to banish the infinitely small from mathematics. However, he recognized that the use of differentials was deeply ingrained in mathematical practice, so he sought to explain them here through formal operations rather than through an appeal to some vague notion of infinitely small quantities.

Servois' formula (4) gives dz in terms of a constant increment in the independent variable, which he denoted by a Greek letter such as α. If we call it $d\zeta$ instead and formally divide it through both sides of (4), we get Euler's formula (3). It's extremely unlikely that Servois gleaned his definition of the differential directly from E401, although his publication record makes it quite clear that he was very familiar with Euler's works. Rather, the line from formula (3) to definition (4) passes through the works of Lagrange, Arbogast (1759–1803) and Jacques Français (1775–1833). Arbogast and Français established the calculus of operations, in which operators were abstracted from the functions to which they were applied, so that a formula like (3) could be re-written as

$$D = \Delta - \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 - \frac{1}{4}\Delta^4 + \frac{1}{5}\Delta^5 - \cdots$$

where D is the derivative operator. Because the right hand side has the form of the power series for the natural log, Français wrote Euler's formula as

$$D = \ln(1 + \Delta) \tag{5}$$

and Servois said that the differential is the logarithm of what he called the "varied state," i.e. the forward increment operator that maps z to $z + \Delta z$.

Ivor Grattan-Guinness traces this evolution in [Grattan-Guinness 1990, pp. 161–163, 211–219]. The next step after E401 was taken by Lagrange, who succeeded Euler at the Berlin Academy. In [Lagrange 1774], he not only generalized Euler's formula (3) to the multivariable case, but he produced its dual, by showing that

$$\Delta z = f(\zeta + h) - f(\zeta) = e^{h\frac{dz}{d\zeta}} - 1$$

where h, the increment for the difference operator, was taken by Euler to be 1 in E401. Français could derive the corresponding $\Delta = e^D - 1$ by formally solving relation (5) for Δ, but Lagrange derived his result from the Taylor series expansion of the function z.

We should note carefully that none of these later theoretical consequences of E401 were foreshadowed in any way by Euler himself. He must have observed the elegance of the expression (3) of the derivative in terms of differences and noticed the analogy with the logarithm series. However, in this paper it was just a means to a practical end: the construction of accurate lunar tables and, by extension, progress on the Longitude Problem.

To illustrate the use of his newly discovered tool, Euler gathered astronomical data from the lunar tables of Jérôme LaLande (1732–1807) for Paris on six consecutive days, July 31 through August 5 of 1765. His coordinate system takes the center of the earth as the origin, with the plane of the ecliptic as the xy-plane and the positive x-axis pointing in the direction of the vernal equinox. With units chosen so that the mean distance from the earth to the sun is 100,000, the position (x, y, z) of the moon at noon, ζ, days after August 1, 1765, is approximately given by the quadratic formulas

$$x = 166.970 + 36.090\zeta - 5.104\zeta\zeta$$

$$y = -184.039 + 40.316\zeta + 5.618\zeta\zeta$$

$$z = -9.7545 + 5.1982\zeta + 0.3020\zeta\zeta$$

As another consequence of his methods, Euler calculates the ratio of the sun's mass to that of the earth to be 309,108, assuming solar parallax to be $9''$ of arc. This compares reasonably well with the currently accepted figure of 332,830 and would have been much closer had he used the currently accepted value of 8.794 seconds of arc for solar parallax.

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Part V

Applied Mathematics

26

Density of Air

(July 2009)

Leonhard Euler did an immense amount of work in optics, but that work is not very well known among mathematicians. Seven volumes in Series III of the *Opera omnia* are devoted to Euler's optics, two volumes to his 1769 book the *Dioptricae* and five volumes containing the 56 papers he wrote on the subject. All but six of those papers were published during Euler's lifetime, evidence of how important his work was considered at the time. Several of Euler's *Letters to a German Princess* were devoted to optics as well.

The two volumes of the *Dioptricae* and the 56 papers fill 2522 pages of the *Opera omnia*. By comparison, Euler's work on number theory consists of 96 papers, no books, taking up 1955 pages in four volumes of Series I of the *Opera omnia*.

One of Euler's earliest optics papers is also one of his best known, *Nova theoria lucis et colorum*, "A new theory of light and color" [E88] in which he criticized Isaac Newton's "projectile" theory of light, whereby luminous sources emitted streams of rapidly moving corpuscles and proposed instead that light was rather like sound. Just as sound is a disturbance transmitted by waves in the air, so also, Euler suggested, light is a disturbance in the aether, which Euler and the other scientists of his times believed was a subtle substance filling the universe. [Sandifer Feb 2008] Euler was one of the first to propose a wave theory of light, and his theory was quite influential up until the time of Einstein. [Hakfoort 1995, Home 2007]

The *Dioptrica* is a textbook on optics that leads the reader through the theory of refraction and the function of lenses and how to use the theory to design telescopes and microscopes. Except for being written in Latin and having no problem sets at the end of each chapter, it is much like a modern text on the subject.

In our next two columns, we are going to examine one of Euler's papers on how the Earth's atmosphere refracts light, *Sur l'effet de la réfraction dans les observations terrestres*, "On the effect of refraction on terrestrial observations," [E502] written about 1777 and published in 1780. The paper is quite similar to one of Euler's earlier papers, *De la réfraction de la lumière en passant par l'atmosphere selon les divers degrés tant* *de la chaleur que de l'élasticité de l'air*, "On the refraction of light passing through the atmosphere due to the different degrees of heat and elasticity of the atmosphere," [E219] written in 1754 and published in 1756. The earlier article applies to celestial observations, for which the light passed through the full depth of the atmosphere. The later article applies to terrestrial observations passing from one altitude to another, say observing a taller mountain from a lower one or observing a ship at sea from a mountain on shore.

Euler begins his article by explaining that,

> "Because of refraction, the stars appear to us higher above the horizon than they really are. We also encounter this same phenomenon in terrestrial observations, where objects always appear higher to us than they would if it were not for refraction. The reason for this is that the rays of light do not always go in straight lines to our eyes, as we ordinarily suppose, but they are found to be a little bit curved, and their concavity is turned downward. . . . I propose to explain here this effect of refraction and to determine all the phenomena that result from it."

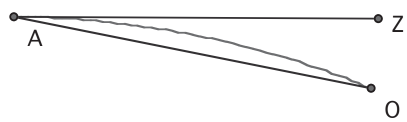

Fig. 1

Euler asks us to look at Fig. 1, showing an object at point O from which a ray arrives at point A following the curved line OA. The direction of this ray at A is the straight line AZ, which is tangent to the curve at A. To the observer's eye, it appears that the object is somewhere along the tangent AZ and as a consequence it appears to be higher or taller than it actually is.

From here, Euler's article divides naturally into two parts, the first about how the density of the atmosphere varies with altitude and the second about how that variation on density causes the refraction Euler describes. We divide our two columns along the same lines. In this month's column, we will see how Euler sets up and solves the differential equation that describes the density of the atmosphere as a function of altitude. Next month we will learn about refraction and see how Euler develops tables that correct for that refraction.

Euler tells us that because the curvature of light rays "is caused by the different densities of the atmosphere between the object at Z and the eye at A, it is necessary to start by determining the law followed by the density of the air as it decreases with altitude." Note that he now locates the object at Z instead of at O, as he had in Fig. 1. He makes the simplifying assumption that "at equal heights above the surface of the earth, the density of the air is always the same." Because he means to apply this model to line-of-sight observations on the surface of the earth, the errors introduced by this assumption will probably compensate. It will turn out that the *change* in air density will have a bigger impact than the absolute density itself.

In what follows, Euler will use the symbol q to denote two different things, the density of air at a point Q and also a point infinitely close to that point Q. This might be inattentiveness on the part of Euler or his assistants, or he may think that the meaning of the symbol is clear enough from its context and there's no need to introduce a different symbol.

Referring to Fig. 3 (we skip over Fig. 2), Euler takes c to be the density of air at A, the observer's location. He lets q denote the density of air at some point Q directly above A and he takes x to be the height AQ. He takes the air pressure at A, measured by the height of mercury in a barometer, to be k and the pressure at Q to be p. Next he cites what we now call Boyle's Law, that the density of the atmosphere is proportional to the air pressure and gets the formula

$$\frac{q}{c} = \frac{p}{k}.$$

Now Euler plans to do calculus. Following his usual 18th century procedures, he takes another point q infinitely close to Q and takes $Qq = dx$. Then the density at q will be $q + dq$ and the height of the barometer, that is to say the air pressure, will be $p + dp$, where, in Euler's words, "it is clear that the differentials dq and dp will have negative values." Now the problem is to find the relations among these three differentials.

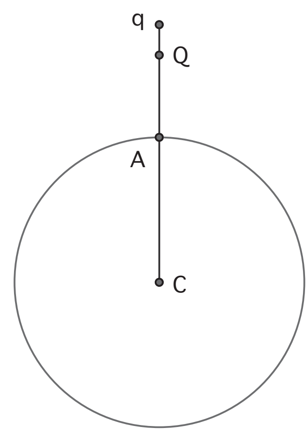

Fig. 3

Because the height of the barometer falls by the quantity dp when the altitude increases by dx, it follows that a column of mercury of height dp must weigh the same as a column of air of height dx. Euler knew that measurements showed that, near the surface of the earth, mercury is about 10,000 times as dense as air. He denoted this ratio of densities at the altitude of the observer by m. At the altitude of the observer, this makes $dx = -mdp$.

However, the density of air varies with altitude and the density of air at Q is less than c, the density of air at A. Thus at Q, if the altitude increases by dx, the corresponding column of air will be less dense in the ratio of c to q. This gives us the differential equation

$$dx = -\frac{mcdp}{q},$$

or equivalently,

$$dx = -\frac{mkdp}{p}.$$

Euler easily solves the second equation to get

$$x = -mk \ln p + C.$$

When $x = 0$ we have $p = k$ so $0 = -mk \ln k + C$. This tells us that $C = mk \ln k$, thus $x = -mk \ln p + mk \ln k$. We can rewrite this last formula either as

$$x = -mk \ln \frac{k}{p} \text{ or as } x = mk \ln \frac{c}{q}.$$

Euler was writing for an 18th century European audience, so he gives the values of these constants in units that would have been familiar to his readers. The value of k, he says, is about 28 *pouces* or $2\frac{1}{3}$ *pieds de Paris*. Today, these are about 30 inches, 750 mm or 1000 millibars. He adds that the product mk expresses a height of about 23,333 *pieds* or 4000 *toises*. A *toise* was a French unit of measure, usually equal to six feet, but sometimes six and a half, and the length of a foot varied across different parts of France. The unit became obsolete with the establishment of the metric system after the French revolution of 1789, but people continued to use it for many years in Louisiana and Quebec. Euler's *toises* seem to be about 6.25 of our modern feet, so 4000 *toises* are about 25,000 in today's feet.

Euler notes that his equation can be rewritten in a number of forms, starting with $\frac{x}{mk} = \ln\frac{k}{p} = \ln\frac{c}{q}$. He can exponentiate to get either $\frac{c}{q} = e^{\frac{x}{mk}}$ or $\frac{q}{c} = e^{-\frac{x}{mk}}$. This last version expands into a Taylor series approximation to give

$$\frac{c}{q} = 1 - \frac{x}{mk} + \frac{x}{mmkk} - \frac{x^3}{m^3k^3}.$$

He uses the second version because it leads to an alternating series and he knows that it will converge more quickly.

Euler tells us that for all but the highest mountains, x is considerably less than mk, so even this approximation converges very quickly, and that for most observations only the first two terms are necessary, so it is usually accurate enough to take

$$\frac{q}{c} = 1 - \frac{x}{mk}.$$

With this, Euler has a good formula giving q, the density of air in terms of c, the density at the altitude of the observer and x, the altitude above the observer, as well as in terms of known or observable constants.

Euler opens the next section of his paper with the words, "Having determined the density of air for all heights above the level of the level of the observer at the point A, it remains to determine the refraction suffered by a ray of light as it passes from air of one density into air of another density." This will require another set of differential equations that incorporate Snell's law, and will be the subject of next month's column.

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27

Bending Light

(August 2009)

In our last column we began a study of one of Euler's papers on how the Earth's atmosphere refracts light, *Sur l'effet de la réfraction dans les observations terrestres*, "On the effect of refraction on terrestrial observations," [E502] written about 1777 and published in 1780. We learned "that the rays of light do not always go in straight lines to our eyes, as we ordinarily suppose, but they are found to be a little bit curved, and their concavity is turned downward" and that this phenomenon is due in part to refraction as the rays pass between the rarified air at higher altitudes and the denser air at lower altitudes.

As we saw last month, in Euler's Fig. 3[1], C represents the center of the earth and A the location of an observer. The circle is not the surface of the earth but all the points at the level of the observer. The point Q is some point above the observer and x is the vertical distance AQ. The density of the air at A and Q is denoted by c and q respectively (though q also denotes a point infinitely close to Q). The air pressure, what Euler calls its elasticity, is denoted by k and p respectively. Because he used a mercury barometer to measure air pressure, he also used m to denote the relative density of mercury to air and told us that m was approximately 10,000. Then, by what we now call Boyle's law,

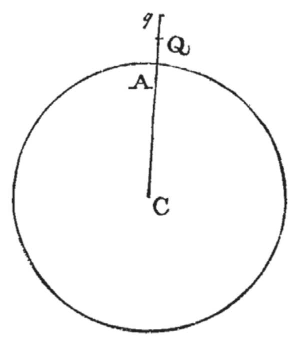

Fig. 3

$$\frac{q}{c} = \frac{p}{k}.$$

The main result of the first part of Euler's paper was an equation that describes the density of air at different altitudes. He found that

$$\frac{q}{c} = e^{-\frac{x}{mk}}.$$

[1] Our numbering of figures may be confusing here. We are using Euler's figure numbering throughout. We've added an illustration between his Figs. 3 and 4 and named it Fig. 3.5.

He uses this formula in various forms, including its logarithmic forms, $\frac{x}{mk} = \ln\frac{k}{p} = \ln\frac{c}{q}$. He also notes that his formula for $\frac{c}{q}$ is well approximated by the first few terms of a Tayler series expansion of its right-hand side, so for most practical purposes he would be able to use

$$\frac{c}{q} = 1 - \frac{x}{mk} + \frac{xx}{mmkk} - \frac{x^3}{m^3k^3}.$$

Indeed, because m is so large, "we could content ourselves for most observations to the first two terms, $1 - \frac{x}{mk}$, at least when we do not have to measure mountains of very considerable height."

Knowing the formulas for density, Euler is ready to move on to refraction. Euler asks us to consider a ray of light that passes directly from a vacuum into ordinary air of density c, which continues to denote the density of air at the point A, the level of the observer. Then there is a physical constant δ such that the ratio of the sine of the incidence to the sine of the refraction is as 1 is to $1 - \delta$, and that the value of δ is about $\frac{3}{10,000}$. In writing this, Euler is implicitly using Snell's law, illustrated in our Fig. 3.5.

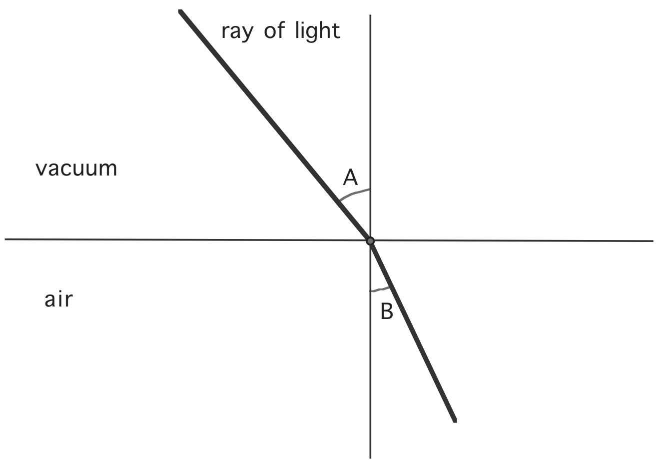

Fig. 3.5. Snell's law

In Fig. 3.5, a ray of light is shown passing from a vacuum into air of density c. The angle A is the angle of incidence, or simply the incidence, and Euler calls the angle B the angle of refraction, or just the refraction. Then Euler's version of Snell's law says that $\frac{\sin A}{\sin B} = \frac{1}{1-\delta}$. This version is formulated relative to the optical density of air at the observer. Modern versions use either the speed of light in the two media or they use tables of optical densities. They are all equivalent.

In Euler's problem, though, the density of air varies with altitude, so he has to adapt his formula a bit to describe light passing from a vacuum into air of density q. He claims that then the ratio of the sines will be "as 1 to $1 - \frac{\delta q}{c}$", that is to say, $\frac{\sin A}{\sin B} = \frac{1}{1-\frac{\delta q}{c}}$

Likewise, if the same ray of light passes from a vacuum into air of density r, the ratio of the corresponding sines will be as 1 is to $1 - \frac{\delta q}{c}$. Moreover, the process reverses, so that for light passing the other direction, from air of density r into a vacuum, the corresponding ratio will be the inverse.

Euler claims that as a consequence, the ratio of the sines for a ray passing from air of density q to air of density r will be as $1 - \frac{\delta q}{c}$ to $1 - \frac{\delta q}{c}$. But because δ is so very small, this is almost the same as the ratio 1 to $1 + \frac{\delta(q-r)}{c}$.

Now, using a standard technique of physics pioneered by Euler himself, Euler takes r to be infinitely close to q, which amounts to taking $r = q + dq$. Then as the ray passes from air of density q to air of density $q + dq$, the ratio of refraction will be as 1 is to $1 - \frac{\delta dq}{c}$. Reciprocally, as the light passes in the other direction, the ratio will be the reciprocal, $1 - \frac{\delta dq}{c}$ to 1.

As we learned in last month's column, [Sandifer Jun 2009] the density of the atmosphere at an altitude x above the level of the observer at A is described by the equation $\ln \frac{c}{q} = \frac{x}{mk}$. Differentiating this gives $\frac{dq}{q} = -\frac{dx}{mk}$, which can be rewritten as $dq = -\frac{q\,dx}{mk}$. Substituting this into the last form of his ratio of refraction makes that ratio as 1 is to $1 - \frac{\delta q\,dx}{cmk}$.

Now that he has formulas for the density of air and for the ratio of refraction, Euler is ready to set up and solve his main problem. He refers us to his rather complicated Fig. 4, where the circle represents the surface of the earth, A, X, x and D are points on the surface and C is the center. Then the radius of the earth is $AC = a$. Let O be an object directly above the point D that is seen by an observer at A by means of a ray of light that travels along the curve $OzZA$. He takes Z and z infinitely close together and assumes that they are directly above the points x and X, respectively.

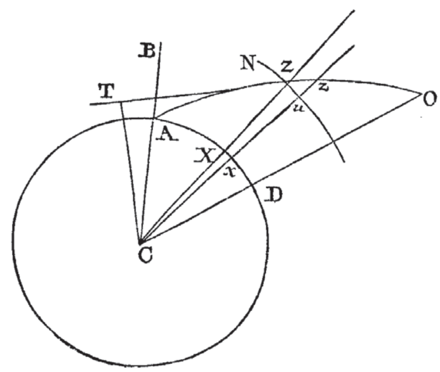

Fig. 4

Euler takes B to be a point directly above A for the sole purpose of being able to measure the angle $BAZ = \zeta$. He takes the angle $ACZ = \varphi$ and the distance $CZ = z$. Let x be the height of the point Z above X be x so that $XZ = x$ and $Z = a + x$ [taking Z to be both a distance and a point]. Thus for the point z we have the angle $ACz = \varphi + d\varphi$. This makes the angle $ZCz = d\varphi$ and the distance $CZ = z + dz$. Also he draws the tangent ZT at the point Z and chooses the point T so that CT is perpendicular to the tangent. Let $CT = t$ and take the angle $CZT = \omega$. Then $\sin \omega = \frac{t}{z}$.

Now let NZu be the arc of a circle through Z with center C. It represents the layer of the atmosphere that passes through the point Z. Euler needs a new point s, not shown in Fig. 4, located beyond the point Z on the ray CZ, so that he can measure the angle of incidence of the ray $OzZA$ as it passes through the layer NZu. Then angle zZs is the angle of incidence. We've just named this angle $CZT = \omega$, so angle $CzZ = \omega + d$. Because angle zZs is an external angle of triangle CZz, it equals the sum of the other two angles CzZ and ZCz. Consequently, the angle of incidence is

$$zZs = \omega + d\omega + d\varphi,$$

Because $d\omega$ and $d\varphi$ are infinitesimals, Euler safely assumes that $\cos(d\omega + d\varphi) = 1$ and $\sin(d\omega + d\varphi) = d\omega + d\varphi$, so by the angle addition formula for the sine function, the sine of the angle of incidence is

$$\sin = (\omega + d\omega + d\varphi) = \sin\omega + (d\omega + d\varphi)\cos\omega.$$

Meanwhile, the sine of the angle of refraction is sin ω. To apply Snell's law, we need the ratio of the sine of the incidence to the sine of the refraction. As fractions, this ratio is

$$\frac{\sin\omega + (d\omega + d\varphi)\cos\omega}{\sin\omega} = 1 + \frac{(d\omega + d\varphi)\cos\omega}{\sin\omega} = 1 + (d\omega + d\varphi)\cot\omega.$$

Euler will find it more convenient if he can rewrite this fraction with a 1 in its numerator. Because the numerator of the fraction on the middle is an infinitesimal, he rewrites the ratio as

$$1 : 1 - \frac{(d\omega + d\varphi)\cos\omega}{\sin\omega}.$$

When it comes time to use this ratio, though, he will use it in the form

$$1 : 1 - (d\omega + d\varphi)\cos\omega.$$

That's one part of Snell's law. Now Euler must consider the effect of the density of air. Earlier he denoted the density of air at Z as q, so the density at z is $q + dq$. From his earlier work on refraction, we get that the ratio of the sines, what Euler calls the "ratio of refraction" is 1:1 + $\frac{\delta dq}{c}$. Now, by Snell's law, the ratio of the sines equals the ratio of refraction, so we get

$$\frac{\delta dq}{c} = -(d\omega + d\varphi)\cot\omega.$$

The first part of the article, the part we dealt with in last month's column, told us what we need to know to find dq. For the given height $XZ = x$, we have $\ln\frac{c}{q} = \frac{x}{mk}$, from which it follows that

$$dq = -\frac{qdx}{mk} = -\frac{qdz}{mk}.$$

Substituting this value for dq gives the equation

$$\frac{\delta qdz}{mck} = (d\omega + d\varphi)\cot\omega.$$

Because $\frac{q}{c} = e^{-\frac{x}{mk}}$, this is the same as

$$\frac{\delta e^{-\frac{x}{mk}}dz}{mk} = (d\omega + d\varphi)\cot\omega.$$

That's a bit of a mess, but Euler, ever a genius at substitution, offers us a way to simplify it. Take $CZ = z$ and $CT = t$. Then $\sin\omega = \frac{t}{z}$ and $\cos\omega\frac{\sqrt{zz-tt}}{z}$. Taking the differential gives $d\omega = \frac{zdt - tdz}{z\sqrt{zz-tt}}$. Because triangles CZT and Zzu are similar and because $Zu = zd\varphi$ and $uz = dz$, we have $CT : ZT = Zu : zu$, that is to say $\frac{t}{\sqrt{zz-tt}} = \frac{zd\varphi}{dz}$, from which we get $d\varphi = \frac{tdz}{z\sqrt{zz-tt}}$. As a consequence we have $d\omega + d\varphi = \frac{dt}{\sqrt{zz-tt}}$. Because $\cot\omega = \frac{\sqrt{zz-tt}}{t}$, Euler's messy equation reduces to

$$\frac{\delta e^{\frac{-x}{mk}}dz}{mk} = \frac{dt}{t},$$

Because $x = z - a$, this really involves just two variables, z and t, and Euler tells us it is easy to solve.

But Euler doesn't solve it until ten pages later in the paper. Instead, he tells us that there is an easy way to use this form to determine the radius of curvature of at the point Z and then to use that to answer some of his questions about refraction.

Euler tells us that "as we know," the radius of curvature is $\frac{zdz}{dt}$. This seems completely unreasonable to those of us who learned formulas for radius of curvature that involved three-halves powers and second derivatives. However, t and z don't form a rectangular or a polar coordinate system. In some of his papers on differential geometry, Euler has shown that for these particular variables, the radius of curvature really is $\frac{zdz}{dt}$.

Given this, Euler finds that the radius of curvature at the point Z is

$$\frac{mkze^{\frac{x}{mk}}}{\delta t} = \frac{mke^{\frac{x}{mk}}}{\delta \sin \omega}.$$

Now Euler takes the point Z to coincide with the point A, so that $z = a$, $x = 0$ and $\omega = \zeta$. There, the radius of curvature becomes $\frac{mk}{\delta \sin \zeta}$. Euler plans to make calculations that do not involve great heights, so he feels safe in assuming that the radius of curvature will not change a great deal between the source of the light at O and the observer at A, and he concludes that the curve $OzZA$ is approximately the arc of a circle of radius $\frac{mk}{\delta \sin \zeta}$. He calls this radius g, then does a bit of calculation, again using *toises* as his unit of length (see last month's column for more about *toises*) and estimating that $\delta = \frac{3}{10\,000}$. He finds that $g = AG$ is about 13.3 million *toises*, or about 4.08 times the radius of the earth. Fig. 5 is not drawn to scale; the arc AD is much too large and the radius of curvature AG is too small.

Euler is ready to use this approximation to do an example. Following Fig. 5, he supposes that the observer at A sees an object at O that is apparently exactly on the horizon, so the angle ζ is 90°. Let OA be the path of the ray of light from O to A and let AG be the radius of curvature of that path. Because $\zeta = 90°$, we know that AG passes through the center of the circle.

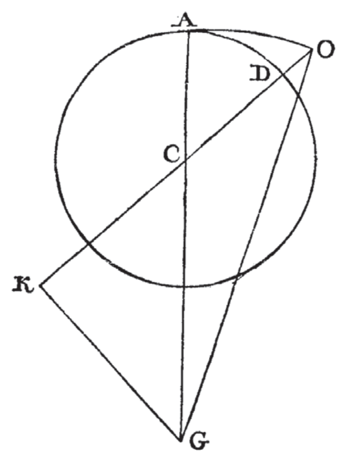

Fig. 5

Further, take angle $ACO = \varphi$. Then let D to be the point on the surface of the earth directly beneath O and let $AD = s$, which makes $s = a\varphi$, where $a = CA$ is the radius of the earth. Now, Euler asks how high O must be above the point D for it to be visible to the observer at A?

This is a fairly elementary geometry problem that a modern reader might solve using polar coordinates. Euler uses a different strategy, extending the radius OC to the point K that makes OKG a right angle. Then he finds OK in terms of a, g and φ to be

$$OK = \sqrt{gg \cos \varphi^2 + 2ag \sin \varphi^2 - aa \sin \varphi^2}.$$

From there, he finds, because φ must be rather small, that DO is approximately $\frac{3}{4} \cdot \frac{ss}{2a}$. All this is fairly routine, so we won't give details.

Euler concludes the main part of this article with a table of vertical adjustments that must be made for observations over various distances from 100 *toises* (about 200 yards) to 40,000 *toises* (almost 50 miles). Then he does an example where the ray of light is elevated from the horizon when it arrives at the observer at the point A, and finally, almost as an afterthought, he gives an exact solution to the differential equation he had found earlier.

In this E502, we've seen quite a few aspects of Euler's work in applied mathematics that don't often get a chance to shine in this series of columns. We see how Euler often put a problem down for a while, only to pick it up and extend it more than 20 years later. He had solved a similar problem [E219] in 1754, but he had not studied the optics of the atmosphere in the years in between. We see that even after he became nearly totally blind, Euler still worked on optics. It reminds us of the deaf Beethoven, still composing his Ninth Symphony

On a technical level, we see how Euler makes the transition from the algebraic formulation of physical laws, in this case Boyle's law and Snell's law, to a calculus-based formulation. He was the first to make this a standard technique in science. Euler also shows us how to use good approximations to the trigonometric and exponential functions to get accurate, though not exact, solutions to practical problems.

Euler spent as much time on practical problems that were on the cutting edge of the technological and theoretical frontier of his day as he did on his pure mathematics.

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28

Saws and Modeling

(November 2008)

Euler seemed to be interested in everything, and when he was interested in something, he sought to understand it with mathematics. Somehow, he got interested in saws, and in 1756, while working at the Berlin Academy, he wrote a 25-page paper, *Sur l'action des scies*, "On the action of saws" [E235].

I can only speculate on why Euler decided to write this paper. In their Editors' Introduction to the volume of the *Opera omnia* where this paper is reprinted, Charles Blanc and Pierre de Haller suggest that "its main purpose was to show one more time the possibility of putting into play the laws of mechanics and the techniques of mathematical analysis for measuring out the best advantage for concrete situations. One does not find in fact any truly new idea about the use of mathematics in practice."

I would suggest a more practical and timely motivation. When Euler first arrived in St. Petersburg in 1728, his actual position was as Physician to the Russian Navy. He mostly worked at the St. Petersburg Academy, but the Navy had some claim on his time and efforts. Reportedly, as part of a tour of military facilities, Euler visited some of the sawmills that provided lumber to the Navy. There he learned both the critical importance of lumber to military operations and the actual operation of sawmills.

Almost 30 years later, Euler was working in the academy of the Prussian King Frederick II, who was about to embark on what is often called the Seven Years War, except in America where we learn of it as the French and Indian War. It would also make sense if Euler, remembering the strategic importance of lumber supplies, were trying to help his King in the forthcoming war effort.

Regardless of his motivations, Euler begins his analysis of saws by describing the scope of his problem. First, he is interested in vertical saws being moved by a machine in a steady, repeatable way, not in horizontal saws or saws that are being moved by the hands of real people. He has two reasons for this restriction. First, the actual motion of people is hard to describe, and second, this reflects the design of actual sawmills.

The machines he studies will lift the saw blade and advance the timber, and then the saw blade will fall under its own weight, making the cut as it descends. Such saws were used for centuries to make the long cuts that slice trees into boards. Euler wanted to calculate the best values for the various measurements in the design of the saw and blade, the blade's length, its number of teeth, the width and depth of those teeth, and the number of men necessary to operate the saw most efficiently.

Euler describes a saw ABCD, as shown in Fig. 1, attached to move constantly along a vertical line EF, alternately ascending, when it does no work, and descending under its own weight. There are a lot of parts in this system, and he starts with the smallest one he can think of, the action of a single tooth as it cuts into the wood to a certain depth. He tells us that the resistance to this action depends on:

1. the hardness of the wood,
2. the size of the tooth, and
3. the depth to which it penetrates the wood.

For this third item, the depth ought be neither too large nor too small, for if it is too small, the tooth won't cut any wood, and if it is too large the tooth won't go through the wood. Euler uses ρ to denote the resistance to the tooth, but he doesn't give us any particular units for that resistance. Further, he lets α be the depth of the cut, item 3 on his list, and he assumes, perhaps from experience or from experiment, but more likely from some thought experiment, that ρ is proportional to the square of α.

It is clear, at least to Euler, that each tooth of the saw should act equally on the wood. Thus, each tooth should penetrate to the same depth and the teeth should not be arranged parallel to AC, for then the first tooth would do all the cutting and the rest of them would not touch the wood. Hence the line of teeth should be slanted, being farther from AC at C than at A, and the teeth should be arranged in arithmetic sequence, first tooth at k, next at $k + \alpha$ (where α is the depth of each cut), etc.

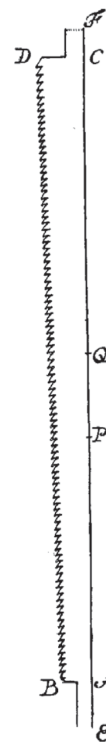

Fig. 1

Taking the length of the saw $AC = f$, the number of teeth to be n, and the depth of each penetration to be α, he calculates the angle for the saw to be ζ, where

$$\tan \zeta = \frac{(n-1)\alpha}{f}.$$

Solving this for α gives

$$\alpha = \frac{f \tan \zeta}{n-1}$$

so the difference between CD and AB will be

$$n\alpha = \frac{nf \tan \zeta}{n-1}.$$

Now let the total number of teeth be ν and let the interval PQ be the part of the saw actually touching the wood. Call the length of that interval z. The resistance the wood makes on the saw, taking into account z, the thickness of the wood and f, the total number of teeth in the saw, works out to be $\frac{nz\rho}{f}$. Euler introduces the parameter $R = n\alpha$, which represents what the resistance would be if all the teeth were in contact with the wood at once, and rewrites the actual resistance as $\frac{z}{f}R$. Because $R = n\rho = \frac{f\rho\tan\zeta}{\alpha}$, Euler rewrites it again as

$$\frac{z\rho\tan\zeta}{\alpha}.$$

To end his description of the saw blade, Euler takes c to be the depth to which one complete stroke of the saw cuts. Then $c = n\alpha \approx f\tan\zeta$. This last formula is only an approximation because of the way Euler measured ζ above when he wrote $n\alpha = \frac{nf\tan\zeta}{n-1}$.

So far, Euler has used ten different, though related measurements to describe his saw and its action: α, ρ, k, n, ζ, f, z, R, and c. He is now ready to consider the motion of the saw as it is descending. The saw is supposed to be attached to a weight heavy enough so that the saw will fall under its own weight without any other force being applied. He plans to calculate the total time of the descent of the saw.

Euler decides that the saw will fall from a height a at the beginning to build up speed before the teeth start to cut the wood. In the 18th century, it was difficult to measure small intervals of time accurately, and different places used different units of length. To account for this, Euler took as his standard length a unit g, the distance through which a falling body will fall in one second. Then he tells us that the free fall will last $\sqrt{\frac{a}{g}}$ seconds. We would do that calculation a bit differently today, and we would use the symbol g to be the acceleration due to gravity, not the (closely related) distance it represents here.

Speeds were also difficult to measure accurately. Euler and his contemporaries used a similar idea to describe speed. They measured it as the height from which a freely falling body would have to fall to achieve the same speed. We will see this principle in action a little later.

Euler divides the rest the motion of the saw as it descends into three phases. The first phase lasts from when the first saw tooth at B reaches the wood until it exits the other side. During this interval, the resistance to the saw steadily increases as more and more teeth cut into the wood. The second phase lasts from the end of the first phase until the last saw tooth at D reaches the wood. During this second phase, the number of teeth touching the wood is constant, so the resistance to the saw is constant as well. Finally, the third phase lasts from the end of the second phase until the last saw tooth at D exits the other side of the wood, at which time the saw should stop falling, lest its energy be wasted. Euler calculates the times for each of these three phases.

For the first phase, Euler refers to Fig. 2, where GH $=$ MN $= b$ is the thickness of the wood and GB $= z$ is the length of the part of the saw in contact with the wood instead of the thickness of the wood as it was before. During the first phase, z increases from 0 to b. Let us consider the action at the moment that z has some particular value. Take v to be the speed of the saw at that instant, measured, as we mentioned above, as the distance from which a freely falling body must drop to achieve that speed. Let P be the total weight of the saw. Following the analysis of the saw that Euler did above, the resistance to the saw at that point will be

$$\frac{z\rho\tan\zeta}{\alpha} = \frac{c\rho}{\alpha}\cdot\frac{z}{f}.$$

Take $N = \frac{\rho \tan \zeta}{\alpha}$, so that the resistance can be simplified to Nz. Euler tells us that now "the principles of Mechanics furnish us with the following equation:"

$$Pdv = (P - Nz)dz \quad \text{or} \quad dv = dz\left(1 - \frac{Nz}{p}\right).$$

This is $F = ma$ in disguise. Integrating this and using the initial condition $v = a$ when $z = 0$, we get

$$v = a + z + \frac{Nzz}{2P}.$$

Thus, at the end of the first time interval, when the tooth B arrives at the point H, we have $z = b$, and the speed of the saw corresponds to a freefall from the height $a + b - \frac{Nbb}{2P}$.

Knowing the speed at every point z, Euler does a difficult calculation to find the time that elapses in the first phase. He introduces a new constant, $e = \frac{P}{N}$, not to be confused with that other constant e that he sometimes used. This makes

$$v = a + z - \frac{zz}{2e} = \frac{2ae + 2ez - zz}{2e}.$$

Now, because of the way Euler and his friends measured time and speed, the quantity Euler denoted by v, is not the same as velocity, the quantity we denote by v today. For our v, we have $\frac{dz}{dt} = v$, but, confusingly, for Euler's v, we have instead $\frac{dz}{dt} = \sqrt{v}$, Given this, the previous expression becomes

$$dt = \frac{dz\sqrt{2e}}{\sqrt{2ae + 2ez - zz}}.$$

When we integrate this from $z = 0$ to $z = b$, we find that the duration of the first phase is

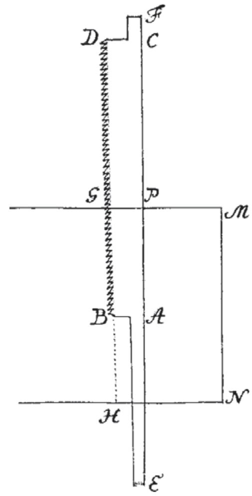

Fig. 2

$$\frac{\sqrt{e}}{\sqrt{2g}}\left(\sin^{-1}\frac{e}{\sqrt{2ae + ee}} - \sin^{-1}\frac{e - b}{\sqrt{2ae + ee}}\right) \text{seconds.}$$

Similar analyses with correspondingly similar figures lead us to find that the duration of the second phase will be

$$\frac{\sqrt{2e}}{2(e - b)\sqrt{g}}\left(\sqrt{2ae + bb + 2(e - b)f} - \sqrt{2ae + 2be - bb}\right) \text{seconds,}$$

and that of the third will be

$$\frac{\sqrt{e}}{\sqrt{2g}} \ln\left(\frac{e + \sqrt{2ae + 2be + 2f(e - b)}}{e - b + \sqrt{2ae + bb + 2f(e - b)}}\right) \text{seconds.}$$

Thus, the total time it takes the saw to descend is the sum of these three durations plus the interval of free fall at the beginning. Euler writes it out as a formidable sum. We omit it here.

By now, the King, for whom I propose Euler was writing, has probably given up and stopped reading this paper. That's too bad, because Euler is finally going to draw some conclusions. In particular, all of the quantities involved in this sum have to be real, so that none of the quantities under the radical signs can be negative. Euler does not mention that the argument of the logarithm function cannot be negative or zero, and the quantities of which we take the arcsine must be between -1 and $+1$, so he only gets two conditions:

$$\text{I.} \qquad a + b > \frac{bb}{2e} \text{ and}$$

$$\text{II.} \qquad a + b > \frac{1}{2}e + f\left(\frac{b}{e} - 1\right).$$

Unless these two conditions are satisfied, the saw will stop part way through its descent. This might happen because the tree is too thick (b is too large) or if the saw is too long (f is too large). The saw could also stop if the weight P is too small or the parameter N, which measures the resistance of the wood and the size of the teeth, is too large. Cryptic as these formulas may be, they furnish actual constraints on the design of the saw itself. Euler provides several pages of discussion on what these formulas mean and how to use them.

Finally, Euler turns to personnel issues. Apparently, for the saws that he had seen, the saw blade was lifted by one or several men working, rather than by the action of a water wheel or by teams of animals, as is sometimes shown in old pictures. Euler supposes that there are m men working to lift the saw, and that each of them lifts the saw through a distance s each second, and that each of them can lift a weight of S pounds. He calculates that the m men working together can lift the saw at a rate of $\frac{mSs}{P}$ feet per second. The total distance the saw must be lifted is $a + b + f$ feet, where still a is the distance of free fall at the beginning of the saw's descent, b is the thickness of the wood and f is the length of the saw. Given this, it will take the m men $\frac{P(a+b+f)}{mSs}$ seconds to lift the saw the required $a + b + f$ feet.

Euler pauses to confess some of the errors of approximation that he knows he is making. He argues that he can neglect what he hopes is a small distance that the saw goes past the bottom of the piece of wood. He knows that he is also ignoring friction and the effort necessary to advance the tree the distance c in each cycle of the saw.

To begin an example, take T to be the total time of one cycle of the saw, adding the time it takes to lift the saw to the time he found earlier for the saw to descend. If $a = 0$, $b = 1$, $e = 1$ and $f = 3$, all values that describe the saw itself, then $g = 15.625$, and he finds that

$$T = 0.79643 + \frac{4P}{mSs}\text{seconds.}$$

Let's think about how big this saw is. It has a three-foot blade, and it is cutting a tree that is a foot thick, so the action of the saw takes the top end of the saw to a height of 7 feet above the ground. Euler is about to assume that the saw itself weighs $P = 189$ pounds, so it appears that Euler's saw, though fairly large and heavy, is still small enough that it could be moved around on a cart or a wagon.

He makes some further guesses about the men operating the saw and the weight of the saw itself. He supposes that $s = 2$ feet per second and that $S = 30$ pounds. Given this, he estimates the amount that various numbers of men can saw in one hour:

1 man	3.82 feet per hour
2	7.23
3	10.30
4	13.06
5	15.57

Euler notes that there are diminishing returns here, and that infinitely many men could saw 67.5 feet of timber an hour. He verifies this phenomenon with a few more examples and concludes that "it is advantageous to employ as small a number of men as possible."

He further notes that a three-foot saw is far more efficient than a two-foot saw, and gives the advice that the saw ought to be as long as possible as well. He suggests four feet.

With this, Euler ends the paper rather abrubtly.

Euler's many books and papers range from the entirely abstract to the eminently practical. It seems to me that his most abstract work, especially that in number theory, he did for the sheer joy and beauty of the mathematics, but that almost all the rest of it was rooted in the hope that it would have practical applications in the real world. He only wrote a few papers that were as explicitly concerned with the mechanics of machinery as this one, but some of those were quite important and involved difficult mathematics.

The best known of these is his work on the strength of columns, where he combines the calculus of variations and the theory of elastic curves to calculate the buckling strength of columns and to design columns with the greatest possible strength. He also wrote on the shape of gear teeth, energy from windmills, on earthen dams and on caring for winter wheat crops. There seems to be no end to his interests or to his confidence in the applications of mathematics.

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29

PDEs of Fluids

(September 2008)

For his whole life Euler was interested in fluids and fluid mechanics, especially their applications to shipbuilding and navigation. He first wrote on fluid mechanics in his Paris Prize essay of 1727, [E4], *Meditationes super problemate nautico, de implantatione malorum* . . . , (Thoughts about a navigational problem on the placement of masts), an essay that earned the young Euler an *acessit*, roughly an honorable mention, from the Paris Academy. Euler's last book, *Théorie complete de la construction et de la manoeuvre des vaisseaux*, (Complete theory of the construction and maneuvering of ships) [E426], published in 1773, also dealt with practical applications of fluid mechanics. We could summarize Euler's contribution to the subject by saying that he extended the principles described by Archimedes in *On floating bodies* from statics to dynamics, using calculus and partial differential equations. Indeed, he made some of the first practical applications of partial differential equations.

Euler's work is very well known among people who study fluid mechanics. Several of the fundamental equations that describe non-turbulent fluid flow are known simply as "the Euler equations," and the problem of extending those equations to turbulent flow, the Navier-Stokes equations, is one of the great unsolved problems of our age.

This month, we are going to look at [E258], *Principia motus fluidorum,* (Principles of the motion of fluids), in which Euler derives the partial differential equations that describe two of the basic properties of fluids:

1. the differential equations for the continuity of incompressible fluids, and
2. the dynamical equations for ideal incompressible fluids.

We will examine the first of these derivations in detail.

Euler begins by warning us how much more complicated fluids are than solids. If we know the motions of just three points of a rigid solid, then we can determine the motion of the entire body. For fluids, though, different parts of the fluid can have very different motions. Even knowing the flows of many points still leaves infinitely many possible flows.

This is not to say that there are no laws regarding fluid flow, and one of Euler's favorite is the Law of Impenetrability, that two objects cannot occupy the same space at the same time. This will be Euler's main tool in his analysis. He will describe what we would call "volume elements" and regard them as individual objects, which therefore must obey the Law of Impenetrability. Moreover, Euler will assume that the fluid flow is continuous and incompressible. Euler's notion of continuity was a bit different than ours is today, and his notion implied that the flow also be differentiable. This kind of flow is now called incompressible laminar flow. This rules out phenomena like breaking into droplets, forming cavities and flowing around obstacles. Thus Euler seeks to describe what he calls *possible* motions, those that are both incompressible and continuous.

For the first part of his paper, the part in which he derives the PDEs of incompressible fluids, he additionally assumes that the fluid body is subject to no forces or pressures. In the end, this last assumption does not change Euler's conclusions.

Euler begins with the two-dimensional case and asks us to consider an arbitrary point l in the fluid body. (See Fig. 1.) He takes his axes to be the lines AQ and AB, so the coordinates of the point l are $AL = x$ and $Ll = y$.

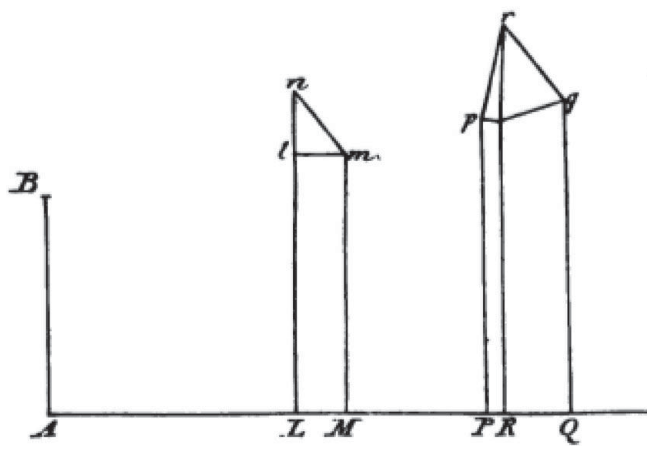

Fig. 1

With respect to these axes, we resolve the motion of the point l into its two components, $u = lm$ parallel to the x-axis and $v = ln$ parallel to the y-axis. He notes that the speed of flow at point l is thus $\sqrt{uu + vv}$ and its direction, given as an angle relative to the x-axis is arctan $\frac{u}{v}$.

Now, these velocity components u and v are not constant, but they vary with x and y. Euler introduces functions L, l, M and m so that he can describe these variations as differentials, writing

$$du = ldx + Ldy \quad \text{and}$$
$$dv = mdx + Mdy.$$

We would write $L = \frac{\partial u}{\partial x}$, $l = \frac{\partial u}{\partial y}$, $M = \frac{\partial v}{\partial x}$ and $m = \frac{\partial v}{\partial y}$ Both pairs, L and l, M and m, are themselves partial derivatives, so $\frac{\partial L}{\partial y} = \frac{\partial l}{\partial x}$ and $\frac{\partial M}{\partial y} = \frac{\partial m}{\partial x}$. When Euler wrote this paper in 1752, this fact that "mixed partial derivatives are equal" was still a fairly recent result. [E44, Sandifer 2004]

If a new point P (again, see Fig. 1) is located at distances dx and dy relative to the point l, then the velocity components at the point P will be

$$u + Ldx + ldy \quad \text{and}$$
$$v + Mdx + mdy.$$

Euler cleverly reuses his points *m* and *n* by letting lmn be a triangular element of water, and takes $lm = dx$ and $ln = dy$. He reassures us that,[1] "The whole mass of the fluid can be mentally divided up into elements like this, so that what we prescribe for one element will apply equally well to all." Euler seeks to describe the points p, q and r to which the points l, *m* and *n* respectively are moved by the flow during the time interval dt. He calls this time interval a *tempusculo*, or "tiny time interval." I don't recall seeing this word before, and I wonder if it is simply uncommon, if it was a bit of "math slang" popular in the 1750s, or if Euler constructed it just for the occasion.

To locate the points p, q and r, Euler begins by giving us table of the velocities in the x and y directions (i.e., parallel to AL and AB, respectively) at the points l, *m* and *n*:

point:	l	m	n
speed in the x direction	u	$u + Ldx$	$u + ldy$
speed in the y direction	v	$v + Mdx$	$v + mdy$

This lets him write the coordinates of each of the points p, q and r. First,

$$AP - AL = udt \quad \text{and}$$

$$Pp - Ll = vdt.$$

This gives the coordinates of p as $A = AL + udt$ and $Pp = Ll + vdt$. Similarly, the coordinates of q are $AQ = AM + (u + ldx)dt$ and $Qq = Mm + (v + Mdx)dt$, and those of r are $AR = AL + (u + ldy)dt$ and $Rr = Ln + (v + mdy)dt$.

Euler gives a brief argument that pqr is still triangle because triangle lmn and the time interval dt are infinitely small. Then, "[s]ince the element lmn ought not be extended into a greater area, nor to be compressed into a smaller one, its motion must be so composed that the area of triangle pqr equals the area of triangle *lmn*." This is a key observation, for it allows Euler to describe his notion of incompressibility by giving conditions that preserve the area of the element lmn. Euler does not yet have access to the Divergence theorem that many of us learn in our third semester calculus course, because vector fields and their accompanying notions of gradient, divergence and curl, did not arise until the 19th century. Hence Euler has to do all the work directly, "from scratch."

Triangle lmn is a right triangle, so it is easy to find its area, $\frac{1}{2}dydx$.

To find the area of triangle pqr, we refer again to Fig. 1, and see that the area is the sum of the areas of trapezoids $PprR$ and $RrqQ$, less the area of trapezoid $PpqQ$. In formulas,

$$PprR = \frac{1}{2}PR(Pp + Rr),$$

$$RrqQ = \frac{1}{2}RQ(Rr + Qq) \quad \text{and}$$

$$PpqQ = \frac{1}{2}PQ(Pp + Qq).$$

[1] Here and elsewhere in this column, when I quote Euler's words, I use the translation graciously provided by Enlin Pan.

Substituting and collecting terms gives

$$pqr = \tfrac{1}{2}PQ \cdot Rr - \tfrac{1}{2}RQ \cdot Pp - \tfrac{1}{2}PR \cdot Qq.$$

Now, let $PQ = Q$, $PR = R$ so that $RQ = Q - R$. Also let $Qq = Pp + q$ and $Rr = Pp + r$, and the area formula simplifies to

$$pqr = \tfrac{1}{2}Q \cdot r - \tfrac{1}{2}R \cdot q.$$

From what came earlier, we can rewrite the elements of the right hand side using differentials to get

$$Q = dx + Ldxdt, \quad q = mdxdt,$$
$$R = ldydt \quad \text{and} \quad r = dy + mdydt.$$

Now, the details are interesting, so we'll include them.

Substitution, then factoring gives

$$pqr = \tfrac{1}{2}dxdy(1 + Ldt)(1 + Mdt) - \tfrac{1}{2}Mldxdydt^2,$$

which combines to give

$$pqr = \tfrac{1}{2}dxdy(1 + Ldt + mdt + Lmdt^2 - Mldt^2).$$

Because we know that $\Delta pqr = \frac{1}{2}dxdy$ and that $dxdy \neq 0$, and that we substitute, then subtract and cancel to get

$$Ldt + mdt + Lmdt^2 - Mldt^2 = 0$$

or

$$L + m + Lmdt - Mldt = 0.$$

As *dt* vanishes, this gives

$$L + m = 0.$$

In terms of partial derivatives, Euler writes

$$\frac{du}{dx} + \frac{dv}{dy} = 0,$$

but we would write

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

or even

$$\nabla \cdot \mathbf{u} = 0.$$

This last notation was at least 100 years in Euler's future.

An analogous argument, based on Fig. 2, seven pages long instead of four, leads to the analogous conclusion for three dimensions, which Euler writes

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0.$$

It is interesting to note that, except for the relative positions of points p, q and r, Fig. 1 is a proper subset of Fig. 2. Likewise, the argument leading to Euler's two dimensional formula is a proper subset of the analogous three dimensional argument as well.

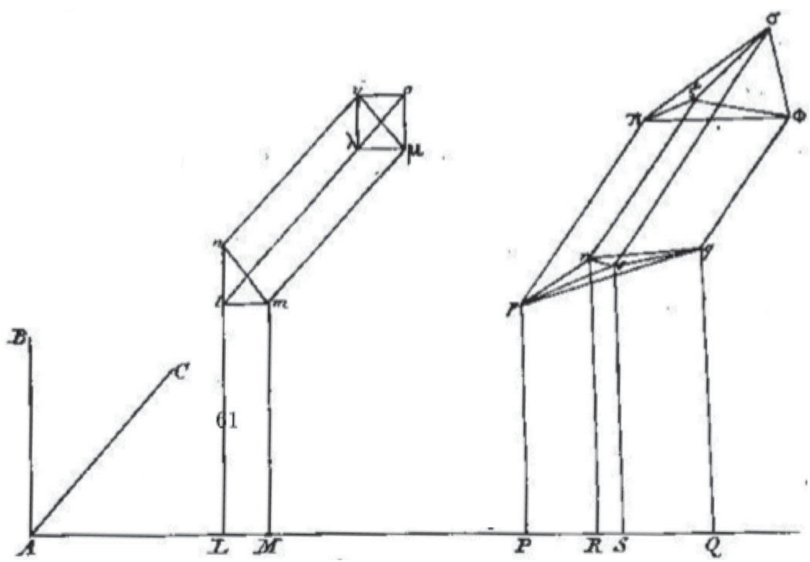

Fig. 2

At this point, we've described only the first part of Euler's paper. In the first part, Euler had not concerned himself with how the flow might change over time. The key new idea in the second part is that forces, internal and external, might cause the flow itself to change with time. Thus, his equations of motion in two dimensions have an extra term to describe how the flow changes with time. In particular, and in modern terms, he begins with

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial t}dt \quad \text{and}$$

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial t}dt.$$

This is clearly analogous to his starting point in the first section, where his first equations were

$$du = ldx + Ldy \quad \text{and}$$

$$dv = mdx + Mdy.$$

In fluids, change in velocity is related to pressure, denoted by p. As in the first part of the paper, Euler uses his initial equations to derive the differential equations, first in two dimensions, then in three, that describe the pressures in fluid flow. For three dimensions, and in modern terms, they are

$$\frac{\partial p}{\partial x} = -2\left(\frac{\partial u}{\partial x}u + \frac{\partial u}{\partial y}v + \frac{\partial u}{\partial z}w + \frac{\partial u}{\partial t}\right),$$

$$\frac{\partial p}{\partial y} = -2\left(\frac{\partial v}{\partial x}u + \frac{\partial v}{\partial y}v + \frac{\partial v}{\partial z}w + \frac{\partial v}{\partial t}\right) \quad \text{and}$$

$$\frac{\partial p}{\partial z} = -1 - 2\left(\frac{\partial w}{\partial x}u + \frac{\partial w}{\partial y}v + \frac{\partial w}{\partial z}w + \frac{\partial w}{\partial t}\right).$$

In the interests of brevity, we will not give more details of these derivations. The interested reader is encouraged to consult [Truesdell 1954] and Enlin Pan, in his English translation of [E258].

Finally, sharp-eyed readers who read the references first may note three other articles that Euler wrote about fluids, [E225], [E226] and [E227]. From their titles “General principles on the state of fluid equilibrium,” “General principles on the movement of fluids,” and “Continuation of the researches on the theory of the movement of fluids,” they seem to cover much of the same material as [E258]. From their length, a total of 131 pages, compared to 36 pages for E258, they seem to cover the material in more depth, and from their Eneström numbers, all less than 258, it is evident that they were all published before E258. Why, then, did we describe E258 instead of E225, E226 and E227?

E258 was actually written first, in 1752, but did not appear in the journals of the St. Petersburg Academy until its volume for papers presented in the years 1756 and 1757. Typical publication delays delayed the actual printing of that volume until 1761. The other three were written in 1753, 1755 and 1755 respectively in French for the journal of the Berlin Academy and published in that academy’s volume for the year 1755, printed in 1757. Euler’s colleagues in Berlin would have learned of the results in 1753 and 1755, and scientists elsewhere would have learned them in 1757. Hence, if we are studying the influence of Euler’s ideas, then we should have looked at the three papers in French. Moreover, they represent a more refined and complete treatise on the subject.

However, if we wish to see the growth of Euler’s ideas, to see “how Euler did it,” then we should look at his earlier work on the subject, and that is E258. Readers interested in the differences between E258 and the three treatises in French should consult [Truesdell 1954]. Truesdell’s “Introduction” itself is an outstanding essay, and it’s in English!

Special thanks to Enlin Pan for allowing me to use his English translation of E258. It was most helpful.

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30

Euler and Gravity

(December 2009—A Guest Column by Dominic Klyve)

The popular myth of the discovery of gravity goes something like this: one day, an apple fell on the head of a young Isaac Newton. After pondering this event, Newton wrote down an equation describing an invisible force, which he called gravity. This equation united ideas about the paths of cannon balls and apples (terrestrial motion) with the paths of moons and planets (celestial motion). Once it was written down, it elegantly and easily explained the motion of all the planets and moons, and remained unquestioned, revered, and perfect for centuries (at least until Einstein).

Readers who are familiar with the history of science know better; nothing is ever so simple. In fact, Newton's theory was accepted, tested, clung to, retested, questioned, revised and un-revised, and finally accepted again during the 75 years after the publication of Newton's *Principia*. More importantly for our purposes, Euler was at the center of all of it.

Prelude to a crisis

De Causa Gravitatis

Euler had been thinking about gravity even before the worst of the chaos mentioned above. In 1743, he published an anonymous essay, *De Causa Gravitatis* [Euler 1743]. The essay itself is remarkable for two reasons: first, it was published anonymously. This seems to have been a technique that Euler used to get ideas out to the public without having to publicly defend them, which gave him greater leeway to experiment with bold claims. Furthermore, it was published only two years after he arrived at his new job in Berlin, and written even closer to the time of his move, and we might guess that Euler was especially keen not to publish anything controversial before he had settled in. The second remarkable thing about this paper is that it does not have an Eneström number. The man who cataloged all of Euler's works early in the 20th century missed this one completely. We only know today that Euler is the author because Euler admitted it in a letter written 22 years later to Georges-Louis Lesage (see [Kleinert 1996] for an interesting introduction to *de Causa* and its discovery).

In this paper Euler examines the standard Newtonian view that gravity is a fundamental property of all bodies, and can't be derived from any more basic principle. Euler is not convinced that this is true, and gives an example of how an inverse square law might be the result of some other property of the universe. Euler takes as a given that the universe is permeated by the *ether*, a subtle fluid existing everywhere that permits light (among other things) to travel the distances between astronomical bodies. Next, since Bernoulli's principle holds that fluid moving quickly relative to a body has lower density than fluid that is stationary, it may be reasonable to assume that the density of the ether near the Earth is less than that far away from the Earth.

If we further assume that the pressure drops off with the reciprocal of the distance to the Earth, we find something interesting. Euler writes:

"Let the absolute compression of the ether (when it is not lessened) be denoted c. And let the distance from the center of the Earth be equal to c minus a quantity reciprocally proportional to x. And so let a compression of the ether in distance x from the center of the Earth be $C = c - cg/x$. From this, if body $AABB$ is in position around the Earth, the surface of it above AA will be pressed downwards by force $= c - cg/CA$, moreover the surface below BB will be pressed upwards by a force $c - cg/CB$, which force, since it is less than before, the body will be pressed downwards by force $= cg(1/CA - 1/CB) = cg.AB/(ACBC)$.

And so if the magnitude of body AB is incomprehensibly less than distance CB, we will have $AC = BC$, from this the gravity of such a body in whatever distance from the center will be as $1/AC^2$; this is reciprocally as a square of the distance from the center." (translation by DeSchmidt and Klyve [Euler 1743])

This is, frankly, rather bizarre. Euler here suggests that the thing that holds us to the ground is a pushing force from the (comparatively) dense ether above us. We should note here that there's no evidence that Euler took this explanation seriously (probably the reason for his anonymous publication in the first place). Rather, he seems to want to use this simple explanation as a reminder that we shouldn't stop looking for a "first cause" of gravity, and that we may be able to derive the inverse square law from an earlier principle.

The gravity crisis

A few years after the publication of *De Causa Gravitatis*, Newton's inverse square law met with challenges considerably greater than those provided by philosophical speculations. New developments in observational technology allowed astronomers to measure the location of the moon and planets to a precision not before achieved. There was just one problem—the new accurate measurements failed to conform to those predicted by the theory. In fact, there were three significant problems with the predictions of Newton's laws, and three attempts to fix them. We turn next to these.

Problem 1: Lunar apsides

It was well known that Newton's laws implied that bodies orbiting Earth would move in an ellipse. When carefully applied, in fact, Newton's laws also show that this ellipse will itself slowly revolve around the Earth, as a consequence of the gravitational force of the sun. The pattern of revolution of the "lunar apsis" is complicated, but averages about 3° per month (see Figure 2), a fact well established through observations in the early 18th century.

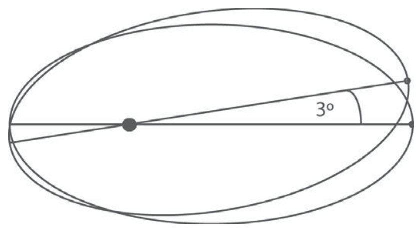

Figure 2. Precession of the apsidal line—exaggerated (figure courtesy of Rob Bradley)

The problem with this was that the revolution calculated using Newton's inverse square law for gravitation did not give a figure of 3° per month; it gave only half that. The three leading mathematicians of the time—Jean d'Alembert, Alexis Clairaut, and Euler—had attempted the calculation, and each of them came inescapably to the same conclusion that Newton himself had in the *Principia*: that Newton's laws suggested a monthly apsis revolution of only $1{:}5^\circ$. What was going on? (For an engaging and more detailed account of the issue of the lunar apsis, readers are encouraged to read Rob Bradley's article [Bradley 2007] on this subject.)

Problem 2: Jupiter and Saturn

The interactions of Jupiter and Saturn are devilishly complicated. The basic idea is that when Jupiter starts to catch up to Saturn, it pulls "backward" on Saturn, causing the ringed planet to lose kinetic energy. Saturn then falls to a closer orbit and, a bit surprisingly, speeds up (as a consequence of Kepler's third law). When Jupiter passes Saturn, it pulls "forward," giving Saturn more kinetic energy, moving it to a farther orbit, and slowing it down. The mathematics of all of this is made difficult by the planets' elliptical orbits (see Wilson [Wilson 2007] for details), but it's possible to work out, using calculus, just what the change in Jupiter's and Saturn's orbits should be. As was the case with the moon, however, the theoretically calculated difference did not match the observed difference.

Problem 3: The eclipse of 1748

The first two problems described above were very public problems—all of the major players of the day were aware of them, and historians of science have often discussed them. Another issue that arose in the context of 18th century mechanics, however, is less well known, and was an issue only for Euler.

In 1746, when 39 years old, he published the *Opuscula varii argument* [E80], a fascinating book containing eight essays on such topics as light and colors, the material basis for thought, studies on "the nature of the smallest parts of matter," and a discourse describing the frictional force the planets feel as a result of passing through the ether. Also included was his new collection of tables describing the motion of the moon. Deriving the motion as carefully as he could using the calculus, he refined and calibrated the results using observations [Wilson 2007]. These tables became the most accurate available, predicting the location of the moon to an accuracy of $\frac{1}{2}$ arc minute.

Flushed with the success of his work, he anxiously awaited a solar eclipse which he had calculated to occur on July 25, 1748. When the eclipse finally happened, Euler found he had correctly predicted the length and the start time of the eclipse, but that he had gotten the length of the "annular" phase (when the sun can be seen as a ring around the moon) completely wrong. Euler wrote "reections" about this agreement and disagreement during the weeks following the eclipse [E117], and a more detailed paper trying to explain the disagreement a few months later [E141]. In each of these papers he tries, but never fully succeeds, to account for his error, and he must have begun to wonder: is the problem with his calculation the same problem that everyone was finding with the moon, Jupiter, and Saturn?

Desperate for a Solution

At the end of the 1740's, Euler, D'Alembert, and Clairaut, were rapidly running out of ideas. If their mathematics, together with Newton's law of gravity, failed to explain the motion of bodies in the cosmos, something must be wrong. And failing to find any errors in their calculations, they reluctantly turned elsewhere.

Attempted Solution 1: Questioning the inverse square law

One of the boldest attempts to reconcile the observed and theoretical descriptions of the moon's motion was made not by Euler, but Clairaut, who announced in November 1747 at a public session in the French Academy of Sciences that Newton's theory of gravity was wrong. That one of the leading mathematicians in the world would publicly make such a claim is evidence of just how dire the situation had become. Clairaut suggested that the strength of gravity was proportional not to $\frac{1}{r^2}$, but the more complicated

$$\frac{1}{r^2} + \frac{c}{r^4}$$

for some constant c. Over large distances, the c/r^4 term would effectively disappear, accounting for the utility of the inverse square law over large distances. He then began trying to find a value of c which could account for the moon's motion. He would continue to pursue this idea until May 17, 1749, when he made an equally dramatic announcement in which he claimed that Newton was right after all [Hankins 1985].

Attempted Solution 2: Changing the shape of the moon

During the period when Clairaut was entertaining the change to Newton's law, Euler continued to try to explain the motion of the moon without accepting that Newton may have been wrong. In a surprising letter of January 20, 1748, D'Alembert wrote to Euler [Euler 1980] to suggest a new theory: perhaps the moon (or at least its distribution of mass) was not spherical. If, after all, we only see one side from the Earth, we can't know how far back it truly extends. And perhaps if it extends far enough back, the apsidal motion would indeed be 3°, as observed. In an even more surprising response written less than four weeks later [Euler 1980], Euler says that he too had considered this idea, and had worked out the details! He found that moon would have to extend back about $2\frac{1}{2}$ Earth diameters in the direction away from us, which seemed untenable (translations and more discussion of this correspondence can be found in Bradley [Bradley 2007]).

Attempted Solution 3: Moving Berlin

A third attempt to fix the discrepancy between prediction and measurements was employed by Euler in his second paper on the solar eclipse. In E141, Euler tries to explain the failure of this prediction about the eclipse by writing

> "Yet, toward correcting the error in the duration of the annulus, I must note that in my calculations I had assumed the latitude of Berlin was $52^\circ, 36'$; now in fact the last observations that Mr. Kies made with the excellent Quadrant that Mr. de Maupertuis gave to the Academy only gave its elevation to be $52^\circ, 31', 30''$, so I had placed Berlin too far North by $4', 30''$. Just glancing at the chart of this Eclipse published at Nurnberg offers the assurance that if Berlin were situated $4'$ more to the north, the duration of the annulus would have considerably lengthened and would have been very close to my calculation." [E141]

In other words, like lost drivers around the world, Euler blames his maps. Today it is easy, using a software program like Starry Night, to recreate the eclipse, and look at the effect of moving about $4'$ north from Berlin. An arc minute on the surface of the Earth roughly corresponds to a mile, and moving four miles can be seen to have essentially no effect. That is, the discrepancy in Euler's maps cannot be the problem. Euler himself doesn't redo his calculations with the new value, but merely reworks them using a different method, and—hindsight really is easier than foresight—gets the right answer. In the end, though, this is no answer at all. Euler's calculations still fail to predict the motion of the moon.

A Solution at last

When Clairaut announced in 1749 that Newton's inverse-square law was right after all, Euler was anxious to know his solution. He could find no errors in his own calculations, and knew that either Clairaut had erred, or else had found a dramatic new method of calculation. In order to see Clairaut's work, Euler arranged for the annual prize of the Saint

Petersburg Academy to be given to the paper which could best "demonstrate whether all the inequalities observed in lunar motion are in accordance with Newtonian theory" (quoted in [Wilson 2007]). Euler was on the prize committee, and copies of all the submissions were sent to him. One can imagine him tearing open the envelope with the "anonymous" submissions, shuffling through them to find Clairaut's (Euler would have recognized his style immediately), and eagerly reading it. After satisfying himself of the correctness of Clairaut's solution, he wrote two long and glowing letters to the French mathematician, praising his work and congratulating him on his success.

What, then, had Euler been doing wrong? The problem turned out to be that he (and Clairaut and d'Alembert) had failed to take into account second-order effects. One way the calculus of the time was done used differentials. Given a variable x, its value an infinitesimal moment later was written $x + \mathrm{d}x$. If $y = x^2$, then a small change in x would change y's value to

$$y + dy = (x + \mathrm{d}x)^2 = x^2 + 2x\,\mathrm{d}x + (\mathrm{d}x)^2.$$

Since dx is infinitesimal, $(\mathrm{d}x)^2$ is *really* infinitesimal, and was routinely treated as 0 and eliminated from future calculations (indeed, it is only the coefficient of the dx term that today we call the derivative of y). It turned out, though, that for 3-body systems (Earth-Moon-Sun or Sun-Jupiter-Saturn), second-order effects are very important. When Clairaut redid his calculations, attempting the tedious task of keeping all the second-order terms, it turned out that Newton's laws precisely matched the observed behavior of the moon, Jupiter, and Saturn.

Back to Philosophy

Letters to a German Princess

By 1750, then, the perceived problems with the law of gravity had been resolved, and Euler could turn his attention back to more philosophical questions. Mathematicians familiar with Euler's work only through today's textbooks are often surprised to discover that Euler was also a writer of "popular science" books—a sort of 18th Century Isaac Asimov or Stephen Jay Gould. His *Letters to a German Princess* [E343] contain explanations of scores of topics spanning the science of the day, from logic to optics, from music to the origin of evil. The letters are available online (in English translation) and are well-worth reading even today. In letters 45–53 (among others), Euler treats gravity.

It is in this work that we see a more mature Euler. He wrote these letters in 1760, when the problems that beset Newton's theory in the 1740's had long since been resolved. Nevertheless, he still shows some of the reservations he had 25 years earlier with accepting gravity as a first principle. Indeed, in letter 46 he writes that "Philosophers have warmly disputed, whether there actually exists a power which acts in an invisible manner upon bodies: or whether it be an internal quality inherent in the very nature of the bodies . . ."

Euler's work is meant to be read by a lay audience, and he gives several visual images for the reader to think about. In one wonderful image, Euler writes: "There is a cellar under my apartment, but the floor supports me, and preserves me from falling into it. Were the floor suddenly to crumble away, and the arch of the cellar to tumble in at the same time, I must infallibly be precipitated into it, because my body is heavy, like all other bodies with which we are acquainted."

Little shows Euler to be a product of his time more than his explanation of this last qualifier: "I say, *with which we are acquainted*, for there may, perhaps, be bodies destitute of weight: such as, possibly, light itself, the elementary fire, the electric fluid, or that of the magnet: or such as the bodies of angels which have formerly appeared to men." At the time, light, fire, electricity, and magnetism all were thought by many people to consist of weightless (imponderous) bodies which diffused through other matter, granting it properties like heat or electricity. Euler's claim that angels are weightless (and, for that matter, that they have "appeared to men") was less grounded by scientific evidence.

Euler's later work

Euler would continue to work on problems concerning gravity until the end of his life, even famously working on the orbit of Uranus on the day he died. In 1772 he published a third Lunar Theory, in which he used his new understanding of gravitational calculations to find more accurate descriptions of the motion of the moon. He never did fully satisfy his own curiosity about the cause of gravity but since the scientific community had to wait until Einstein to get an explanation better than Newton's, he can perhaps be forgiven this failure. Had he somehow lived to the 20th century, however, we can be sure that Euler would have found fertile ground for further research in our modern views on gravity as well.

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Dominic Klyve (dklyve@carthage.edu) is Assistant Professor of Mathematics at Carthage College in Kenosha, WI, and is a member of the Executive Board of the Euler Society. Dominic is also co-founder (with Lee Stemkoski) and director of the Euler Archive. Dominic has written several papers about Euler's life and work, and has worked on translations of several of Euler's papers.

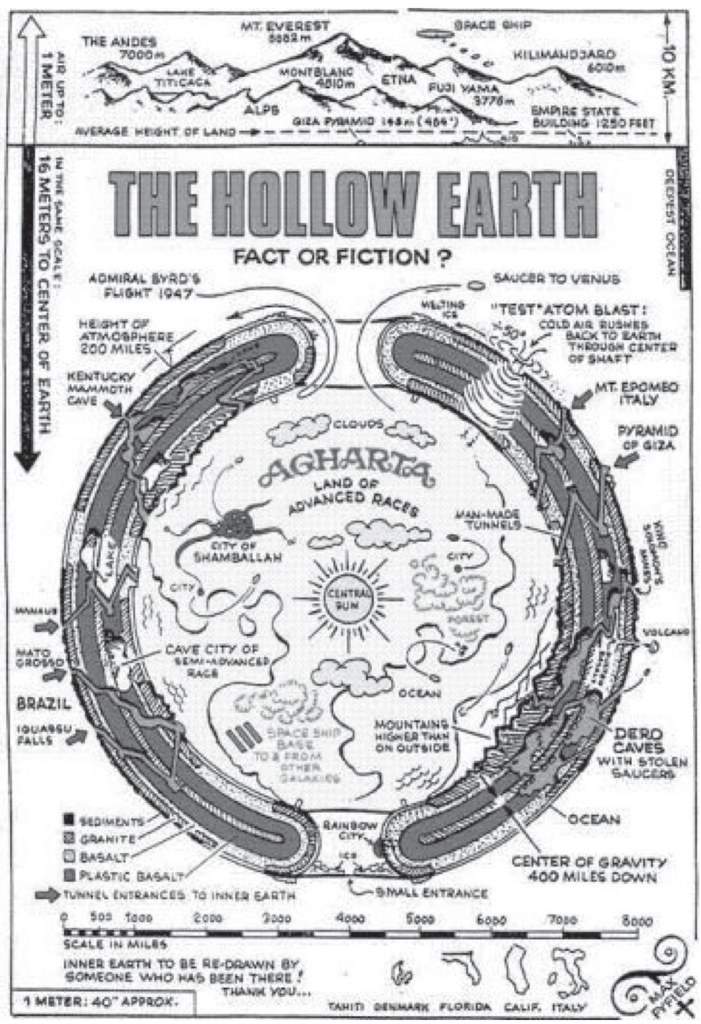
Part VI

Euleriana

31

Euler and the Hollow Earth: Fact or Fiction?

(April 2007)



Is the earth hollow? Is there a sun 600 miles in diameter at the center of the hollow earth? Is the inside of the shell of the hollow earth covered with mountains larger than the ones we see on the outside? Is there a hole in the shell of the hollow earth through which flying saucers from Venus and space ships from other galaxies fly to get to their bases inside the hollow earth? Are there secret passages from the bases of the Great Pyramid and other locations around the earth that connect the outside to the inside of the earth? Is the mushroom cloud of an atomic bomb really caused when the bomb pokes a hole through the shell and the gasses inside the earth rush through to escape?

Some people claim to believe all of this, and they even give us detailed maps of what is inside the earth. See, for example, the extravagant map above, drawn by Max Fyfield and available on scores of pages on the web. [Fy] Imagine my surprise when I found apparently reputable sources that said that Euler also endorsed a hollow earth theory, and that Fyfield's map was based on Euler's theories. This, of course, piqued my curiosity, so I decided to look into the question of Euler and the Hollow Earth.

Here are a few excerpts from some websites I found that credit Euler with a hollow earth theory.

Leonhard Euler

Later theorists came up with variations to Halley's [sic] model. In the 17th century, **Leonhard Euler** proposed a single-shell hollow Earth with a small sun (1.000 km across) at the centre, providing light and warmth for an inner-Earth civilisation. Others proposed *two* inner suns, and even named them: Pluto and Proserpine.

http://strangemaps.wordpress.com/2007/03/01/85-inside-the-hollow-earth/

Leonhard Euler

In the 18th century Swiss mathematician Leonhard Euler took the multiple spheres theory and replaced it with a single hollow sphere that contained a sun 600 miles wide. He said the sun maintained heat and light for an advanced civilization that he said lived there. A Scottish mathematician Sir John Leslie suggested that there was not one sun but in fact two he named these Pluto and Proserpine.

http://tinwiki.org/wiki/Hollow_Earth

Leonard Euler

Leonard Euler (1707–1783), noted mathematician and one of the founders of higher mathematics. He stated that "mathematically the Earth has to be hollow." He also believed there was a center sun inside the Earth's interior, which provided daylight to a splendid subterranean civilization.

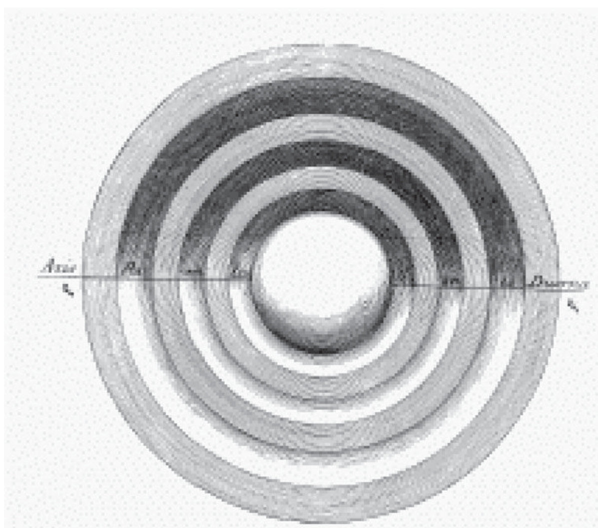
http://www.xenophilia.com/zb0008d.htm

Even the usually reliable John Lienhard, a historian of science at the University of Houston and the creator of the NPR feature Engines of our Ingenuity, got in on the act, or was taken in by the deception: [L]

No. 2180:
HOLLOW EARTH
by John H. Lienhard

> One person who picked up on that idea was Leonhard Euler, the great mathematician of the 18th century. Euler proposed that Earth was completely hollow (no concentric shells) with a 600 mile diameter sun in the center. His hollow interior could be reached through holes at the North and South Poles.
>
> http://www.uh.edu/engines/epi2180.htm

The earliest hollow earth theory seriously set forth by an important scientist seems to be by Edmond Halley. [H] In about 1691, Halley was trying to explain why the earth has a magnetic field, and why it varied. He proposed that the earth might be composed of concentric shells, separated by fluids so that one shell could move relative to another. It was this relative motion, he said, that caused the magnetic field, and variations in the motion caused the variations in the field. Simanek [Si] offers the illustration at the right, which he says comes from Halley's 1681 paper.



Modern readers might be tempted to mock Halley's gullibility and naivety in proposing such a theory, but they would be unfair to do so. It was a well-reasoned effort to explain an observed scientific phenomenon. The theory conformed to the facts as the scientists of his time knew them, and, as new facts emerged that contradicted the theory, they abandoned the theory. That is the way the scientific method is supposed to work.

Note that Halley's theory involved no flying saucers, hidden central sun, or secret tunnels from the base of the Pyramid of Giza.

Let us move on to the next century. In the 1730s, one of the open questions of science concerned the shape of the earth. Some people thought that the earth would bulge at the poles, and be narrower at the equator so that it could spin more efficiently. Euler joined

Newton and others in believing that the earth would bulge at the equator and be flatter at the poles. In 1738 he published a paper [E32] "On the shape of the Earth" in which he considered the earth as a fluid mass and predicted that it would bulge at the equator rather than at the poles. Cassini had made measurements a few years earlier that suggested that there was a bulge at the poles. At about the same time as Euler was writing E32, Maupertuis was planning a pair of expeditions, one to Peru and the other to Lapland, to make more accurate measurements that would show that Euler and Newton were correct. For this, Maupertuis became famous as "the Man who Flattened the Earth."

Through this, Euler never suggested that the earth was hollow. He considered it as a fluid with a crust on it, not too different from modern theories.

Euler passed up another opportunity to propound a hollow earth theory in the early 1750s. The topic of interest was the precession of the orbit of the moon. The moon has an elliptical orbit around the earth, and each month the axis of that ellipse moves about 3°. That wandering of the axis is called *precession*. Euler, D'Alembert and Clairaut studied this, but their analyses could only explain about half of the observed precession. Being faithful to the scientific method, they examined their assumptions to try to find ways to explain the other half of the precession. We'll mention two of their efforts.

They considered the possibility that Newton's inverse square law for gravity was not quite right, and that for shorter astronomical distances, gravity was a little stronger than predicted. Perhaps the force of gravity between two masses M and m, was not $Mm\frac{G}{r^2}$, where G is the gravitational constant. Perhaps it was a little more than that, say $Mm(\frac{G}{r^2} + \frac{g}{r^4})$, where g is another gravitational constant.

They also considered the possibility that the moon was somehow more massive than they thought. Perhaps it even had two parts and was shaped like a dumbbell, with the nearer part always hiding the more distant part from our view.

They *could* have tried to explain the phenomenon by finding a way to make the earth less massive than they thought, perhaps by being hollow, but from the evidence that is currently available, it seems that they did not consider this possibility.

Eventually, Clairaut found a way to improve the analysis by considering more terms in certain series expansions, and he explained the other half of the precession. Like Halley, everyone involved followed good scientific method. When the predictions didn't fit the observations, they tried to improve their theories and to refine their analysis until they could explain the discrepancies.

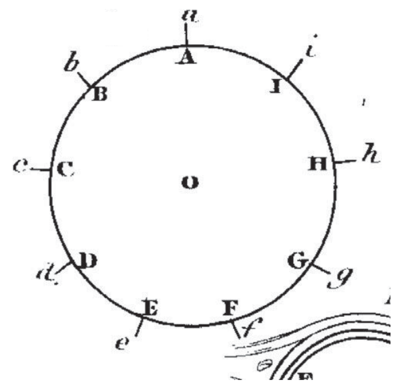
Simanek [Si] points us to a third place people might think they find a Hollow Earth theory in Euler's work:

> Some books and websites say that Leonhard Euler proposed a simpler hollow earth model. Some give details, but few provide a reference. One that does cites: Euler, Leonhard; **Letters of Euler on Natural Philosophy**, Vol 2, Letter LVIII, pp. 202–203, 1835. However, that references his comments on an interesting mathematical problem: "If you drilled a hole all the way through the earth, and dropped a stone in the hole, what would happen?" It's a "thought experiment" and someone may have misread Euler, supposing Euler really thought there was hole all the way through the earth. Then others picked it up without checking sources.
>
> http://www.lhup.edu/~dsimanek/hollow/morrow.htm

Though Simanek apparently means Volume I, letters XLIX and L (and I think there may be problems with his citation of Halley as well), he is accurate in his account of Euler's *Lettres à un Princess d'Allemagne* [E343] and in his warning against believing such things "without checking sources."

What did Euler really say? In Letter XLIV, he gives us the illustration at the right that he calls "Fig. 30."[1] Here, he explains that the direction "down" changes at different locations on the earth. At one point, it may be in the direction aA, another bB, etc., but always, "down" means "towards the center of the earth," marked O. He writes:

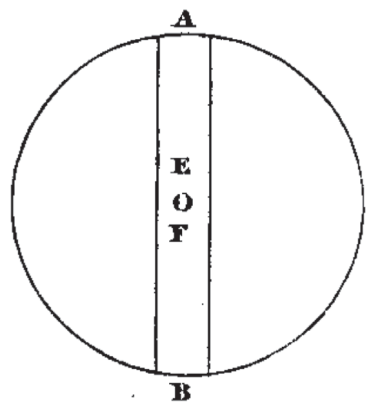
> "In fact, were you to dig a hole in the earth, at whatever place, and to continue your labour incessantly, digging always downward and downward perpendicularly, you would at length reach the centre of the earth.... It is true, such a project could never be executed, as it would be necessary to dig to the depth of 3956 English miles; but there is no harm in supposing it, in order to discover what would be the result.



Having explained what "down" means at the surface of the earth, he invites us into his hypothetical hole to see what "down" means inside the earth. He gives us "Fig. 31," with his hole clear through the earth from A to its antipode at B. He explains that, whether one falls from point A, B, E or F, one would always fall towards the center of the earth, that is, towards point O.

Euler continues his discussion in Letter L, writing:

> "Let us now return to the aperture made in the earth through its centre; it is clear, that a body at the very centre must entirely lose its gravity, as it could no longer move in any direction, all those of gravity tending continually toward the centre of the earth....
>
> "Having travelled, in idea, to the centre of the earth, let us return to its surface, and ascend to the summit of the loftiest mountains."



Thus it is clear that Euler is, indeed, doing a thought experiment. There is no real hole to the center of the earth, and he isn't even considering a hollow earth, just one with a hole in it.

[1] In Euler's time, it was difficult to include figures in the text of a book, so usually all the illustrations were gathered together and printed on just a few sheets, which were bound in the back of the book. The lines in the lower right of this illustration are part of a different figure, one about the anatomy of the eye, not part of his figure of the earth.

We conclude that Euler did *not* propose a theory that the earth is hollow.

This column has relied rather too heavily on web resources, so some readers might think that the pseudo-scientific ideas set forth here circulate only because of the Internet. This is not the case, as the remarkable bibliography [Fr] that Ruth Freitag prepared for the Library of Congress demonstrates. Among over sixty books and articles on her list, there are five from the 1820's that endorse the Hollow Earth theory. The list further suggests that the theory enjoyed resurgences in the 1880's and again in the 1930's. The Internet did not create these theories. It only makes them easier to find.

It is quite difficult to separate fact from fiction when trying to write about the history of scientific fantasies and hoaxes.[2] I believe that three of the sources cited here, [Fr, L, Si], should be taken particularly seriously, and they should be forgiven if a band of hoaxters were able to mislead them about Euler's role in the hollow earth theories. Of course, maybe I was fooled, too. I hope that we've set the record straight, and we can treat it all as a good April Fool's Day joke.

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[Si] Simanek, Donald E., "Turning the Universe Inside-Out: Ulysses Grant Morrow's Naples Experiment," http://www.lhup.edu/~dsimanek/hollow/morrow.htm. 211, 212

[2] I'm trying to make a distinction here. I am taking a "fantasy" to be something that a person makes up and believes. A "hoax," on the other hand, the person who makes it up does not believe it, but wants other people to believe it. If he does not mean other people to believe it, then it is just "fiction."

32

Fallible Euler

(February 2008)

By now, regular readers of this column might have come to believe that, except for occasional computational errors beyond the 15th decimal place, and except for a regular and flagrant disregard of the issues of convergence when dealing with series, Euler was always right about everything. Now that 2007, the so-called "Euler year" is over and the celebrations of the 300th anniversary of his birth are winding down, perhaps we will be forgiven if we admit an uncomfortable fact: Euler was sometimes wrong. We are devoting this month's column to a few of the things Euler was wrong about.

Lunar atmosphere

Euler thought that the moon had an atmosphere. In [E142], *Sur l'atmosphere de la Lune prouvée par la dernier eclipse annulaire du Soliel*, (On the atmosphere of the Moon, proved by the recent annular eclipse of the Sun), Euler describes the observations made in Berlin of the eclipse of July 25, 1748. Euler says that he himself took part in the observations, and this would be a rare example of Euler taking his own data. Other sources indicate that sisters Christine and Margarethe Kirsch assisted him. Christine is known for carefully keeping a diary of the weather for many years.

Euler and his assistants set up a telescope in a darkened room, making what we call a *camera oscura*. This allowed the image of the sun to be projected onto a white screen. The details of the eclipse had been calculated in advance by Johann Kies, and they had used Kies's calculations to draw a circle on the screen in the position where the eclipse was predicted. If the calculations were accurate, the image of the eclipse at its maximum would exactly coincide with the circle at exactly the time predicted.

They didn't. Though the time and position of the images were as predicted, the sizes were observed to differ in two significant ways.

First in order of occurrence, but probably second in importance, the crescent of the sun as the eclipse approached its maximum (an annular eclipse doesn't reach *totality*) did not behave as expected. When an annular eclipse occurs, the moon is a little too far from the earth for totality, so the disk of the moon is not large enough to cover the whole disk of the sun. As a result, as the disk of the moon almost covers the sun, the horns of the crescent of the sun become very sharp as they apparently move around the disk of the sun. This is shown in Euler's Fig. 1.

As the eclipse approached its maximum, the tips of the horns were observed to move *outside* the circle that Euler and his assistants had drawn on their screen. That circle is *DFEAEF* in the diagram, while the disk of the moon is seen as arc *GFBFG*. The tips of the horns are seen outside the predicted disk of the sun at the two points marked *G*.

The second way in which the observations deviated from the predictions was that at the maximum, the apparent diameter of the disk of the sun was clearly larger than it should have been.

For this particular eclipse, Kies had calculated the apparent radius of the sun to be 952 seconds of arc, and that of the moon to be 898″. Moreover, Kies calculated that at the maximum of the eclipse, the apparent centers of the sun and the moon would not quite coincide, but would be 53 seconds of arc apart. Since $952 - 898 = 54$, the annulus at the maximum should have been obviously eccentric, only 1″ at its thinnest, and 107″ at its thickest. Euler observed the thick part of the annulus to be 107″, but at its thinnest it was 26″ instead of 1″.

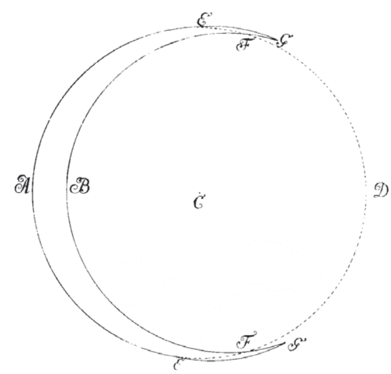

Fig. 1

Colin Maclaurin had made a similar observation of an annular eclipse on February 18, 1737, and had concluded that the image of the moon appeared smaller than expected, but Euler's observations, along with the carefully drawn circle on the screen, showed that it was the sun's image that was too large, not the moon's that was too small.

Armed with these observations, and with the theory of refraction of light, Euler concluded that the image of the sun was being magnified by the refraction effects of an atmosphere on the moon.

Euler continued his calculations and found that to produce the observed refraction, the atmosphere of the moon was about 1/200 as dense as the atmosphere of the earth.

He is not entirely confident of his "discovery" of a lunar atmosphere, though, for he writes that "this celebrated question has agitated astronomers for a long time: *whether or not the moon has an atmosphere*, has not yet been decided.

Euler proposed further research on the lunar atmosphere. He suggested that people could measure the apparent angles between stars as the moon occulted one of them. I do not know whether these experiments were ever performed.

Now that men have walked on the moon, we are sure that Euler was wrong; the moon has no atmosphere. The phenomena Euler observed are optical effects of light passing close to a sharp edge, and not the refraction of a lunar atmosphere.

Aether and planetary friction

From our 21st century point of view, it is easy for us to respect Euler's error about the atmosphere of the moon. He followed good scientific methodology in making his observations. He formulated a plausible theory consistent with well-established scientific facts, and he proposed experiments to attempt to test his theories.

It is a little more difficult for us to set aside our modern prejudices to accept the 18th century ideas of so-called *aether*. Not to be confused with the soporific solvent "ether," the aether was a kind of subtle fluid that filled the spaces between material substances and provided a medium for transmitting phenomena like gravity, light, magnetism and electric fields.

The 18th century view of the world resisted the idea of "action at a distance." Natural philosophers of the time thought that for things to affect one another, there must be some kind of connection between them to carry the effects. Sound behaves like light in many ways, and vacuum experiments in the 17th century had shown that sound does not travel in a vacuum. It must be carried by some kind of physical medium, usually air. It stood to reason that the other physical phenomena required a physical medium to transmit them. The theory of aether provided that medium.

The physical properties of aether were hard to pin down. It had to be light, flexible and porous. It had to be strong enough to hold things apart, but thin enough to allow things to move. It had pores that allowed light to move in any direction in a way that one beam of light never interfered with another beam of light. It had to be so subtle that it never got in the way of any physical phenomenon, but so powerful and omnipresent that it was always there when it was needed.

Scientists, philosophers and theologians of the time needed aether, for without it, phenomena like magnetism, electrical fields and gravity would be "action at a distance," and they could not accept that. Action at a distance was an occult phenomenon, in both senses of the word "occult," "hidden" and "ungodly." They generally agreed with Leibniz that the world was a beautiful and righteous creation, and at the same time agreed with Newton that the workings of the universe could be studied by analysis and experimentation. The existence of occult phenomena was at odds with both outlooks. Hence, they needed aether to avoid action at a distance.

It requires a deep understanding both of modern science and of 18th century natural philosophy to understand whether this is really different from some science today, when people spend careers and fortunes in a search that transcends galaxies and dimensions, looking for gravity waves and gravity particles.

Having stirred up that hornets' nest, we'll quickly move on.

Euler, like most scientists of his day, believed in the aether and he would appeal to its properties when he tried to explain natural phenomena. He, and others, hypothesized that the aether might exert friction on the planets and comets in their orbits. This phenomenon might be too subtle to observe on earth, perhaps because of the interference of other matter, or perhaps because the planets move so much more quickly than anything moves on earth.

Euler wrote one paper [E89] and three published letters [E183, E184, E218], and his son Johann Albrecht wrote two versions of one paper [A8, A8^2] on the resistance the aether exerted on the motion of planets and comets.

In the late 1740s and early 1750s, astronomers and mathematicians were struggling to explain minute discrepancies in orbits of the moon and planets and in the earth's rotation that had been observed using the improved and more accurate scientific instruments that had been developed. New instruments and new mathematics combined, for example, to help Euler understand how the fact that the orbit of Saturn is slightly inclined to the orbit of Jupiter made their influences on each other's orbits different than the simpler co-planar models predicted. [W] Similar progress enabled Clairaut to explain irregularities in the moon's orbit around the earth. [K]

When 18th century observations suggested that the length of the year might be shorter than it had been in the three previous centuries, Euler noted also that the orbital period of Halley's comet had also been decreasing, and he set out to offer an explanation for these apparently related observations.

It may seem counterintuitive, especially to those who have not studied physics, but a shorter orbital period means that the body is slowing down. A slower body moves into a lower orbit, hence has a shorter distance to travel each time it orbits. The shorter distance more than compensates for the slower speed, and the time it takes for the slower body to orbit is actually *less* than the time it takes for a faster body.

Thus, a shorter orbital period would mean that the earth and Halley's comet, and presumably other bodies orbiting the sun as well, were slowing down.

On the other hand, comparisons with ancient observations made by Ptolmey about 150 CE seemed to suggest that the earth's year was instead getting longer, not shorter. This was consistent with the theory of vortices, an alternative to the theory of aether and the point of view favored by Descartes. I confess that I find vortex theory even more confusing than the theory of the aether. It seems that according to Descartes, all of space is filled with fine particles, moving in complex, swirling paths. In their larger and longer swirling paths, the vortices carry the planets and comets in their orbits, while in on a smaller scale they help manage trajectories, currents and other phenomena.

The property of vortices that we care about here is that sometimes a moving body could build up vortices behind it that help to push the body along. Thus, vortices could provide what some have called "negative resistance."

The theory of vortices was not entirely inconsistent with the theory of aether, but they were rival theories on many points. In particular, if the year were getting longer, as suggested by Ptolmey's data, then that would tend to support the vortex theories, but if were getting shorter, as suggested by Halley's comet and other 18th century data, then it tended to support planetary resistance due to the aether.

In 1746, Euler wrote *De relaxione motus planetarum* (On the enlarging of the motion of the planets) [E89] and also exchanged some letters with the astronomer Joseph Nicolas Delisle, (1688–1768) [R517][1] to make some speculations, backed with calculations, about these theories and observations.

[1] Over 3000 items of Euler's correspondence are cataloged and indexed in Series IV-A volume 1 of his *Opera omnia*, edited by Juskevic, Smirnov and Habicht and published by Birkhäuser in 1975. They assigned each item an "R-number," similar to the way Eneström assigned index numbers, now called E-numbers, to Euler's published works. As we shall see later in this column, some of Euler's letters have been published individually, and some of those items have both R-numbers and E-numbers. We will give both.

In his letter to Delisle, Euler proposed that a day had been lost in counting the days since Ptolmey's observations, so that rather than lengthening, the length of the year had decreased from 365 days 5 hours 55 minutes in Ptolmey's time to 365 days 5 hours 48 minutes in 1700, and that "this diminution of the year is the effect of the resistance of the aether, which I have explained in a piece that will appear." So, with Euler's proposed correction Ptolmey's data supported a shortening year rather than a lengthening one.

Of course, E89 was the paper that Euler said would appear. In that paper, Euler explained how the resistance of the aether would shorten the year, rather than lengthening it as intuition would suggest. In addition, Euler knew a lot about friction on spherical objects as they move through resisting media because of the work he'd done on ballistics and the trajectories of cannonballs. [S Dec06, S Jan07] He reversed those same calculations to try to calculate how dense the aether must be in order to provide the resistance necessary to shorten the year enough to fit the observations. He concluded that air must be about 4×10^8 times as dense as aether, and that that density would also account for the observed shortening of the period of Halley's comet.

Later, in 1749, Euler wrote two letters to Caspar Wettstein, Chaplain to the Princess of Wales and a friend of Euler from their days in St. Petersburg, asking first if Wettstein could help Euler locate some Arabic astronomical observations to corroborate those of Ptolmey [E183=R2763], and the second letter [E184=R2765] to warn that, even if ancient observations didn't confirm that the number of days in a year was decreasing, then it was also possible that the friction of the aether was also changing the length of the day, and the length of the year could be decreasing without our being able to detect it.

In 1754, Euler also wrote a letter to the Norwegian philosopher Erich Pontoppidan [E218=R2021] in which Euler agreed that there could be no life as we know it on a planet as distant as Saturn, hence the date of Creation had to be more recent than when the Earth's orbit was as far from the sun as that of Saturn is today.

Modern theory holds that the experiments of Michelson and Morley, performed in 1887, proved that there was no aether, that the period of the earth's year has not changed measurably, and that the period of Halley's comet has shortened because of gravitational effects of the outer planets. Euler was chasing a false theory. On the other hand, most of us agree that people don't live on Saturn.

The nature of the atmosphere

Air is a tricky thing. It is hard to see and hard to hold on to, but for some reason the air around the earth doesn't just float away into space. Euler lived in a time after Robert Boyle showed that the pressure of a gas is inversely proportional to its volume in 1660 and before Émile Clapeyron unified many properties of gasses into the Combined Gas Law in 1834. Even so, this progress in thermodynamics only described the properties of air, and not the nature of air and why it had these properties. Euler sought to provide a model for the mechanics of air that could explain its observed properties. He first made his speculations in one of his very early works, *Tentamen explicationis phaenomenorum aeris*, (Tentative explanations of the phenomena of air) [E7] and he added to his ideas late in his life in *Conjectura circa naturam aëris, pro explicandis phaenomenis in atmosphaera observatis*

(Conjectures about the nature of air, for explaining phenomena observed in the atmosphere) [E527], written in 1780 and published in 1782.

Euler proposed that in its most compressed state, air consisted of bubbles of water filled with aether, as shown in Euler's fig. 1 below, taken from the 1727 volume of the journal of the St. Petersburg Academy.

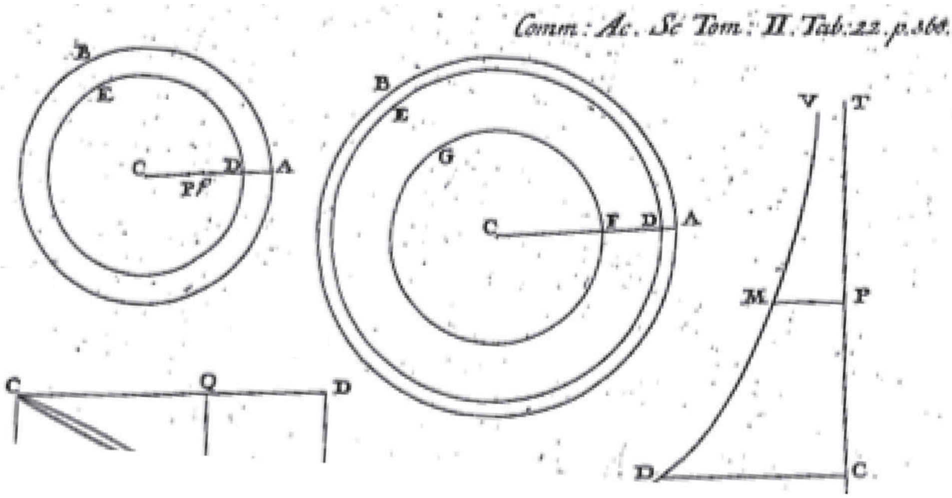

Fig: 1 Fig: 2

This shows the bubble centered at C, with inner radius CD and outer radius CA.

Euler's fig. 2 shows what happens when air gets warmer. The aether inside the bubble begins to spin, and the centrifugal force opens a vortex inside the aether, shown as the circle with radius CF. This allows the bubble to expand, thus providing a mechanism for air to expand when it gets warmer.

The vortex also gives a way for pressure to compress the bubble, providing a means for Boyle's Law to apply.

Finally, the surface of the bubble is made of water. When water evaporates, it simply thickens the surface of the bubble, simultaneously explaining where water goes when it evaporates and why moist air is heavier than dry air.

Other scientists of the era apparently did not find Euler's speculations very convincing, and, like most ideas in the scientific ritual of trial and error, these ideas of Euler were relegated to the "error" group.

Other Euler errors

Euler had a number of other misconceptions that we don't have room to discuss here. He thought that the tails of comets had the same origins as the lights of the aurora borealis, and they were both related to a fascinating but relatively unknown phenomenon called "zodiacal light." In fact, they are three entirely different phenomena, and Euler was completely wrong about their cause. He also used hypothesized properties of the porosity of the aether to explain the phenomena of heat transfer, magnetism and electricity, and briefly flirted with the idea that Newton's inverse square law was only an approximation, and that there had to be another term, inversely proportional to r^4, introduced by the pressure of the aether.

As we look at these errors of Euler, we see that they are all rooted in the scientific worldview of his times, and that we should probably forgive him for not knowing things that would not be discovered for a century or more after he died. It serves to remind us that Euler was not a 21st century scientist trapped in an 18th century timeline, but rather he was a citizen of his own times, some of whose accomplishments are still interesting and relevant today.

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33

Euler and the Pirates

(April 2009)

We sometimes celebrate the first of April with a column on the lighter side of Euler scholarship. We continue that occasional tradition with some stories intended to help perpetuate the idea that no matter where we look, we can find a connection with Euler.

Once is amusing. Twice is a coincidence. Three times is worth remarking about.

We've recently come across a third, and maybe a fourth, person with connections both to Euler and to piracy or privateering. For our collective amusement, we thought we'd share them with you.

Maupertuis

The first, of course, was Maupertuis (1698–1759), President of the Berlin Academy for much of the time Euler was there. According to Mary Terrall's award winning biography, *The Man Who Flattened the Earth*, [Terrall 2002] his family fortune was built on piracy. His father, René Moreau, had been a merchant and ship owner working out of the French port of San Malo. When he got a license from the King and agreed to prey mainly on British ships, he was quite successful and made a fortune in the 1690s by attacking English shipping. He married well enough to be a candidate for the aristocracy, and after he retired from seafaring, he was awarded the hereditary title "sieur de Maupertuys" for Services to the Crown.

As a scientist, Maupertuis is best known for leading an expedition to Lapland to take measurements on the shape of the earth, and then combining those measurements with data from other expeditions that he had organized to determine that the Earth was not a perfect sphere, but instead it bulges at the equator, as Newton had predicted.

Maupertuis is also credited with the discovery of the so-called Principle of Least Action, which he stated as, "In all the changes that take place in the universe, the sum of the products of each body multiplied by the distance it moves and by the speed with which it moves is the least possible." Though Maupertuis was not the clearest of writers, the Principle of Least Action is a great and basic principle of physics. It leads to several important laws of conservation and explains why so many problems in physics involve maximization and minimization.

Euler used the Principle of Least Action to great advantage, and it provided much of the impetus for his work in the calculus of variations. Indeed, some people suspect that Euler had formulated the Principle of Least Action before Maupertuis did, but ceded priority to avoid disputes at the Academy. They might argue that René Moreau had not been the only pirate in the family.

Henry Watson

The second privateer was Henry Watson (1737?–1786), the Englishman living in India who translated and published Euler's *Théorie complette* [E426] from its original French in 1776, just three years after Euler published it. Watson's friends published a second edition of the translation in 1790, to which they added a "Sketch of the life and character of the late Col. Henry Watson." There we learn that "though *Holland* may boast a *Coehorn*, and *France* a *Vauban*, yet *England* can boast their superiors in a *Robins* and a *Watson*." Menno Coehorn (1641–1704) and the Marquis de Vauban (1633–1707) were important military engineers, and Benjamin Robins (1707–1751) wrote the book on artillery that Euler subsequently translated from English into German. [E77, Sandifer December 2006] All three were more famous in 1790 than they are today.

The Late
COLL. HEN^Y. WATSON.
Chief Engineer.

Watson's friends also tell us that "as early as 1753, he cut a conspicuous figure as a mathematician in the Ladies Diary," a popular forum in England at the time for exchanging recreations and problems in mathematics, and whose readership was not confined to women. [Peterson 2009] "He gave signal proofs of his superior abilities as an engineer; particularly at the siege of Belleisle in 1761." That siege was the subject of a Mother Goose nursery rhyme. In 1762, during the siege of the Morro Castle in Havana, he was carried to his tent, thought dead, but he revived and returned to the field to see the fort captured.

Watson lost a fortune reported to be £100,000 (worth about $25 million today) to the British East India Company when they let his project to modernize the docks in Calcutta go bankrupt, then bought the nearly-completed project for pennies on the pound. He angrily went back to England to try to sue the BEIC to recover his fortune, but died of a fever before he arrived.

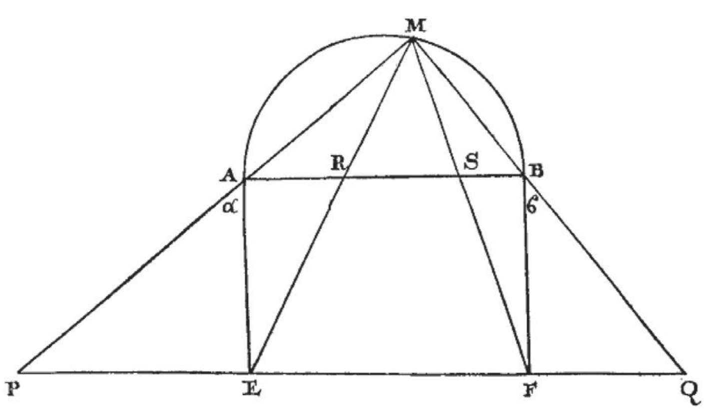
Shortly before that, he'd borrowed money to build three ships based on Euler's innovations, completing only two of them. THEN he got around to applying to the Crown for a privateer's license, intending to advance England's interests near the Philippines. He didn't get the license and instead used them as merchant ships. Some sources say he used them to transport opium, but others say that he put them into the tea trade where, his (very sympathetic) eulogists tell us that they were among the swiftest and best-handling ships on the seas.

Watson was an important and mostly-successful man of his times, and though he was never actually a pirate, he'd wanted to be one.

Kenelm Digby

A third privateer has more tenuous connections to Euler. Euler read Fermat rather systematically, and used Fermat's problems as a source of inspiration, especially in number theory, but also in geometry. One interesting problem in geometry [Sandifer December 2008] was from a letter from Fermat to Kenelm Digby (1603–1665). [Fermat 1658, Fermat 1894] The original copy of that letter is now lost, but as was common in that era before photocopying machines, Digby transcribed the entire letter and sent it on to John Wallis, and that transcription survives among Wallis's correspondence.

That problem involved the figure below, in which AMB is a semicircle and $ABFE$ is a rectangle with sides in the ratio $\sqrt{2} : 1$. Then for any point M on the circumference of the semicircle, if M is joined to the points E and F and if the line segments ME and MF cut the diameter at R and S respectively, then the segments AS, SB and AB satisfy the relation $AS^2 + RB^2 = AB^2$. It suggests a Pythagorean theorem, but none is evident. Euler solved the problem in [E135].



A second connection might be made through a substance called the "powder of sympathy." This alchemical concoction was made from copper sulfate and rainwater. According to Digby, he could cure a wound by rubbing the powder of sympathy on the weapon that had inflicted the wound or even by immersing bandages that had covered the wound in a solution containing the powerful material. Digby explained that it worked by "attraction and by the small material particles given off by all objects." Moreover, and this is a key detail, Digby observed that his patients would swoon and cry out as he applied his cure to the weapons or bandages.

A hundred years later, in Euler's time, the greatest technological problem of the era was to find a way of calculating longitude at sea. Untold thousands of lives and, perhaps worse in some eyes, valuable treasures had been lost when ships were lost at sea for want of more accurate navigational tools. In the 17th century, the Paris Academy had offered a generous prize for a means of finding longitude at sea, but had squandered the prize on a fairly simple water anemometer that didn't work if there were any currents involved and turned out to be useless. In 1714, the British Parliament offered an even more lavish prize, up to £20,000 (worth about $5 million today.) That prize was eventually awarded for a clock invented by John Harrison, but the Longitude Board was so slow in making its decision that Harrison had died before collecting the full prize. The Board also awarded a £300 consolation prize to Euler for work he had done on accurate calculations of the orbits of the moon and 4 planets that were, in themselves, very useful and *could* have led to a different solution to the problem. Therein lies the Euler connection.

One of many unsuccessful solutions, a method that Dava Sobel describes as "canine vivisection," [Sobel] would have used Digby's powder of sympathy. The plan was that before a ship was to set out, they would wound a dog and then bandage the wound. [Brown 1949] As the ship was about to set sail, they would remove the bandage. Then, every hour on the hour, someone back home would immerse the dog's bandage in a solution containing the powder of sympathy. The dog was supposed to swoon and cry out, and the mariner at sea would know what time it was back in his homeport. By comparing the time back home to the local time, observed by measuring the angle of the sun or some other reliable astronomical object, and using the conversion factor 1 hour = 15 degrees, the mariner could calculate his longitude. The solution wasn't that much different from Harrison's scheme, but it involved dogs and the powder of sympathy instead of a clock. Also, unlike Harrison's clock, canine vivisection didn't work.

Those are the two Euler connections. What about Digby as a pirate?

Kenelm Digby was everything. His father had been executed following the Gunpowder Plot. Digby distinguished himself as an alchemist and was one of the founders of the Royal Society.

In 1627, Digby led an expedition of privateers to the Mediterranean, hoping to gain wealth and fame by capturing and plundering French merchant ships. All the way east across the Mediterranean, he and his crews sighted nothing but fishing boats and neutral shipping. But when they got to the far northeastern corner of the Sea, off a Turkish port named Scanderoon, he found and attacked a group of French and Venetian merchant ships and managed to fill his two ships with loot and return to England. Depending on whose side you're on, this was either a daring attack on enemy shipping or it was senseless aggression on what had previously been a friendly port. Either way, Digby became wealthy and he was knighted for his success on behalf of the English King.

John Paul Jones

We might make an equally tenuous connection with John Paul Jones, ex-slave trader and "Father of the American Navy." Jones served in the navy of Catherine the Great for a few months in 1788. He was stationed in St. Petersburg, but apparently never actually sailed under the Russian flag. Euler had died there five years earlier. During the American Revolution, John Paul Jones had been charged with piracy in the Netherlands for attacking ships "under an unknown flag," the flag of the new United States of America. When the American flag was duly entered into the appropriate Dutch records, the charges were dropped.

JPJ's months in St. Petersburg were tarnished with scandal. The account of this episode on Wikipedia is highly sanitized. A letter from one of Euler's sons in St. Petersburg to relatives in Berlin survives in the archives at Yale, and the "gossip" there seems to be true.

So, does this connection between Euler and JP Jones count? Does Jones count as a pirate, since the charges were soon dropped?

Conclusions

Euler's name does not loom large in the history of piracy and privateering. Indeed, our four "pirates" are all better known for something other than piracy, if, indeed, they are known at all.

In recent years, it has become popular in historical analysis to emphasize how things are connected to other things, what Lovejoy calls the "essentiality of relations." [Lovejoy 1950, p. 10] Indeed, in the internet age, connections are a modern way to understand things, including history. But sometimes, as with Euler and the Pirates, there are connections, but they don't mean anything. They're just fun. We hope this was fun.

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34

Euler as a Teacher – Part 1

(January 2010)

What would it have been like to be a student of Leonhard Euler? Until we invent time machines, it will be impossible to answer this question. Still, it is almost impossible not to ask it anyway.

By all accounts, Euler was regarded as an excellent teacher. His eulogists all mention that he was a kind and pious man, a great genius and wonderful teacher. There are stories that students from abroad, particularly from France and Russia, studying in Berlin would rent rooms in Euler's house and that they would talk about science and mathematics at meal times. Accounts of the last day of Euler's life, in September 1783, include the detail that he spent part of that day teaching mathematics to his grandson.

Despite these flowery tributes, we have very few accounts beyond the snippets cited above of Euler actually in the act of teaching.

When Peter the Great founded his Academy of Sciences in St. Petersburg, he intended it to be a comprehensive scholarly institution, doing the latest in scientific research, educating the youth of Russia and promoting the sciences among the general public. Euler supposedly had responsibilities in all three points of this academic mission, but he seems to have left few footprints outside his work in basic research.

The archives of the Academy [SPA 1886, pp. 222–224] reprint a catalog that tells us that in 1732, Euler would teach physics on Monday and Tuesday mornings, and on Saturdays he "will confirm its truth by experiments." His course seems to be the only one in the catalog (which is only two pages long) that tells us that he will be using a textbook. "In this matter, he will follow the esteemed s'Gravesand's book *Physices Elementa Mathematica.*"

In 1734, he taught his Course in Mathematics from 2:00 to 3:00 in the afternoon. The catalog [SPA 1886, pp. 554–555] does not tell us which days that course met, nor does it mention a textbook.

These were the only catalogs I could find from Euler's time in St. Petersburg. I found nothing to describe the outcomes of these courses. They apparently didn't do student evaluations, nor did deans or directors observe and evaluate the quality of teaching. Indeed, I came across a contract for a French teacher named Comble. It says

> Comble is engaged for six consecutive years, and he promises not only to acquit himself faithfully to his duties in the capacity of teacher of the language, but also to all other functions which the directors of the Academy find appropriate to charge to him. If after the term of six years has expired, he no longer wishes to continue in his service, he is obligated to declare that in writing a year in advance, and he will be accorded his departure without any difficulty.

The contract says nothing at all about the quality of his work, and has no provisions by which he might be fired, though perhaps he could lose his job if he did not "acquit himself faithfully."

Euler's classes in the Academy's Gymnasium could not have been very large. The archives give us the number of students admitted each year for the years 1726 to 1738. Students generally aged between 10 and 19 years old when they were admitted, though I came across one student, the son of one of the Academy's typesetters, who was only five and a half years old. Most were sons of military families, though there were occasional sons of merchants, and one was the son of an attendant to a princess. Education was for the upper middle class, and was only for boys.

New students at the Academy's Gymnasium, by year

1726	112	1732	21
1727	57	1733	46
1728	28	1734	34
1729	74	1735	23
1730	14	1736	22
1731	28	1737	16
		1738	12

I find occasional announcements of public lectures in St. Petersburg, intended to promote the sciences among the general public, following the dictates of Peter the Great. Lectures concerned then popular subjects. One described the dissection of an elephant. Another addressed the delicate question of whether it was possible to tell if the earth moved around the sun. The question was delicate because the Orthodox Church already had a strong opinion on the subject. I can find no mention that Euler ever gave any of these lectures, though in on January 31, 1732, he "responded on behalf of the Academy" when Gmelin gave a lecture "On the origin and progress of Chemistry." It is not clear what Euler's responsibilities were here.

Euler was well respected as a textbook author as well. [Katz 2007] He wrote a series of textbooks that extended from elementary arithmetic (the *Rechenkunst* of 1738 and 1740), through algebra, precalculus and differential and integral calculus. Moreover, we could consider the *Methodus inveniendi* as a textbook on the calculus of variations and his *Letters to a German Princess* as a textbook on science and natural philosophy.

We know something about Euler's own education. In his autobiography, [1] he tells us that he believed that he had the best of all possible educations at the feet of Johann Bernoulli. His Bernoulli lessons are also recounted in the Fuss eulogy [Fuss]:

> It did not take long before he was noticed by Johann Bernoulli, the greatest of the living geometers. He soon distinguished himself from his fellow students and since Bernoulli was not able to provide all that his young mathematician asked of him, he told him to bring all the problems that he encountered when studying and every Saturday he would help him work through them. This instilled an excellent process; but only one that can succeed with an extremely talented genius which Mr. Euler possessed. He was destined to exceed his teacher who at the time was unsurpassed in mathematics.

Little is known about Euler's teaching activities during his Berlin period (1741–1766). One of the few facts about this period comes from a footnote in the Fuss eulogy, from which it appears that Euler made some effort to recreate his own education with Russian students

> (6) Euler opened his home to students that the Academy sent to Berlin to study mathematics. Mssrs Kotelnikov and Rumovsky spent a number of years in this situation and enjoyed the teachings of their incomparable master.

In an upcoming column, we'll consider Euler as a teacher during his second St. Petersburg period.

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[1] "My father's life as dictated to me by him. Recorded [by J. A. Euler] in St. Petersburg on the 1st of December 1767." [Fellmann 2007, p. 5–7]

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Euler as a Teacher – Part 2

(February 2010)

Let us start with the Great Quotation, dubiously attributed to Laplace by Guglielmo Libri[1] about 1846

Lisez Euler, lisez Euler, c'est notre maître à tous.

We traditionally translate this as

Read Euler, read Euler. He is the master of us all.

This gave Bill Dunham a title befitting his most excellent book, [Dunham 1998] but there are other ways to translate it. Because maître $= \begin{cases} \text{master} \\ \text{teacher} \end{cases}$ and "*notre … à tous*" can mean "of us all" or "*notre*" can be assigned to modify "*maître*", leaving "*à tous*" to mean "all things", other valid translations include:

Read Euler, read Euler. He is our master in all things.
Read Euler, read Euler. He is the teacher of us all.
Read Euler, read Euler. He is our teacher in all things.
etc.

St. Petersburg 1766–1783

In Part 1 of this column [Sandifer Jan 2010] we looked at what is known about Euler the Teacher during his first St. Petersburg period (1727–1741) and his time in Berlin (1741–1766). Condorcet [Condorcet 1786] gives us some accounts of Euler's teaching in his second St. Petersburg period:

[1] Libri was a scoundrel, a forger, a book thief, and an indifferent mathematician, [Rice 2003] but he did write a decent history of mathematics. In Libri's defense, note that he claims that he heard these words "*de sa propre bouche*," from Laplace's own mouth, not that Laplace actually wrote them down. [WikiQuote]

> His sons and students copied his calculations and wrote by dictation the remaining *Mémoires*.
>
> It can be seen that he much preferred the education of his students than the small satisfaction derived from astonishment; he never believed that he had truly done enough for Science if he did not feel that that he had added new truths to enrich it and the exposure of the simplicity of the idea which lead him there.
>
> ...
>
> Of the sixteen professors attached to the Saint Petersburg Academy eight were trained under him and all are known through their works and have been awarded various academic distinctions and are proud to add the title of Euler's disciples.

Condorcet goes on to mention some of the people who studied under Euler, his two sons, Lexell, and Fuss in particular, and that Fuss married one of Euler's granddaughters.

Except for disclaiming "the small satisfaction derived from astonishment," this does little to tell us about how Euler taught or why he was effective. The stories that Condorcet relates about:

- Euler's students taking dictation,
- Euler reading things written large on a tablet or chalkboard,
- Euler wearing a shiny track in the table as he used it to guide himself while he paced around it and talked to his students, tell us very little about how Euler actually taught.

Euler last attended a meeting of the St. Petersburg Academy on January 16, 1777, after which he sent his papers in to the Academy with his assistants. One of the portraits of Euler, shown above, has a sub-portrait, a smaller rectangle beneath the oval of the main portrait. The sub-portrait shows two men, one with pen and paper, sitting at a table. Apparently it pictures Euler dictating to one of his assistants, probably his son, Johann Albrecht, because Euler himself could no longer read or write.

Internal evidence

I want to cite two kinds of evidence about Euler's teaching in St. Petersburg: 1° data from the *Adversariis mathematicis*, or Mathematical daybook [E806]; and 2° subjective observations from reading several of Euler's late papers.

The *Adversariis mathematicis*

The *Adversariis mathematicis* was a series of three notebooks kept in the foyer of the St. Petersburg Academy. Members used the *Adversariis* as a kind of chat room or virtual seminar to show their colleagues what they were working on and to announce preliminary results. Eventually, the *Adversariis* filled three notebooks totaling 776 pages. Less than 30% of their contents, amounting to 111 entries, appeared in the *Opera posthuma* in 1862. [E805] They are sprinkled about several volumes of the *Opera omnia* according to the subjects of the notes.

Most of the notes are dull and technical. Many are wrong. Some are dead ends, sometimes intriguing, but ultimately doomed. For example, Note 67 is a 17-page joint effort by J. A. Euler, Lexell, Fuss and Krafft to solve Fermat's Last Theorem. They get stuck on the same technical points of unique factorization that befuddled 19th century mathematicians.

Note 24 is a contribution by Krafft, noting that both $x^2 + x + 17$ and $x^2 + x + 41$ give nothing but prime numbers for small values of x.

Note 104 is by J. A. Euler, and essentially he rewrites Ptolmey's theorem about the sides and diagonals of a cyclic quadrilateral in terms of sines, and then extends it to "infinitely large" circles, i.e. straight lines. It is not clear if he noticed that what he gets is a theorem in geometric algebra found in Euclid's *Elements*, book II.

Note 96 is a proof by Nicolas Fuss of the elementary properties of the Euler φ function. Notes 82 and 83 are about magic squares and Greco-Latin squares. Dozens are about Diophantine equations, especially those related to quadratic reciprocity.

The full text of the *Adversariis mathematicis* has not been published, and I've not seen any description of what fills the more than 70% of the notebooks that were not published in 1862. We can only speculate.

By the time these notes were published in 1862, 79 years after Euler's death, the results were all quite stale, but only a couple would have been interesting even if they had been disseminated earlier.

I tabulated who contributed to the *Adversariis mathematiciis*:

	Number	Pages
Fuss	26	68
Lexell	23	70
JA Euler	17	51
Krafft	11	42
Golovin	3	6
Euler	1	2
Unsigned	56	

Note that the number of signatures doesn't add up to 55, because several notes, like Note 67, were joint efforts with more than one signature. Half the notes were unsigned.

What does this say about Euler's teaching? It looks to me like Euler used the *Adversariis* as a proxy for a graduate seminar. It is as if his five students would read papers from earlier in Euler's career, work through the difficulties, either alone or together, and demonstrate their mastery of the material (or ineptitude) by proving old results in new ways, working through examples, filling in details and extending and generalizing the results. The style closely resembles the way Johann Bernoulli had taught Euler some sixty years earlier.

Late papers

When Euler died in 1783, he left a legacy of over 200 unpublished papers, virtually all of which he wrote after returning to St. Petersburg in 1766. Only a few of them were important. Let's look at a couple of examples.

In *Investigatio quarundam serierum quae ad rationem peripheriae circuli ad diametrum vero proxime definiendam maxime sunt accommodatae*, "Investigation of certain series which are designed to approximate the true ratio of the circumference of a circle to its diameter very closely," [E705, Sandifer Feb 2009] Euler repeats work from *De variis modis circuli quadraturam numeris proxime exprimendi*," On several means of expressing the quadrature of area of a circle very accurately." [E74] The first paper showed how to use the Machin equations,

$$\arctan 1 = \arctan \frac{1}{a} + \arctan \frac{1}{b}$$

to give fast converging approximations to π. The second paper does the same thing, but engineers the series so they have easy denominators that are powers of 2, 5 and 10.

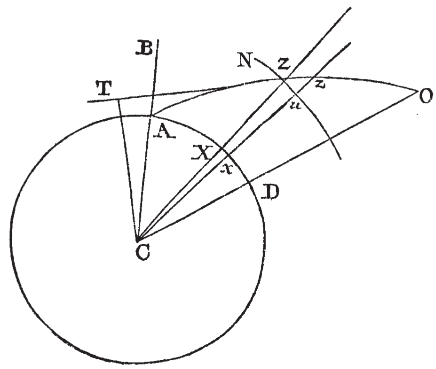

Fig. 4

Sur l'effet de la réfraction dans les observations terrestres, "On the effect of refraction on terrestrial observations," [E502] shows how to correct for the way the varying density of the atmosphere bends light and affects surveying the heights of mountains. It repeats much of the material from *De la réfraction de la lumière en passant par l'atmosphère selon les divers degrés tant de la chaleur que de lélasticité de l'air*, "On the refraction of light passing through the atmosphere due to the different degrees of heat as well as the elasticity of the air," [E219] where he studied the same refraction for astronomical observations.

Many of his other late papers are like this, especially his late number theory, revisiting earlier problems with a new twist or complication, a different proof (not necessarily better) or a longer example. The papers are often a bit choppy and the examples less interesting than in his earlier papers. For a long time, I thought that these were signs of Euler declining as he aged. That may be true, but now I think that these are also the fingerprints of Euler's students. It seems likely that Euler would revisit an old paper and ask his students to see what they could do with it. They could have asked him questions, harvested ideas for extensions, and then, with his guidance, write papers under Euler's name. This would not be much different from a master artist who had his students fill in the backgrounds of his paintings.

Punch line

Euler learned at the feet of Johann Bernoulli, who had Euler "read the masters." Euler read difficult mathematics and Bernoulli helped him when he got stuck.

We now know some details of Euler's classroom teaching during his first St. Petersburg years, but we have no evidence or testimonies about what kind of teacher he was.

We only know a few details about Euler's teaching in Berlin. The setting wasn't formal, but it foreshadowed his later teaching style.

> In his late years, I propose that Euler was able to teach in the style under which he himself had learned. He had learned guided by the principle
>
> Read the masters.
>
> He taught under the style
>
> Read me, read me. I am Euler and I am your teacher in all things.

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About the Author

Ed Sandifer is Professor Emeritus of Mathematics at Western Connecticut State University. He earned his PhD at the University of Massachusetts under John Fogarty, studying ring theory. He became interested in Euler while attending the Institute for the History of Mathematics and Its Uses in Teaching, IHMT, several summers in Washington DC, under the tutelage of Fred Rickey, Victor Katz and Ron Calinger. Because of a series of advising mistakes, as an undergraduate Ed studied more foreign languages than he had to, and so now he can read the works of Euler in their original Latin, French and German. Occasionally he reads Spanish colonial mathematics in its original as well. Ed was the secretary of The Euler Society, and he wrote a monthly on-line column, "How Euler Did It," for the MAA—this volume is a collection of some of those columns. He has also written *The Early Mathematics of Leonhard Euler* and *How Euler Did It*, both also published by the MAA, and edited, along with Robert E. Bradley, *Leonhard Euler: Life, Work and Legacy*. He and his wife, Theresa, live in a small town in western Connecticut. Ed used to be an avid runner and he has over 35 Boston Marathons on his shoes.