Louis Laurencelle
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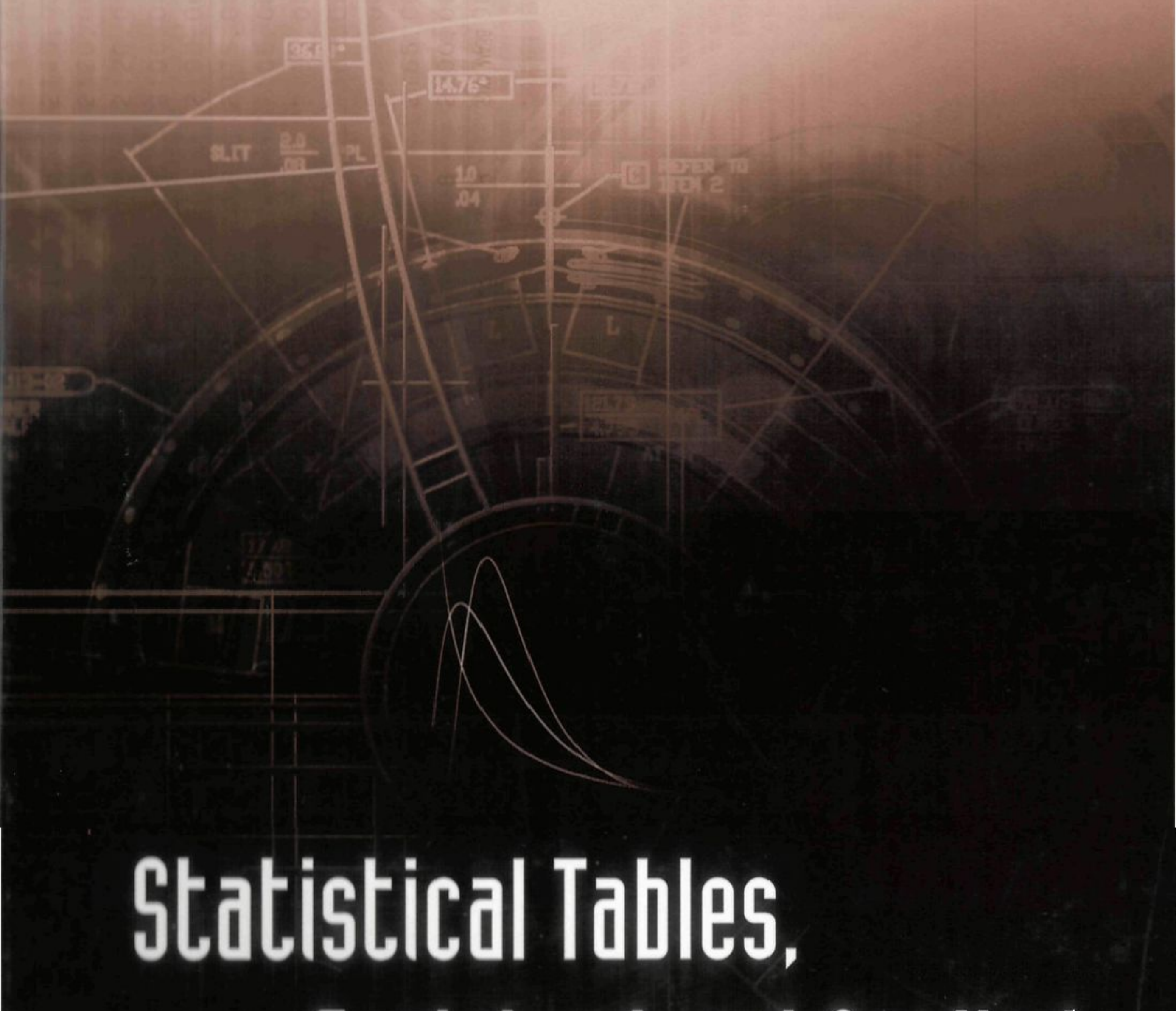

Statistical Tables, Explained and Applied

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STATISTICAL TABLES, EXPLAINED AND APPLIED

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Contents

Introduction

While preparing this book for publication, we had in mind three objectives: (1) to make available, in a handy format, tables of areas, percentiles and critical values for current applications of inferential statistics; (2) to provide, for each table, clear and sufficient guidelines as to their correct use and interpretation; and (3) to present the mathematical basis for the interested reader, together with the recipes and computational algorithms that were used to produce the tables. As for our first objective, the reader will find several "classical" tables of distributions like those of the normal law, Student's t, Chi-square and F of Fisher-Snedecor. All values have been re-computed, occasionnaly with our own algorithms; if our values should disagree with older ones, ours should prevail! Moreover, many other tables are new or made available for the first time; let us mention those of centiles for the $\overline{\mathrm{E}}^2$ statistic concerning non-linear monotonic variation in analysis of variance (ANOVA), of coefficients for the reconversion of orthogonal polynomials, and an extensive set of critical values for the binomial and the number-of-runs distributions. To meet our second objective, we provide, for each distribution, a section on how to read off and use appropriate values in the tables, and another one with illustrative examples. Supplementary examples are presented in a separate section, thus covering most common situations in the realm of significance testing procedures. Finally, our third objective required us to compile more or less scattered and ill-known published documents on the origins, properties and computational algorithms (exact or approximate) for each selected distribution or probability law. For the most important distributions (normal, χ^2, t, F, binomial, random numbers), we present computer algorithms that efficiently generate pseudo random values with chosen characteristics. The reader should benefit from our compiled information and results, as we have tried to render them in the simplest and most easy-to-use fashion.

The selection of our set of tabled statistical distributions (there are many more) has been partly dictated by the practice of ANOVA. Thus, statistics like Hartley's $F_{\max}$ and Cochran's C are often used for assessing the equality of variance assumption generally required for a valid significance test with the F-ratio distribution. Also, Dunn-Šidák's t test, Studentized range q statistic, the $\overline{\mathrm{E}}^2$ statistic and orthogonal polynomials all serve for comparing means in the context of ANOVA or in its sequel. Apart from Winer's classic *Statistical principles in experimental design* (McGraw-Hill 1971, 1991), we refer the reader to Hochberg and Tamhane's (Wiley, 1987) treatise on that topic.

Briefly, our suggestion for the interpretation of effects in ANOVA is a function of the research hypotheses on the effects of the independent variable (I.V.). We distinguish two global settings:

> 1) *If there are no specific or directional hypotheses on the effects of the I.V.*, the global F test for main effect may suffice. When significant at the prescribed level, that test indicates that the I.V. succeeded in bringing up real modifications in the measured phenomenon.
>
> Should we want a detailed analysis of the effects of I.V., we may compare means pairwise according to the levels of I.V.: the usually recommended procedure for this is the HSD test of Tukey; some may prefer the less conservative Newman-Keuls approach. If the wished-for comparisons extend beyond the mean *vs.* mean format to include linear combinations of means (*i.e.* group of means *vs.* group of means), Scheffé's procedure and criterion may be called for, based on the F distribution.
>
> 2) *If planned or pre-determined comparisons are in order, justified by specific or directional research hypotheses or by the structure of the I.V.*, the test to be applied depends on such structure and/or hypotheses. For comparing one level of I.V. (*e.g.* mean $\overline{X}_k$) to every other level (*e.g.* $\overline{X}_1$, $\overline{X}_2$, ..., $\overline{X}_{k-1}$), Dunnett's t test may be used. To verify that a given power of the I.V. (or regressor variable) has a linear influence on the dependent (or measured) variable, orthogonal polynomials analysis is well suited, except when the research hypothesis does not specify a particular function or the I.V. is not metric, in which cases tests on monotonic variation, using6 the $\overline{\mathrm{E}}^2$ statistic, may be applied. On the other hand, if specific hypotheses concern only a subset of parwise comparisons, an appropriate procedure is Dunn-Šidák's t test [akin to the Bonferroni probability criterion].

Such simple rules as those given above cannot serve to cope adequately with every special case or situation that one encounters in the practice of ANOVA. Controversial as they may be, we propose these rules as a starting point to help clarify and make better the criteria and procedures by which means are to be compared in ANOVA designs.

The "Mathematical complements", at the end of the book, is in fact a short dictionnary of concepts and methods, statistical as well as mathematical, that appear in the distribution sections. The purpose of the book being mostly utilitarian, we limit ourselves to the presentation of the main results, theorems and conclusions without algebraic demonstration or proof. However, as one may read in some mathematical textbooks, "the reader may easily verify that..."

Common abbreviations and notations

d.f.	distribution function (of r.v. X), also denoted $P(x)$
df	degrees of freedom (of a statistic), related to parameter ν in some p.d.f.
E(X)	mathematical expectation (or mean) of r.v. X, relative to some p.d.f.
e	Euler's constant, defined by $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots \approx 2.7183$
$\exp(x)$	value of e ($e \approx 2.7183$) to the x^{th} power, or e^x
ln	natural (or Napierian) logarithm
p.d.f.	probability density function (of r.v. X), also denoted $p(x)$
p.m.f.	probability mass function (of X, a discrete r.v.), also denoted $p(x)$
π	usually, area of the unit circle ($\pi \approx 3.1416$). May also designate the true probability of success in a trial, in binomial (or Bernoulli) sampling.
r.v.	random variable, random variate
s.d.	standard deviation (s or s_X for a sample, σ or σ_X for a population or p.d.f.)
var	variance, usually population variance (σ^2)

Normal distribution

✓ Graphical representations

✓ Values of the standard normal distribution function (integral), $P(z)$ (table 1)

✓ Percentiles of the standard normal distribution, $z(P)$ (table 2)

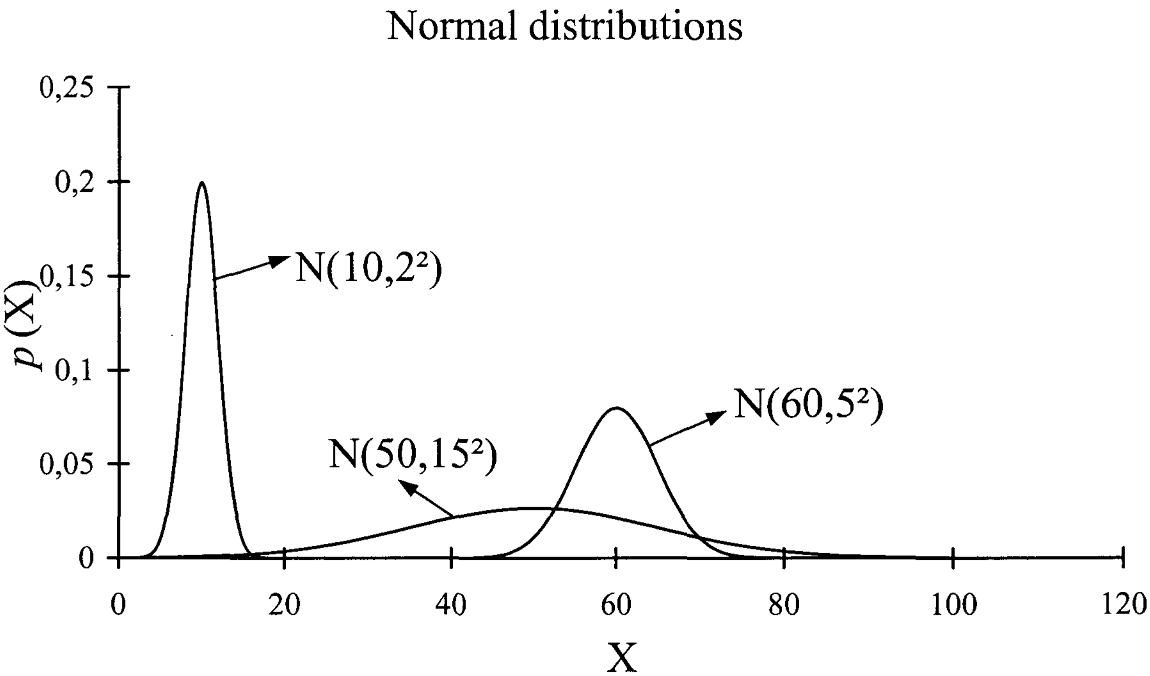
✓ Reading off the table

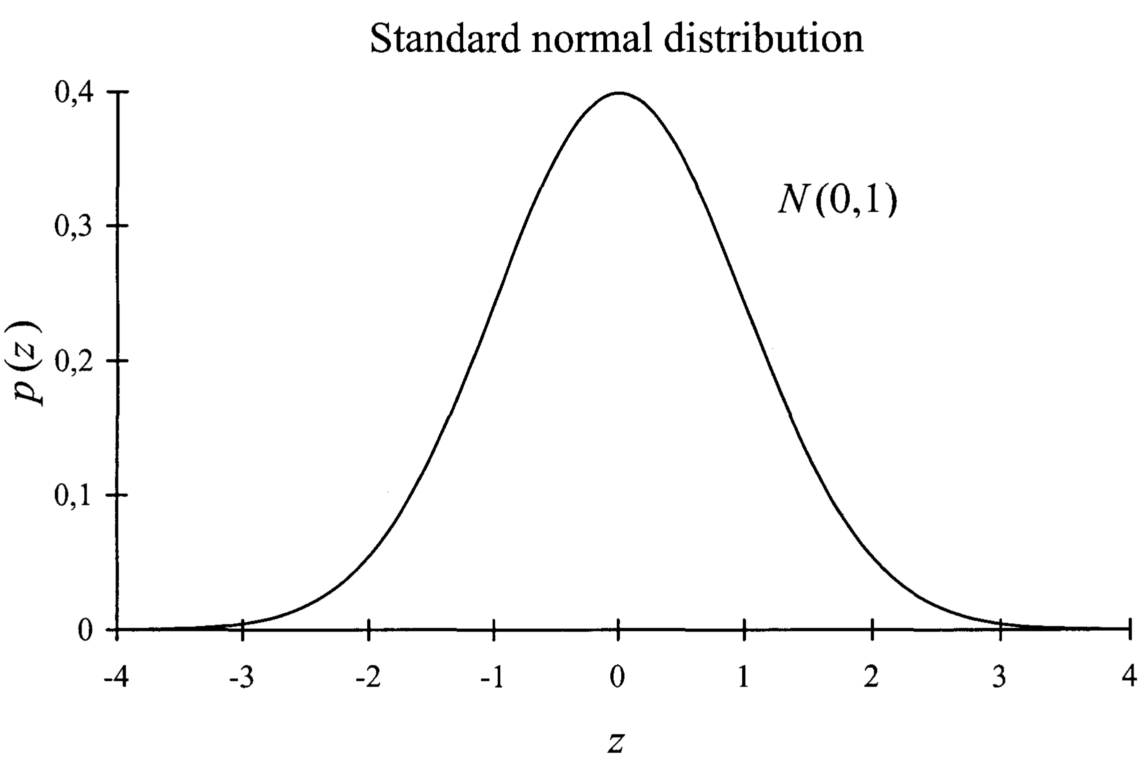
✓ Full examples

✓ Mathematical presentation

Calculation and moments

Generation of pseudo random variates


Normal distributions
0,25
0,2
0,15
0,1
0,05
0
p (X)
N(10,2²)
N(60,5²)
N(50,15²)
0
20
40
60
80
100
120
X


Standard normal distribution
0,4
0,3
0,2
0,1
0
p (z)
N(0,1)
-4
-3
-2
-1
0
1
2
3
4
z

Values of the standard normal distribution function (integral), $P(z)$ (table 1)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
0.00	50000	50040	50080	50120	50160	50199	50239	50279	50319	50359
0.01	50399	50439	50479	50519	50559	50598	50638	50678	50718	50758
0.02	50798	50838	50878	50917	50957	50997	51037	51077	51117	51157
0.03	51197	51237	51276	51316	51356	51396	51436	51476	51516	51555
0.04	51595	51635	51675	51715	51755	51795	51834	51874	51914	51954
0.05	51994	52034	52074	52113	52153	52193	52233	52273	52313	52352
0.06	52392	52432	52472	52512	52551	52591	52631	52671	52711	52751
0.07	52790	52830	52870	52910	52949	52989	53029	53069	53109	53148
0.08	53188	53228	53268	53307	53347	53387	53427	53466	53506	53546
0.09	53586	53625	53665	53705	53745	53784	53824	53864	53903	53943
0.10	53983	54022	54062	54102	54142	54181	54221	54261	54300	54340
0.11	54380	54419	54459	54498	54538	54578	54617	54657	54697	54736
0.12	54776	54815	54855	54895	54934	54974	55013	55053	55093	55132
0.13	55172	55211	55251	55290	55330	55369	55409	55448	55488	55527
0.14	55567	55607	55646	55685	55725	55764	55804	55843	55883	55922
0.15	55962	56001	56041	56080	56120	56159	56198	56238	56277	56317
0.16	56356	56395	56435	56474	56513	56553	56592	56631	56671	56710
0.17	56749	56789	56828	56867	56907	56946	56985	57025	57064	57103
0.18	57142	57182	57221	57260	57299	57339	57378	57417	57456	57495
0.19	57535	57574	57613	57652	57691	57730	57769	57809	57848	57887
0.20	57926	57965	58004	58043	58082	58121	58160	58200	58239	58278
0.21	58317	58356	58395	58434	58473	58512	58551	58590	58629	58667
0.22	58706	58745	58784	58823	58862	58901	58940	58979	59018	59057
0.23	59095	59134	59173	59212	59251	59290	59328	59367	59406	59445
0.24	59483	59522	59561	59600	59638	59677	59716	59755	59793	59832
0.25	59871	59909	59948	59987	60025	60064	60102	60141	60180	60218
0.26	60257	60295	60334	60372	60411	60450	60488	60527	60565	60604
0.27	60642	60680	60719	60757	60796	60834	60873	60911	60949	60988
0.28	61026	61064	61103	61141	61179	61218	61256	61294	61333	61371
0.29	61409	61447	61486	61524	61562	61600	61638	61677	61715	61753
0.30	61791	61829	61867	61906	61944	61982	62020	62058	62096	62134
0.31	62172	62210	62248	62286	62324	62362	62400	62438	62476	62514
0.32	62552	62589	62627	62665	62703	62741	62779	62817	62854	62892
0.33	62930	62968	63006	63043	63081	63119	63156	63194	63232	63270
0.34	63307	63345	63382	63420	63458	63495	63533	63570	63608	63646
0.35	63683	63721	63758	63796	63833	63871	63908	63945	63983	64020
0.36	64058	64095	64132	64170	64207	64244	64282	64319	64356	64394
0.37	64431	64468	64505	64543	64580	64617	64654	64691	64728	64766
0.38	64803	64840	64877	64914	64951	64988	65025	65062	65099	65136
0.39	65173	65210	65247	65284	65321	65358	65395	65432	65468	65505
0.40	65542	65579	65616	65653	65689	65726	65763	65800	65836	65873
0.41	65910	65946	65983	66020	66056	66093	66129	66166	66203	66239
0.42	66276	66312	66349	66385	66422	66458	66495	66531	66567	66604
0.43	66640	66677	66713	66749	66786	66822	66858	66894	66931	66967
0.44	67003	67039	67076	67112	67148	67184	67220	67256	67292	67328
0.45	67364	67401	67437	67473	67509	67545	67581	67616	67652	67688
0.46	67724	67760	67796	67832	67868	67903	67939	67975	68011	68047
0.47	68082	68118	68154	68189	68225	68261	68296	68332	68367	68403
0.48	68439	68474	68510	68545	68581	68616	68652	68687	68723	68758
0.49	68793	68829	68864	68899	68935	68970	69005	69041	69076	69111

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+0.000	+0.001	+0.002	+0.003	+0.004	+0.005	+0.006	+0.007	+0.008	+0.009
0.50	69146	69181	69217	69252	69287	69322	69357	69392	69427	69462
0.51	69497	69532	69567	69602	69637	69672	69707	69742	69777	69812
0.52	69847	69882	69916	69951	69986	70021	70056	70090	70125	70160
0.53	70194	70229	70264	70298	70333	70368	70402	70437	70471	70506
0.54	70540	70575	70609	70644	70678	70712	70747	70781	70815	70850
0.55	70884	70918	70953	70987	71021	71055	71089	71124	71158	71192
0.56	71226	71260	71294	71328	71362	71396	71430	71464	71498	71532
0.57	71566	71600	71634	71668	71702	71735	71769	71803	71837	71871
0.58	71904	71938	71972	72005	72039	72073	72106	72140	72173	72207
0.59	72240	72274	72307	72341	72374	72408	72441	72475	72508	72541
0.60	72575	72608	72641	72675	72708	72741	72774	72807	72841	72874
0.61	72907	72940	72973	73006	73039	73072	73105	73138	73171	73204
0.62	73237	73270	73303	73336	73369	73401	73434	73467	73500	73533
0.63	73565	73598	73631	73663	73696	73729	73761	73794	73826	73859
0.64	73891	73924	73956	73989	74021	74054	74086	74118	74151	74183
0.65	74215	74248	74280	74312	74344	74377	74409	74441	74473	74505
0.66	74537	74569	74601	74633	74665	74697	74729	74761	74793	74825
0.67	74857	74889	74921	74953	74984	75016	75048	75080	75111	75143
0.68	75175	75206	75238	75270	75301	75333	75364	75396	75427	75459
0.69	75490	75522	75553	75585	75616	75647	75679	75710	75741	75772
0.70	75804	75835	75866	75897	75928	75959	75991	76022	76053	76084
0.71	76115	76146	76177	76208	76239	76270	76300	76331	76362	76393
0.72	76424	76455	76485	76516	76547	76577	76608	76639	76669	76700
0.73	76730	76761	76792	76822	76853	76883	76913	76944	76974	77005
0.74	77035	77065	77096	77126	77156	77186	77217	77247	77277	77307
0.75	77337	77367	77397	77428	77458	77488	77518	77548	77577	77607
0.76	77637	77667	77697	77727	77757	77786	77816	77846	77876	77905
0.77	77935	77965	77994	78024	78053	78083	78113	78142	78172	78201
0.78	78230	78260	78289	78319	78348	78377	78407	78436	78465	78494
0.79	78524	78553	78582	78611	78640	78669	78698	78727	78756	78785
0.80	78814	78843	78872	78901	78930	78959	78988	79017	79045	79074
0.81	79103	79132	79160	79189	79218	79246	79275	79304	79332	79361
0.82	79389	79418	79446	79475	79503	79531	79560	79588	79616	79645
0.83	79673	79701	79730	79758	79786	79814	79842	79870	79898	79927
0.84	79955	79983	80011	80039	80067	80094	80122	80150	80178	80206
0.85	80234	80262	80289	80317	80345	80372	80400	80428	80455	80483
0.86	80511	80538	80566	80593	80621	80648	80675	80703	80730	80758
0.87	80785	80812	80840	80867	80894	80921	80948	80976	81003	81030
0.88	81057	81084	81111	81138	81165	81192	81219	81246	81273	81300
0.89	81327	81354	81380	81407	81434	81461	81487	81514	81541	81567
0.90	81594	81621	81647	81674	81700	81727	81753	81780	81806	81832
0.91	81859	81885	81912	81938	81964	81990	82017	82043	82069	82095
0.92	82121	82147	82174	82200	82226	82252	82278	82304	82330	82356
0.93	82381	82407	82433	82459	82485	82511	82536	82562	82588	82613
0.94	82639	82665	82690	82716	82742	82767	82793	82818	82844	82869
0.95	82894	82920	82945	82970	82996	83021	83046	83072	83097	83122
0.96	83147	83172	83198	83223	83248	83273	83298	83323	83348	83373
0.97	83398	83423	83447	83472	83497	83522	83547	83572	83596	83621
0.98	83646	83670	83695	83720	83744	83769	83793	83818	83842	83867
0.99	83891	83916	83940	83965	83989	84013	84037	84062	84086	84110

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
1.00	84134	84159	84183	84207	84231	84255	84279	84303	84327	84351
1.01	84375	84399	84423	84447	84471	84495	84519	84542	84566	84590
1.02	84614	84637	84661	84685	84708	84732	84755	84779	84803	84826
1.03	84849	84873	84896	84920	84943	84967	84990	85013	85036	85060
1.04	85083	85106	85129	85153	85176	85199	85222	85245	85268	85291
1.05	85314	85337	85360	85383	85406	85429	85452	85474	85497	85520
1.06	85543	85566	85588	85611	85634	85656	85679	85701	85724	85747
1.07	85769	85792	85814	85836	85859	85881	85904	85926	85948	85971
1.08	85993	86015	86037	86060	86082	86104	86126	86148	86170	86192
1.09	86214	86236	86258	86280	86302	86324	86346	86368	86390	86412
1.10	86433	86455	86477	86499	86520	86542	86564	86585	86607	86628
1.11	86650	86672	86693	86715	86736	86757	86779	86800	86822	86843
1.12	86864	86886	86907	86928	86949	86971	86992	87013	87034	87055
1.13	87076	87097	87118	87139	87160	87181	87202	87223	87244	87265
1.14	87286	87307	87327	87348	87369	87390	87410	87431	87452	87472
1.15	87493	87513	87534	87554	87575	87595	87616	87636	87657	87677
1.16	87698	87718	87738	87759	87779	87799	87819	87839	87860	87880
1.17	87900	87920	87940	87960	87980	88000	88020	88040	88060	88080
1.18	88100	88120	88140	88160	88179	88199	88219	88239	88258	88278
1.19	88298	88317	88337	88357	88376	88396	88415	88435	88454	88474
1.20	88493	88512	88532	88551	88571	88590	88609	88628	88648	88667
1.21	88686	88705	88724	88744	88763	88782	88801	88820	88839	88858
1.22	88877	88896	88915	88934	88952	88971	88990	89009	89028	89046
1.23	89065	89084	89103	89121	89140	89158	89177	89196	89214	89233
1.24	89251	89270	89288	89307	89325	89343	89362	89380	89398	89417
1.25	89435	89453	89472	89490	89508	89526	89544	89562	89580	89598
1.26	89617	89635	89653	89671	89688	89706	89724	89742	89760	89778
1.27	89796	89814	89831	89849	89867	89885	89902	89920	89938	89955
1.28	89973	89990	90008	90025	90043	90060	90078	90095	90113	90130
1.29	90147	90165	90182	90199	90217	90234	90251	90268	90286	90303
1.30	90320	90337	90354	90371	90388	90405	90422	90439	90456	90473
1.31	90490	90507	90524	90541	90558	90575	90591	90608	90625	90642
1.32	90658	90675	90692	90708	90725	90741	90758	90775	90791	90808
1.33	90824	90841	90857	90873	90890	90906	90923	90939	90955	90971
1.34	90988	91004	91020	91036	91053	91069	91085	91101	91117	91133
1.35	91149	91165	91181	91197	91213	91229	91245	91261	91277	91293
1.36	91309	91324	91340	91356	91372	91387	91403	91419	91434	91450
1.37	91466	91481	91497	91512	91528	91543	91559	91574	91590	91605
1.38	91621	91636	91651	91667	91682	91697	91713	91728	91743	91758
1.39	91774	91789	91804	91819	91834	91849	91864	91879	91894	91909
1.40	91924	91939	91954	91969	91984	91999	92014	92029	92043	92058
1.41	92073	92088	92103	92117	92132	92147	92161	92176	92190	92205
1.42	92220	92234	92249	92263	92278	92292	92307	92321	92335	92350
1.43	92364	92378	92393	92407	92421	92436	92450	92464	92478	92492
1.44	92507	92521	92535	92549	92563	92577	92591	92605	92619	92633
1.45	92647	92661	92675	92689	92703	92717	92730	92744	92758	92772
1.46	92785	92799	92813	92827	92840	92854	92868	92881	92895	92908
1.47	92922	92935	92949	92962	92976	92989	93003	93016	93030	93043
1.48	93056	93070	93083	93096	93110	93123	93136	93149	93162	93176
1.49	93189	93202	93215	93228	93241	93254	93267	93280	93293	93306

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
1.50	93319	93332	93345	93358	93371	93384	93397	93409	93422	93435
1.51	93448	93461	93473	93486	93499	93511	93524	93537	93549	93562
1.52	93574	93587	93600	93612	93625	93637	93650	93662	93674	93687
1.53	93699	93712	93724	93736	93749	93761	93773	93785	93798	93810
1.54	93822	93834	93846	93858	93871	93883	93895	93907	93919	93931
1.55	93943	93955	93967	93979	93991	94003	94015	94026	94038	94050
1.56	94062	94074	94086	94097	94109	94121	94133	94144	94156	94168
1.57	94179	94191	94202	94214	94226	94237	94249	94260	94272	94283
1.58	94295	94306	94318	94329	94340	94352	94363	94374	94386	94397
1.59	94408	94420	94431	94442	94453	94464	94476	94487	94498	94509
1.60	94520	94531	94542	94553	94564	94575	94586	94597	94608	94619
1.61	94630	94641	94652	94663	94674	94684	94695	94706	94717	94728
1.62	94738	94749	94760	94771	94781	94792	94803	94813	94824	94834
1.63	94845	94855	94866	94877	94887	94898	94908	94918	94929	94939
1.64	94950	94960	94970	94981	94991	95002	95012	95022	95032	95043
1.65	95053	95063	95073	95083	95094	95104	95114	95124	95134	95144
1.66	95154	95164	95174	95184	95194	95204	95214	95224	95234	95244
1.67	95254	95264	95274	95284	95293	95303	95313	95323	95333	95342
1.68	95352	95362	95372	95381	95391	95401	95410	95420	95429	95439
1.69	95449	95458	95468	95477	95487	95496	95506	95515	95525	95534
1.70	95543	95553	95562	95572	95581	95590	95600	95609	95618	95627
1.71	95637	95646	95655	95664	95674	95683	95692	95701	95710	95719
1.72	95728	95737	95747	95756	95765	95774	95783	95792	95801	95810
1.73	95818	95827	95836	95845	95854	95863	95872	95881	95889	95898
1.74	95907	95916	95925	95933	95942	95951	95959	95968	95977	95985
1.75	95994	96003	96011	96020	96028	96037	96046	96054	96063	96071
1.76	96080	96088	96097	96105	96113	96122	96130	96139	96147	96155
1.77	96164	96172	96180	96189	96197	96205	96213	96222	96230	96238
1.78	96246	96254	96263	96271	96279	96287	96295	96303	96311	96319
1.79	96327	96335	96343	96351	96359	96367	96375	96383	96391	96399
1.80	96407	96415	96423	96431	96438	96446	96454	96462	96470	96477
1.81	96485	96493	96501	96508	96516	96524	96531	96539	96547	96554
1.82	96562	96570	96577	96585	96592	96600	96607	96615	96623	96630
1.83	96638	96645	96652	96660	96667	96675	96682	96690	96697	96704
1.84	96712	96719	96726	96734	96741	96748	96755	96763	96770	96777
1.85	96784	96792	96799	96806	96813	96820	96827	96834	96842	96849
1.86	96856	96863	96870	96877	96884	96891	96898	96905	96912	96919
1.87	96926	96933	96940	96947	96953	96960	96967	96974	96981	96988
1.88	96995	97001	97008	97015	97022	97029	97035	97042	97049	97055
1.89	97062	97069	97075	97082	97089	97095	97102	97109	97115	97122
1.90	97128	97135	97141	97148	97154	97161	97167	97174	97180	97187
1.91	97193	97200	97206	97213	97219	97225	97232	97238	97244	97251
1.92	97257	97263	97270	97276	97282	97289	97295	97301	97307	97313
1.93	97320	97326	97332	97338	97344	97350	97357	97363	97369	97375
1.94	97381	97387	97393	97399	97405	97411	97417	97423	97429	97435
1.95	97441	97447	97453	97459	97465	97471	97477	97483	97488	97494
1.96	97500	97506	97512	97518	97523	97529	97535	97541	97547	97552
1.97	97558	97564	97570	97575	97581	97587	97592	97598	97604	97609
1.98	97615	97620	97626	97632	97637	97643	97648	97654	97659	97665
1.99	97670	97676	97681	97687	97692	97698	97703	97709	97714	97720

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
2.00	97725	97730	97736	97741	97746	97752	97757	97763	97768	97773
2.01	97778	97784	97789	97794	97800	97805	97810	97815	97820	97826
2.02	97831	97836	97841	97846	97851	97857	97862	97867	97872	97877
2.03	97882	97887	97892	97897	97902	97907	97912	97917	97923	97927
2.04	97932	97937	97942	97947	97952	97957	97962	97967	97972	97977
2.05	97982	97987	97992	97996	98001	98006	98011	98016	98020	98025
2.06	98030	98035	98040	98044	98049	98054	98059	98063	98068	98073
2.07	98077	98082	98087	98091	98096	98101	98105	98110	98115	98119
2.08	98124	98128	98133	98137	98142	98147	98151	98156	98160	98165
2.09	98169	98174	98178	98183	98187	98191	98196	98200	98205	98209
2.10	98214	98218	98222	98227	98231	98235	98240	98244	98248	98253
2.11	98257	98261	98266	98270	98274	98279	98283	98287	98291	98295
2.12	98300	98304	98308	98312	98316	98321	98325	98329	98333	98337
2.13	98341	98346	98350	98354	98358	98362	98366	98370	98374	98378
2.14	98382	98386	98390	98394	98398	98402	98406	98410	98414	98418
2.15	98422	98426	98430	98434	98438	98442	98446	98450	98454	98457
2.16	98461	98465	98469	98473	98477	98481	98484	98488	98492	98496
2.17	98500	98503	98507	98511	98515	98518	98522	98526	98530	98533
2.18	98537	98541	98545	98548	98552	98556	98559	98563	98567	98570
2.19	98574	98577	98581	98585	98588	98592	98595	98599	98603	98606
2.20	98610	98613	98617	98620	98624	98627	98631	98634	98638	98641
2.21	98645	98648	98652	98655	98659	98662	98665	98669	98672	98676
2.22	98679	98682	98686	98689	98693	98696	98699	98703	98706	98709
2.23	98713	98716	98719	98723	98726	98729	98732	98736	98739	98742
2.24	98745	98749	98752	98755	98758	98762	98765	98768	98771	98774
2.25	98778	98781	98784	98787	98790	98793	98796	98800	98803	98806
2.26	98809	98812	98815	98818	98821	98824	98827	98830	98834	98837
2.27	98840	98843	98846	98849	98852	98855	98858	98861	98864	98867
2.28	98870	98873	98876	98878	98881	98884	98887	98890	98893	98896
2.29	98899	98902	98905	98908	98910	98913	98916	98919	98922	98925
2.30	98928	98930	98933	98936	98939	98942	98944	98947	98950	98953
2.31	98956	98958	98961	98964	98967	98969	98972	98975	98978	98980
2.32	98983	98986	98988	98991	98994	98996	98999	99002	99004	99007
2.33	99010	99012	99015	99018	99020	99023	99025	99028	99031	99033
2.34	99036	99038	99041	99044	99046	99049	99051	99054	99056	99059
2.35	99061	99064	99066	99069	99071	99074	99076	99079	99081	99084
2.36	99086	99089	99091	99094	99096	99098	99101	99103	99106	99108
2.37	99111	99113	99115	99118	99120	99123	99125	99127	99130	99132
2.38	99134	99137	99139	99141	99144	99146	99148	99151	99153	99155
2.39	99158	99160	99162	99164	99167	99169	99171	99174	99176	99178
2.40	99180	99182	99185	99187	99189	99191	99194	99196	99198	99200
2.41	99202	99205	99207	99209	99211	99213	99215	99218	99220	99222
2.42	99224	99226	99228	99230	99232	99235	99237	99239	99241	99243
2.43	99245	99247	99249	99251	99253	99255	99257	99260	99262	99264
2.44	99266	99268	99270	99272	99274	99276	99278	99280	99282	99284
2.45	99286	99288	99290	99292	99294	99296	99298	99299	99301	99303
2.46	99305	99307	99309	99311	99313	99315	99317	99319	99321	99323
2.47	99324	99326	99328	99330	99332	99334	99336	99338	99339	99341
2.48	99343	99345	99347	99349	99350	99352	99354	99356	99358	99359
2.49	99361	99363	99365	99367	99368	99370	99372	99374	99376	99377

Values of the standard normal distribution function (integral), $P(z)$ (table 1, cont.)

z	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
2.50	99379	99381	99383	99384	99386	99388	99389	99391	99393	99395
2.51	99396	99398	99400	99401	99403	99405	99407	99408	99410	99412
2.52	99413	99415	99417	99418	99420	99422	99423	99425	99426	99428
2.53	99430	99431	99433	99435	99436	99438	99439	99441	99443	99444
2.54	99446	99447	99449	99450	99452	99454	99455	99457	99458	99460
2.55	99461	99463	99464	99466	99468	99469	99471	99472	99474	99475
2.56	99477	99478	99480	99481	99483	99484	99486	99487	99489	99490
2.57	99492	99493	99494	99496	99497	99499	99500	99502	99503	99505
2.58	99506	99507	99509	99510	99512	99513	99515	99516	99517	99519
2.59	99520	99522	99523	99524	99526	99527	99528	99530	99531	99533
2.60	99534	99535	99537	99538	99539	99541	99542	99543	99545	99546
2.61	99547	99549	99550	99551	99553	99554	99555	99556	99558	99559
2.62	99560	99562	99563	99564	99565	99567	99568	99569	99571	99572
2.63	99573	99574	99576	99577	99578	99579	99581	99582	99583	99584
2.64	99585	99587	99588	99589	99590	99592	99593	99594	99595	99596
2.65	99598	99599	99600	99601	99602	99603	99605	99606	99607	99608
2.66	99609	99610	99612	99613	99614	99615	99616	99617	99618	99620
2.67	99621	99622	99623	99624	99625	99626	99627	99629	99630	99631
2.68	99632	99633	99634	99635	99636	99637	99638	99640	99641	99642
2.69	99643	99644	99645	99646	99647	99648	99649	99650	99651	99652
2.70	99653	99654	99655	99656	99657	99658	99660	99661	99662	99663
2.71	99664	99665	99666	99667	99668	99669	99670	99671	99672	99673
2.72	99674	99675	99676	99677	99678	99678	99679	99680	99681	99682
2.73	99683	99684	99685	99686	99687	99688	99689	99690	99691	99692
2.74	99693	99694	99695	99696	99697	99697	99698	99699	99700	99701
2.75	99702	99703	99704	99705	99706	99707	99707	99708	99709	99710
2.76	99711	99712	99713	99714	99715	99715	99716	99717	99718	99719
2.77	99720	99721	99721	99722	99723	99724	99725	99726	99727	99727
2.78	99728	99729	99730	99731	99732	99732	99733	99734	99735	99736
2.79	99736	99737	99738	99739	99740	99741	99741	99742	99743	99744
2.80	99744	99745	99746	99747	99748	99748	99749	99750	99751	99752
2.81	99752	99753	99754	99755	99755	99756	99757	99758	99758	99759
2.82	99760	99761	99761	99762	99763	99764	99764	99765	99766	99767
2.83	99767	99768	99769	99769	99770	99771	99772	99772	99773	99774
2.84	99774	99775	99776	99777	99777	99778	99779	99779	99780	99781
2.85	99781	99782	99783	99783	99784	99785	99785	99786	99787	99788
2.86	99788	99789	99790	99790	99791	99791	99792	99793	99793	99794
2.87	99795	99795	99796	99797	99797	99798	99799	99799	99800	99801
2.88	99801	99802	99802	99803	99804	99804	99805	99806	99806	99807
2.89	99807	99808	99809	99809	99810	99810	99811	99812	99812	99813
2.90	99813	99814	99815	99815	99816	99816	99817	99818	99818	99819
2.91	99819	99820	99820	99821	99822	99822	99823	99823	99824	99824
2.92	99825	99826	99826	99827	99827	99828	99828	99829	99829	99830
2.93	99831	99831	99832	99832	99833	99833	99834	99834	99835	99835
2.94	99836	99836	99837	99837	99838	99839	99839	99840	99840	99841
2.95	99841	99842	99842	99843	99843	99844	99844	99845	99845	99846
2.96	99846	99847	99847	99848	99848	99849	99849	99850	99850	99851
2.97	99851	99852	99852	99853	99853	99854	99854	99854	99855	99855
2.98	99856	99856	99857	99857	99858	99858	99859	99859	99860	99860
2.99	99861	99861	99861	99862	99862	99863	99863	99864	99864	99865

For values of z from 3 to 7, see the Mathematical presentation subsection.

Percentiles of the standard normal distribution, $z(P)$ (table 2)

P	+.000	+.001	+.002	+.003	+.004	+.005	+.006	+.007	+.008	+.009
.500	.0000	$.0^2251$	$.0^2501$	$.0^2752$	.0100	.0125	.0150	.0175	.0201	.0226
.510	.0251	.0276	.0301	.0326	.0351	.0376	.0401	.0426	.0451	.0476
.520	.0502	.0527	.0552	.0577	.0602	.0627	.0652	.0677	.0702	.0728
.530	.0753	.0778	.0803	.0828	.0853	.0878	.0904	.0929	.0954	.0979
.540	.1004	.1030	.1055	.1080	.1105	.1130	.1156	.1181	.1206	.1231
.550	.1257	.1282	.1307	.1332	.1358	.1383	.1408	.1434	.1459	.1484
.560	.1510	.1535	.1560	.1586	.1611	.1637	.1662	.1687	.1713	.1738
.570	.1764	.1789	.1815	.1840	.1866	.1891	.1917	.1942	.1968	.1993
.580	.2019	.2045	.2070	.2096	.2121	.2147	.2173	.2198	.2224	.2250
.590	.2275	.2301	.2327	.2353	.2378	.2404	.2430	.2456	.2482	.2508
.600	.2533	.2559	.2585	.2611	.2637	.2663	.2689	.2715	.2741	.2767
.610	.2793	.2819	.2845	.2871	.2898	.2924	.2950	.2976	.3002	.3029
.620	.3055	.3081	.3107	.3134	.3160	.3186	.3213	.3239	.3266	.3292
.630	.3319	.3345	.3372	.3398	.3425	.3451	.3478	.3505	.3531	.3558
.640	.3585	.3611	.3638	.3665	.3692	.3719	.3745	.3772	.3799	.3826
.650	.3853	.3880	.3907	.3934	.3961	.3989	.4016	.4043	.4070	.4097
.660	.4125	.4152	.4179	.4207	.4234	.4261	.4289	.4316	.4344	.4372
.670	.4399	.4427	.4454	.4482	.4510	.4538	.4565	.4593	.4621	.4649
.680	.4677	.4705	.4733	.4761	.4789	.4817	.4845	.4874	.4902	.4930
.690	.4959	.4987	.5015	.5044	.5072	.5101	.5129	.5158	.5187	.5215
.700	.5244	.5273	.5302	.5330	.5359	.5388	.5417	.5446	.5476	.5505
.710	.5534	.5563	.5592	.5622	.5651	.5681	.5710	.5740	.5769	.5799
.720	.5828	.5858	.5888	.5918	.5948	.5978	.6008	.6038	.6068	.6098
.730	.6128	.6158	.6189	.6219	.6250	.6280	.6311	.6341	.6372	.6403
.740	.6433	.6464	.6495	.6526	.6557	.6588	.6620	.6651	.6682	.6713
.750	.6745	.6776	.6808	.6840	.6871	.6903	.6935	.6967	.6999	.7031
.760	.7063	.7095	.7128	.7160	.7192	.7225	.7257	.7290	.7323	.7356
.770	.7388	.7421	.7454	.7488	.7521	.7554	.7588	.7621	.7655	.7688
.780	.7722	.7756	.7790	.7824	.7858	.7892	.7926	.7961	.7995	.8030
.790	.8064	.8099	.8134	.8169	.8204	.8239	.8274	.8310	.8345	.8381
.800	.8416	.8452	.8488	.8524	.8560	.8596	.8633	.8669	.8705	.8742
.810	.8779	.8816	.8853	.8890	.8927	.8965	.9002	.9040	.9078	.9116
.820	.9154	.9192	.9230	.9269	.9307	.9346	.9385	.9424	.9463	.9502
.830	.9542	.9581	.9621	.9661	.9701	.9741	.9782	.9822	.9863	.9904
.840	.9945	.9986	1.0027	1.0069	1.0110	1.0152	1.0194	1.0237	1.0279	1.0322
.850	1.0364	1.0407	1.0450	1.0494	1.0537	1.0581	1.0625	1.0669	1.0714	1.0758
.860	1.0803	1.0848	1.0893	1.0939	1.0985	1.1031	1.1077	1.1123	1.1170	1.1217
.870	1.1264	1.1311	1.1359	1.1407	1.1455	1.1503	1.1552	1.1601	1.1650	1.1700
.880	1.1750	1.1800	1.1850	1.1901	1.1952	1.2004	1.2055	1.2107	1.2160	1.2212
.890	1.2265	1.2319	1.2372	1.2426	1.2481	1.2536	1.2591	1.2646	1.2702	1.2759
.900	1.2816	1.2873	1.2930	1.2988	1.3047	1.3106	1.3165	1.3225	1.3285	1.3346
.910	1.3408	1.3469	1.3532	1.3595	1.3658	1.3722	1.3787	1.3852	1.3917	1.3984
.920	1.4051	1.4118	1.4187	1.4255	1.4325	1.4395	1.4466	1.4538	1.4611	1.4684
.930	1.4758	1.4833	1.4909	1.4985	1.5063	1.5141	1.5220	1.5301	1.5382	1.5464
.940	1.5548	1.5632	1.5718	1.5805	1.5893	1.5982	1.6072	1.6164	1.6258	1.6352
.950	1.6449	1.6546	1.6646	1.6747	1.6849	1.6954	1.7060	1.7169	1.7279	1.7392
.960	1.7507	1.7624	1.7744	1.7866	1.7991	1.8119	1.8250	1.8384	1.8522	1.8663
.970	1.8808	1.8957	1.9110	1.9268	1.9431	1.9600	1.9774	1.9954	2.0141	2.0335
.980	2.0537	2.0749	2.0969	2.1201	2.1444	2.1701	2.1973	2.2262	2.2571	2.2904
.990	2.3263	2.3656	2.4089	2.4573	2.5121	2.5758	2.6521	2.7478	2.8782	3.0902

For extreme percentiles (up to $P = 0.999999$), see the Mathematical presentation subsection.

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Reading off the tables

Table 1 gives the probability integral $P(z)$ of the standard normal distribution at z, for positive values $z = 0.000(0.001)2.999$; a hidden decimal point precedes each quantity. Such quantity $P(z)$, in a normal distribution with mean $\mu = 0$ and standard deviation $\sigma = 1$, denotes the probability that a random element Z lies under the indicated z value, *i.e.* $P(z) = \Pr(Z \leq z)$. For negative z values, one may use the complementary relation: $P(z) = 1-P(z)$.

Table 2 is the converse of table 1 and presents the quantile (or percentage point) z corresponding to each P value, for $P = 0.500(0.001)0.999$; when $P < 0.500$, use the relation: $z(P) = -z(1-P)$.

Illustration 1. What proportion of cases lies under $z = 1.614$, in the standard normal distribution? In table 1, taking line 1.61 and column +.004, we obtain $P = 0.94674$.

Illustration 2. What percentage of area do we find under $z = -2.037$, in the standard normal distribution? Using table 1 and relation $P(z) = 1-P(-z)$, we successively obtain $P(2.037)= 0,97917$ (line 2.03, column +.007), $1-0.97917 = 0.02083$, whence the percentage asked for is 100×0.02083, or nearly 2.

Illustration 3. Which abscissa, or z value, segregates the lower 62.5 % of the cases from the 37.5 % higher in the standard normal distribution? In table 2, integral P of 0.625 points to line 0.620 and column +.005, where value $z = 0.3186$ is given, satisfying the relation $\Pr(Z\leq 0.3186) \approx 0.625$.

Illustration 4. Which z value does divide the lower third (from the upper two thirds) in the standard normal distribution? We may approximate ⅓ with 0,333. Using table 2 and as $0.333 < 0.500$, we first obtain $1-0.333 = 0.667$, then read off $z(0.667) = 0.4316$ and, finally, with a change of sign, -0.4316. For more precision, we could also, in the second phase, interpolate between 0.666 (with $z = 0.4289$) and 0.667 (with $z = 0.4316$): for $P = ⅔$, we calculate:

$$z(⅔) = 0.4289 + \frac{⅔ - 0.666}{0.667-0.666}(0.4316-0.4289)$$

$$= 0.4307 ,$$

or $z(⅓) \approx -0.4307$, a value which is precise up to the fourth decimal digit.

Full examples

Example 1. A test of Intellectual Quotient (IQ) for children of a given age is set up by imposing a normal distribution of scores, a mean (μ) of 100 and a standard deviation (σ) of 16. Find the two IQ values that comprise approximately the central 50 % of the young population. *Solution*: The central 50 % of the area in a standard $N(0,1)$ distribution starts at integral $P = 0.25$ and ends at $P = 0.75$. Using table 2, $z(P{=}0.75) \approx 0.6745$; conversely, $z(P{=}0.25) \approx -0.6745$. The desired values are thus $(-0.6745, 0.6745)$ for the standard $N(0,1)$ distribution. These values can be converted approximately[1] into IQ scores with a $N(100,16^2)$ distribution, using QI = 100 + 16z, whence the interval is (89.208, 110.792) or, roughly, (89, 111).

Example 2. The height of people, in a given population, presents a mean of 1.655 m and a standard deviation of 0.205 m. In a representative area comprising 12 000 inhabitants, how many persons having a height of 2 m or more can one expect to find? *Solution*: In order to *predict* an approximate number, we need to stipulate a *model*; here, we favor the model of a normal distribution with the corresponding parameter values, *i.e.* $N(1.655, 0.205^2)$. Transforming height X = 2 m into a standardized z value, we get $z = (2-1.655)/0.205 \approx 1.683$. In table 1, $P(1.683) = 0.95381$, whence the proportion of cases with a height exceeding X = 2 m, or $z = 1.683$, approaches $1-0.95381 = 0.04619$. Multiplying this proportion by 12 000, the number of inhabitants in the designated area, we predict that there be about 554 persons of a height of 2 m or more in that area.

Example 3. A measuring device for strength in Newtons (N) allows to estimate arm flexion strength with a standard error of measurement (σ_ε) of 2.5 N. Using 5 evaluations for each arm, Robert obtains a mean strength of 93.6 N for his right arm, and of 89.8 for his left. May we assure that Robert's right arm is the stronger? *Solution*: Let us suppose that the estimates of each arm's strength fluctuate according to a normal model with means μ_j (j=1,2) and standard deviation 2.5 ($=\sigma_\varepsilon$). The difference between the two means ($\bar{x}_1-\bar{x}_2$) is itself normally distributed, with mean $\mu_1-\mu_2$ and standard error $\sigma_\varepsilon\surd(n_1^{-1}+n_2^{-1})$, here $2.5\times\surd(5^{-1}+5^{-1}) \approx 1.581$. Assume, by hypothesis, that $\mu_1 = \mu_2$, *i.e.* both arms have equal strength. The observed difference, $\bar{x}_1-\bar{x}_2 = 93.6-89.8 = 3.80$, standardized with:

[1] A more precise conversion would need to consider the discreteness of IQ scores (who vary by units), so that it would be generally impossible to obtain an exact interval of scores. In the same vein, the normal model, which is defined for continuous variables in the real domain, cannot be rigorously imposed to any discrete variable such as a test score.

$$z = \frac{(\bar{x}_1 - \bar{x}_2) - (\mu_1 - \mu_2)}{\sigma_e \sqrt{n_1^{-1} + n_2^{-1}}} ,$$

that is, $z = (3.80 - 0)/1.581 = 2.404$, is located 2.404 units of standard error from 0. Admitting a bilateral error rate of 5 %, boundaries of statistical significance fall at the 2.5 and 97.5 percentage points, which, for the standard normal distribution in table 2, point to $z = -1.960$ and $z = 1.960$ respectively. The observed difference thus exceeds the allowed-for interval of normal variation, leading us to conclude that one arm, Robert's right arm, is truly the stronger.

Mathematical presentation

The normal law, or normal distribution, has famous origins as well as innumerable applications. It first appeared in the writings of De Moivre around 1733, and was re-discovered by Laplace and Gauss. Sir Francis Galton, in view of its quasi universality, christened it "normal", synonymously to natural, expressing order, normative: it is used as a model for the distribution of a great many measurable attributes in a population. The normal model is the foremost reference for interpreting continuous random phenomena, and it underlies an overwhelming majority of statistical techniques in estimation and hypotheses testing.

Calculation and moments

The normal law, or normal distribution, has two parameters designated by μ and σ^2, corresponding respectively to the expectation (or mean) and variance of the distributed quantity. The normal p.d.f. is:

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}(x-\mu)^2/\sigma^2} ,$$

where $\pi \approx 3.1416$ and $e \approx 2.7183$. As shown in the graphs, the p.d.f. is symmetrical and reaches its maximum height at $x = \mu$, μ thus being the mode, median and (arithmetic) mean of the distribution. Integration of $p(x)$ is not trivial. One usually resorts to a standardized form, $z = (x-\mu)/\sigma$, z being a *standard score*, whose density function is the so-called standard normal distribution, $N(0,1)$,

$$p(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} .$$

Maximum p.d.f., at $z = 0$, equals $p(0) \approx 0.3989$, and it decreases steadily when z goes to $+\infty$ or $-\infty$, almost vanishing (≈ 0.0044) at $z = \pm 3$.

Precise (analytic) integration of the normal p.d.f. is impossible; nevertheless authors have evolved ways and methods of calculating the normal integral, or d.f., $P(z)$: most methods use series expansions. The simplest of those is based on the expansion of e^x in a Taylor series around zero, *i.e.* $e^x = 1 + x + x^2/2! + x^3/3! +$ etc. After substitution of $x^2/2$ for x, term-by-term integration and evaluation at $x = 0$ and $x = z$, the standard normal integral is:

$$P(z) = \frac{1}{2} + \frac{z}{\sqrt{2\pi}}\left[1 - \frac{z^2}{6} + \frac{z^4}{40} - \frac{z^6}{336} + \cdots + \frac{(-1)^n z^{2n}}{2^n n!\,(2n+1)}\right],$$

the summation within brackets being pursued until the desired precision is attained.

There exist other formulae for approximating the normal d.f. $P(x)$, with varying degrees of complexity and precision. The following,

$$P(z) \approx \frac{1}{2} + \frac{z}{15.04}\left[1 + 4e^{-z^2/8} + e^{-z^2/2}\right],$$

elaborated according to Simpson's rule with one arc, is precise to 0.0002 for $|z| \leq 1$, to 0.001 for $|z| \leq 1.75$, and to 0.01 for $|z| \leq 2.31$. For higher z values, one may use:

$$P(z) \approx 1 - \frac{e^{-z^2/2}}{\sqrt{2\pi}}\left[\frac{z^2+2}{z^3+3z}\right],$$

whose precision is nearly 0.0001 for $z \geq 2.31$ and which has the advantage of always keeping three significant digits for extreme $|z|$ values. Thus, for $z = 5$, the approximated value is 0.9999997132755, whereas the exact 14-digit integral is 0.99999971334835.

Still another approximation formula, more involved than the preceding ones but fitting for a computer program, is due to C. Hastings. Let $z \geq 0$; then,

$$P(z) \approx 1 - \frac{e^{z^2/2}}{\sqrt{2\pi}} \cdot t(b_1+t(b_2+t(b_3+t(b_4+tb_5)))),$$

where $t = 1/(1+0.2316419z)$ and $b_1 = 0.31938153$, $b_2 = -0.356563782$, $b_3 = 1.781477937$, $b_4 = -1.821255978$, $b_5 = 1.330274429$. For any (positive) z value, the precision of the calculated $P(z)$ is at least 0.000000075.

Values reported in tables 1 and 2 have been computed with great precision (12 digits or more) with the Taylor series expansion aforementioned. The two small tables below furnish some supplementary, extreme, values of the standard normal integral (note that $.9^4 6833$ should be read 0.99996833).

z	3	3.5	4	4.5	5	5.5	6	6.5	7
$P(z)$	$.9^2 8650$	$.9^3 7674$	$.9^4 6833$	$.9^5 6602$	$.9^6 7133$	$.9^7 8101$	$.9^9 0134$	$.9^{10} 5984$	$.9^{11} 8720$

P	.99	$.9^2 5$	$.9^3$	$.9^3 5$	$.9^4$	$.9^4 5$	$.9^5$	$.9^5 5$	$.9^6$
$z(P)$	2.32635	2.57583	3.09023	3.29053	3.71902	3.89059	4.26489	4.41717	4.75342

Moments. The expectation (μ) and variance (σ^2) are the two parameters of a normal distribution. The skewness index (γ_1) is zero. As for the kurtosis index (γ_2), the normal law is stipulated as a criterion, a reference shape for all other distributions, consequently this index is again zero.

For the curious reader, let us note that, for a normal $N(\mu,\sigma^2)$ distribution, the mean absolute difference, $\sum|x_i-\bar{x}|/n$, has expectation $\sigma\times\sqrt{(2/\pi)} \approx 0.79788\sigma$. Also, the mean (or expectation) of variates located in the upper 100α % of a normal population is given by $\mu + \sigma\times p(z_{1-\alpha})/\alpha$, $z_{1-\alpha}$ being the $100(1-\alpha)$ percentage point of distribution $N(0,1)$. For example, for $x = z \sim N(0,1)$, the mean of the upper 10 %, denoted $\mu_{(0.10)}$, uses $z_{[1-0.10]} = z_{[0.90]} \approx 1.2818$ (in table 2), $p(1.2816) \approx 0.17549$, and $\mu_{(0.10)} \approx 0 + 1\times0.17549/0.10 \approx 1.7549$.

Generation of pseudo random variates

Suppose a uniform $U(0,1)$ random variate (r.v.) generator, designated UNIF (*see* the section on Random numbers for information on UNIF). A normal $N(0,1)$ r.v. is produced from two independent uniform r.v.'s using the following transformation.

Preparation : $C = 2\pi \approx 6.2831853072$

Production : **Return** $\sqrt{[-2\times\ln(\text{UNIF})]} \times \sin(C\times\text{UNIF}) \rightarrow x$.

Remarks :

1. Standard temporal cost : 4.0 × t(UNIF), *i.e.* the approximate time required to produce one normal r.v. is equivalent to 4 times t(UNIF), the time required to produce one uniform r.v..

2. This method shown above is due to Box and Muller (Devroye 1986) and has some variants. Each invocation (with the same pair of UNIF values) allows to generate a second, independent x' value, through the substitution of "cos" instead of "sin" in the conversion formula.

3. In order to produce a normal $N(\mu,\sigma)$ r.v. y, one first obtains $x \sim N(0,1)$ with the procedure outlined, then $y \leftarrow \mu + \sigma \times x$.

4. In order to produce pairs of normal $N(0,1)$ r.v.'s z_1, z_2 having mutual correlation equal to ρ, one first obtains independent r.v.'s x and x', then $z_1 \leftarrow x$ and $z_2 \leftarrow \rho \times x + \surd(1-\rho^2) \times x'$. Gentle (1998, p. 187) suggests a more elegant approach. Let $\omega = \cos^{-1}\rho$ (in radian units). Then, one first obtains $t \leftarrow \surd[-2\times\ln(\text{UNIF})]$ and $u \leftarrow 2\pi\times\text{UNIF}$, then $z_1 \leftarrow t\times\sin(u)$, $z_2 \leftarrow t\times\sin(u-\omega)$.

Chi-square (χ^2) distribution

- ✓ Graphical representations
- ✓ Selected percentiles of Chi-square (χ^2)
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation

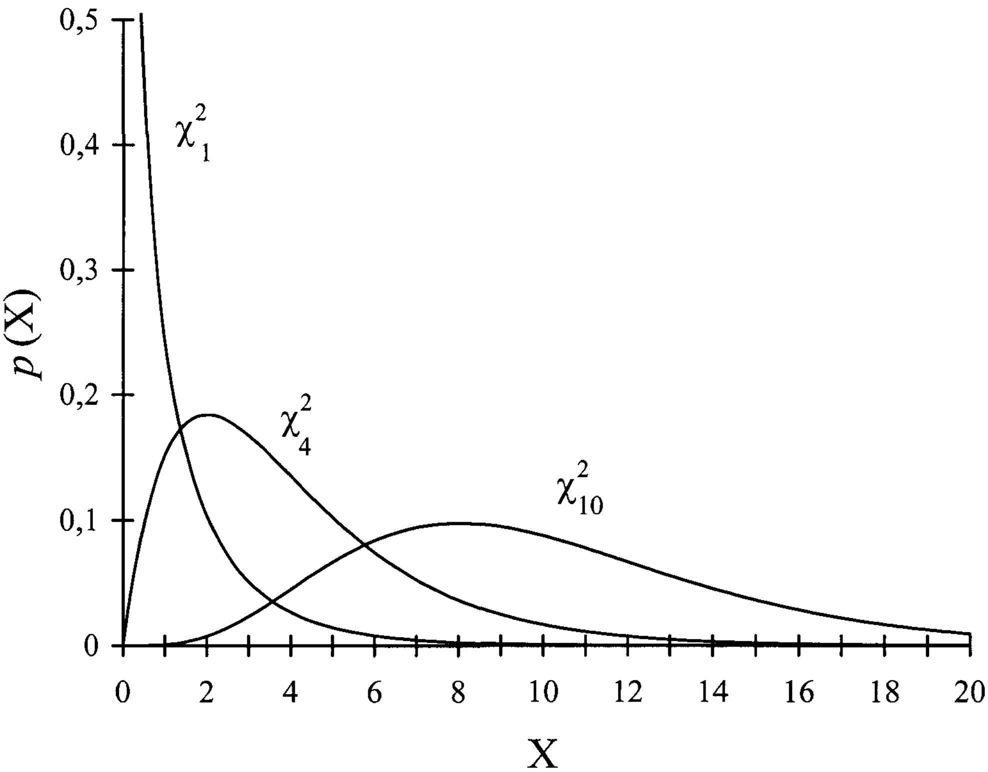
  Calculation and moments

  Generation of pseudo random variates

  The distribution of s, the standard deviation (s.d.)

  Three normal approximations to Chi-square

Chi-square (χ^2) distributions



Selected percentiles of Chi-square (χ^2)

$\nu \setminus P$	.001	.005	.010	.025	.050	.250	.500	.750	.950	.975	.990	.995	.999	ν
1	$.0^{5}16$	$.0^{4}39$	$.0^{3}16$	$.0^{3}98$	$.0^{2}39$	.10	.45	1.32	3.84	5.02	6.63	7.88	10.83	1
2	$.0^{2}20$	.010	.020	.051	.10	.58	1.39	2.77	5.99	7.38	9.21	10.60	13.82	2
3	.024	.072	.11	.22	.35	1.21	2.37	4.11	7.81	9.35	11.34	12.84	16.27	3
4	.091	.21	.30	.48	.71	1.92	3.36	5.39	9.49	11.14	13.28	14.86	18.47	4
5	.21	.41	.55	.83	1.15	2.67	4.35	6.63	11.07	12.83	15.09	16.75	20.52	5
6	.38	.68	.87	1.24	1.64	3.45	5.35	7.84	12.59	14.45	16.81	18.55	22.46	6
7	.60	.99	1.24	1.69	2.17	4.25	6.35	9.04	14.07	16.01	18.48	20.28	24.32	7
8	.86	1.34	1.65	2.18	2.73	5.07	7.34	10.22	15.51	17.53	20.09	21.96	26.12	8
9	1.15	1.73	2.09	2.70	3.33	5.90	8.34	11.39	16.92	19.02	21.67	23.59	27.88	9
10	1.48	2.16	2.56	3.25	3.94	6.74	9.34	12.55	18.31	20.48	23.21	25.19	29.59	10
11	1.83	2.60	3.05	3.82	4.57	7.58	10.34	13.70	19.68	21.92	24.73	26.76	31.26	11
12	2.21	3.07	3.57	4.40	5.23	8.44	11.34	14.85	21.03	23.34	26.22	28.30	32.91	12
13	2.62	3.57	4.11	5.01	5.89	9.30	12.34	15.98	22.36	24.74	27.69	29.82	34.53	13
14	3.04	4.07	4.66	5.63	6.57	10.17	13.34	17.12	23.68	26.12	29.14	31.32	36.12	14
15	3.48	4.60	5.23	6.26	7.26	11.04	14.34	18.25	25.00	27.49	30.58	32.80	37.70	15
16	3.94	5.14	5.81	6.91	7.96	11.91	15.34	19.37	26.30	28.85	32.00	34.27	39.25	16
17	4.42	5.70	6.41	7.56	8.67	12.79	16.34	20.49	27.59	30.19	33.41	35.72	40.79	17
18	4.90	6.26	7.01	8.23	9.39	13.68	17.34	21.60	28.87	31.53	34.81	37.16	42.31	18
19	5.41	6.84	7.63	8.91	10.12	14.56	18.34	22.72	30.14	32.85	36.19	38.58	43.82	19
20	5.92	7.43	8.26	9.59	10.85	15.45	19.34	23.83	31.41	34.17	37.57	40.00	45.31	20
21	6.45	8.03	8.90	10.28	11.59	16.34	20.34	24.93	32.67	35.48	38.93	41.40	46.80	21
22	6.98	8.64	9.54	10.98	12.34	17.24	21.34	26.04	33.92	36.78	40.29	42.80	48.27	22
23	7.53	9.26	10.20	11.69	13.09	18.14	22.34	27.14	35.17	38.08	41.64	44.18	49.73	23
24	8.08	9.89	10.86	12.40	13.85	19.04	23.34	28.24	36.42	39.36	42.98	45.56	51.18	24
25	8.65	10.52	11.52	13.12	14.61	19.94	24.34	29.34	37.65	40.65	44.31	46.93	52.62	25
26	9.22	11.16	12.20	13.84	15.38	20.84	25.34	30.43	38.89	41.92	45.64	48.29	54.05	26
27	9.80	11.81	12.88	14.57	16.15	21.75	26.34	31.53	40.11	43.19	46.96	49.64	55.48	27
28	10.39	12.46	13.56	15.31	16.93	22.66	27.34	32.62	41.34	44.46	48.28	50.99	56.89	28
29	10.99	13.12	14.26	16.05	17.71	23.57	28.34	33.71	42.56	45.72	49.59	52.34	58.30	29
30	11.59	13.79	14.95	16.79	18.49	24.48	29.34	34.80	43.77	46.98	50.89	53.67	59.70	30
31	12.20	14.46	15.66	17.54	19.28	25.39	30.34	35.89	44.99	48.23	52.19	55.00	61.10	31
32	12.81	15.13	16.36	18.29	20.07	26.30	31.34	36.97	46.19	49.48	53.49	56.33	62.49	32
33	13.43	15.82	17.07	19.05	20.87	27.22	32.34	38.06	47.40	50.73	54.78	57.65	63.87	33
34	14.06	16.50	17.79	19.81	21.66	28.14	33.34	39.14	48.60	51.97	56.06	58.96	65.25	34
35	14.69	17.19	18.51	20.57	22.47	29.05	34.34	40.22	49.80	53.20	57.34	60.27	66.62	35
36	15.32	17.89	19.23	21.34	23.27	29.97	35.34	41.30	51.00	54.44	58.62	61.58	67.99	36
37	15.97	18.59	19.96	22.11	24.07	30.89	36.34	42.38	52.19	55.67	59.89	62.88	69.35	37
38	16.61	19.29	20.69	22.88	24.88	31.81	37.34	43.46	53.38	56.90	61.16	64.18	70.70	38
39	17.26	20.00	21.43	23.65	25.70	32.74	38.34	44.54	54.57	58.12	62.43	65.48	72.05	39
40	17.92	20.71	22.16	24.43	26.51	33.66	39.34	45.62	55.76	59.34	63.69	66.77	73.40	40
41	18.58	21.42	22.91	25.21	27.33	34.58	40.34	46.69	56.94	60.56	64.95	68.05	74.74	41
42	19.24	22.14	23.65	26.00	28.14	35.51	41.34	47.77	58.12	61.78	66.21	69.34	76.08	42
43	19.91	22.86	24.40	26.79	28.96	36.44	42.34	48.84	59.30	62.99	67.46	70.62	77.42	43
44	20.58	23.58	25.15	27.57	29.79	37.36	43.34	49.91	60.48	64.20	68.71	71.89	78.75	44
45	21.25	24.31	25.90	28.37	30.61	38.29	44.34	50.98	61.66	65.41	69.96	73.17	80.08	45
46	21.93	25.04	26.66	29.16	31.44	39.22	45.34	52.06	62.83	66.62	71.20	74.44	81.40	46
47	22.61	25.77	27.42	29.96	32.27	40.15	46.34	53.13	64.00	67.82	72.44	75.70	82.72	47
48	23.29	26.51	28.18	30.75	33.10	41.08	47.34	54.20	65.17	69.02	73.68	76.97	84.04	48
49	23.98	27.25	28.94	31.55	33.93	42.01	48.33	55.27	66.34	70.22	74.92	78.23	85.35	49
50	24.67	27.99	29.71	32.36	34.76	42.94	49.33	56.33	67.50	71.42	76.15	79.49	86.66	50

Selected percentiles of Chi-square (χ^2) (cont.)

ν \ P	.001	.005	.010	.025	.050	.250	.500	.750	.950	.975	.990	.995	.999	ν
51	25.37	28.73	30.48	33.16	35.60	43.87	50.33	57.40	68.67	72.62	77.39	80.75	87.97	51
52	26.07	29.48	31.25	33.97	36.44	44.81	51.33	58.47	69.83	73.81	78.62	82.00	89.27	52
53	26.76	30.23	32.02	34.78	37.28	45.74	52.33	59.53	70.99	75.00	79.84	83.25	90.57	53
54	27.47	30.98	32.79	35.59	38.12	46.68	53.33	60.60	72.15	76.19	81.07	84.50	91.87	54
55	28.17	31.73	33.57	36.40	38.96	47.61	54.33	61.66	73.31	77.38	82.29	85.75	93.17	55
56	28.88	32.49	34.35	37.21	39.80	48.55	55.33	62.73	74.47	78.57	83.51	86.99	94.46	56
57	29.59	33.25	35.13	38.03	40.65	49.48	56.33	63.79	75.62	79.75	84.73	88.24	95.75	57
58	30.30	34.01	35.91	38.84	41.49	50.42	57.33	64.86	76.78	80.94	85.95	89.48	97.04	58
59	31.02	34.77	36.70	39.66	42.34	51.36	58.33	65.92	77.93	82.12	87.17	90.72	98.32	59
60	31.74	35.53	37.48	40.48	43.19	52.29	59.33	66.98	79.08	83.30	88.38	91.95	99.61	60
61	32.46	36.30	38.27	41.30	44.04	53.23	60.33	68.04	80.23	84.48	89.59	93.19	100.89	61
62	33.18	37.07	39.06	42.13	44.89	54.17	61.33	69.10	81.38	85.65	90.80	94.42	102.17	62
63	33.91	37.84	39.86	42.95	45.74	55.11	62.33	70.16	82.53	86.83	92.01	95.65	103.44	63
64	34.63	38.61	40.65	43.78	46.59	56.05	63.33	71.23	83.68	88.00	93.22	96.88	104.72	64
65	35.36	39.38	41.44	44.60	47.45	56.99	64.33	72.28	84.82	89.18	94.42	98.11	105.99	65
66	36.09	40.16	42.24	45.43	48.31	57.93	65.33	73.34	85.96	90.35	95.63	99.33	107.26	66
67	36.83	40.94	43.04	46.26	49.16	58.87	66.33	74.40	87.11	91.52	96.83	100.55	108.53	67
68	37.56	41.71	43.84	47.09	50.02	59.81	67.33	75.46	88.25	92.69	98.03	101.78	109.79	68
69	38.30	42.49	44.64	47.92	50.88	60.76	68.33	76.52	89.39	93.86	99.23	103.00	111.06	69
70	39.04	43.28	45.44	48.76	51.74	61.70	69.33	77.58	90.53	95.02	100.43	104.21	112.32	70
71	39.78	44.06	46.25	49.59	52.60	62.64	70.33	78.63	91.67	96.19	101.62	105.43	113.58	71
72	40.52	44.84	47.05	50.43	53.46	63.58	71.33	79.69	92.81	97.35	102.82	106.65	114.84	72
73	41.26	45.63	47.86	51.26	54.33	64.53	72.33	80.75	93.95	98.52	104.01	107.86	116.09	73
74	42.01	46.42	48.67	52.10	55.19	65.47	73.33	81.80	95.08	99.68	105.20	109.07	117.35	74
75	42.76	47.21	49.48	52.94	56.05	66.42	74.33	82.86	96.22	100.84	106.39	110.29	118.60	75
76	43.51	48.00	50.29	53.78	56.92	67.36	75.33	83.91	97.35	102.00	107.58	111.50	119.85	76
77	44.26	48.79	51.10	54.62	57.79	68.31	76.33	84.97	98.48	103.16	108.77	112.70	121.10	77
78	45.01	49.58	51.91	55.47	58.65	69.25	77.33	86.02	99.62	104.32	109.96	113.91	122.35	78
79	45.76	50.38	52.72	56.31	59.52	70.20	78.33	87.08	100.75	105.47	111.14	115.12	123.59	79
80	46.52	51.17	53.54	57.15	60.39	71.14	79.33	88.13	101.88	106.63	112.33	116.32	124.84	80
81	47.28	51.97	54.36	58.00	61.26	72.09	80.33	89.18	103.01	107.78	113.51	117.52	126.08	81
82	48.04	52.77	55.17	58.84	62.13	73.04	81.33	90.24	104.14	108.94	114.69	118.73	127.32	82
83	48.80	53.57	55.99	59.69	63.00	73.99	82.33	91.29	105.27	110.09	115.88	119.93	128.56	83
84	49.56	54.37	56.81	60.54	63.88	74.93	83.33	92.34	106.39	111.24	117.06	121.13	129.80	84
85	50.32	55.17	57.63	61.39	64.75	75.88	84.33	93.39	107.52	112.39	118.24	122.32	131.04	85
86	51.08	55.97	58.46	62.24	65.62	76.83	85.33	94.45	108.65	113.54	119.41	123.52	132.28	86
87	51.85	56.78	59.28	63.09	66.50	77.78	86.33	95.50	109.77	114.69	120.59	124.72	133.51	87
88	52.62	57.58	60.10	63.94	67.37	78.73	87.33	96.55	110.90	115.84	121.77	125.91	134.75	88
89	53.39	58.39	60.93	64.79	68.25	79.68	88.33	97.60	112.02	116.99	122.94	127.11	135.98	89
90	54.16	59.20	61.75	65.65	69.13	80.62	89.33	98.65	113.15	118.14	124.12	128.30	137.21	90
91	54.93	60.00	62.58	66.50	70.00	81.57	90.33	99.70	114.27	119.28	125.29	129.49	138.44	91
92	55.70	60.81	63.41	67.36	70.88	82.52	91.33	100.75	115.39	120.43	126.46	130.68	139.67	92
93	56.47	61.63	64.24	68.21	71.76	83.47	92.33	101.80	116.51	121.57	127.63	131.87	140.89	93
94	57.25	62.44	65.07	69.07	72.64	84.42	93.33	102.85	117.63	122.72	128.80	133.06	142.12	94
95	58.02	63.25	65.90	69.92	73.52	85.38	94.33	103.90	118.75	123.86	129.97	134.25	143.34	95
96	58.80	64.06	66.73	70.78	74.40	86.33	95.33	104.95	119.87	125.00	131.14	135.43	144.57	96
97	59.58	64.88	67.56	71.64	75.28	87.28	96.33	106.00	120.99	126.14	132.31	136.62	145.79	97
98	60.36	65.69	68.40	72.50	76.16	88.23	97.33	107.05	122.11	127.28	133.48	137.80	147.01	98
99	61.14	66.51	69.23	73.36	77.05	89.18	98.33	108.09	123.23	128.42	134.64	138.99	148.23	99
100	61.92	67.33	70.06	74.22	77.93	90.13	99.33	109.14	124.34	129.56	135.81	140.17	149.45	100
z	-3.090	-2.576	-2.326	-1.960	-1.645	-.674	.000	.674	1.645	1.960	2.326	2.576	3.090	z

For degrees of freedom (ν) beyond 100, percentiles of χ^2 may be approximated with:
$\chi^2_{\nu[P]} = ½\left(z_{[P]} + \sqrt{2\nu - 1}\right)^2$, utilizing the normal percentiles $z_{[P]}$ at the foot of the table.

Reading off the table

The table furnishes a set of percentage points of the Chi-square (χ^2_ν) distribution for degrees of freedom (ν) from 1 to 100. For larger ν, the approximation formula printed at the foot of the table is recommended.

Illustration 1. What is the value of $\chi^2_{6[.95]}$, *i.e.* the 95th percentage point of Chi-square with $\nu = 6$? Looking up line 6 (= ν) in the table under column 0.95, we read off 12.59, hence $\chi^2_{6[.95]} = 12.59$. In the same way, we obtain $\chi^2_{13[.99]} = 27.69$ and $\chi^2_{20[.975]} = 34.17$.

Illustration 2. Find $\chi^2_{110[.95]}$. As $\nu = 110 > 100$, it is necessary to calculate some estimate of the required percentage point. Using the recommended formula, with $z_{[.95]} = 1.6449$ as indicated, we calculate $\chi^2_{110[.95]} \approx \frac{1}{2}[1.6449 + \sqrt{(2\times110-1)}]^2 \approx 135.20$. The exact value (when available) is 135.48.

Full examples

Example 1. In a sample containing 50 observations, we obtain $s^2 = 16.43$ as an estimate of variance. What are the limits within which sould lie the true variance σ^2, using a confidence coefficient of 95 %? *Solution*: We must suppose that the individual observations (X_i) obey the normal law, with (unknown) mean μ and variance σ^2. Under that assumption, the sample variance s^2 is distributed as Chi-square with $n-1$ *df*, specifically $(n-1)s^2/\sigma^2 \sim \chi^2_{n-1}$. Using the appropriate percentage points of χ^2 and inverting this formula, we obtain the interval:

$$\left\{\frac{(n-1)s^2}{\chi^2_{n-1[(1+c)/2]}} \; ; \; \frac{(n-1)s^2}{\chi^2_{n-1[(1-c)/2]}}\right\}$$

which comprises σ^2 with probability c. For our data, $n = 50$, $c = 0.95$, $\chi^2_{49[.975]} = 70.22$ and $\chi^2_{49[.025]} = 31.55$; the left boundary is thus $(49\times16.43)/70.22 \approx 11.465$, the right one, $(49\times16.43)/31.55 \approx 25.517$. We may state that $\Pr\{ 11.465 < \sigma^2 < 25.517 \} = 0.95$ or, by taking square roots, $\Pr\{ 3.386 < \sigma < 5.051 \} = 0.95$.

Example 2. In an opinion poll bearing on social and moral issues, 200 people must declare their views as "Against", "Uncertain" or "In favor" relatively to the death penalty. Here are the obtained frequencies of opinion, divided between the two genders:

Gender \ Option	Against	Uncertain	In favor
Men	29	17	51
Women	52	8	43

Can we suppose that, in the entire population, men and women share the same views? *Solution*: The statistical analysis of frequency (or contingency) tables is perhaps the foremost application of Chi-square. Here, the (null) hypothesis according to which the answers are scattered irrespective of gender, *i.e.* the *independence hypothesis*, allows to determine the theoretical frequencies ($ft_{i,j}$), with the multiplicative formula:

$$ft_{i,j} = \mathrm{N}(p_{\mathrm{L}_i} \times p_{\mathrm{C}_j}) \ ;$$

the quantities shown are estimated from the proportions in each line (p_{L_i}) and each colum (p_{C_j}). Other equivalent formulae are possible. The independence hypothesis will be discarded at significance level α if the test statistic $\mathrm{X}^2 = \Sigma_{i,j}[(f_{i,j}-ft_{i,j})^2/ft_{i,j}]$ exceeds $\chi^2_{\nu[1-\alpha]}$, with ν = (nbr. of lines$-$1)$\times$(nbr. of columns$-$1). The table below summarizes the calculations. Note that quantity $ft_{i,j}$ is printed in italics at the lower right corner of each cell, and individual X^2 components, $(f_{i,j}-ft_{i,j})^2/ft_{i,j}$, at the upper left corner.

	Column 1	Column 2	Column 3	Total
Line 1	2.693 29 *39.285*	1.960 17 *12.125*	0.642 51 *45.590*	97 $p_{\mathrm{L}1}$ = 0.485
Line 2	2.536 52 *41.715*	1.846 8 *12.875*	0.605 43 *48.410*	103 $p_{\mathrm{L}2}$ = 0.515
Total	81 $p_{\mathrm{C}1}$ = 0.405	25 $p_{\mathrm{C}2}$ = 0.125	94 $p_{\mathrm{C}3}$ = 0.470	N = 200

Adding all six components $(f_{i,j}-ft_{i,j})^2/ft_{i,j}$, we get $X^2 = 2.693 + 1.960 + 0.642 + 2.536 + 1.846 + 0.605 = 10.282$. The appropriate tabular value significant at 5 % and with $df = (2-1)\times(3-1) = 2$, is $\chi^2_{2[.95]} = 5.99$. As the obtained value (10.282) exceeds the critical value, we may conclude that there is some dependence (or interaction, indeed correlation in a broad sense) between lines and columns, that the frequency profiles vary from one line to the other; in other words, the respondant's gender seems to bias his or her opinion on the death penalty.

Mathematical presentation

A Chi-square variate with ν degrees of freedom is equivalent to the sum of ν independent, squared, standard normal variates, $\sum_{i=1}^{\nu} z_i^2$ and it is denoted χ^2_ν. As an example, the variance (s^2) from a sample of normally distributed observations is distributed as χ^2, the parameter ν being referred to as the *degrees of freedom* (*df*) of the calculated variance. Symbolically, we write:

$$\frac{\nu \cdot s^2}{\sigma^2} \sim \chi^2_\nu \ .$$

In the case of the statistic s^2 based upon n observations from a $N(\mu,\sigma^2)$ distribution, where $s^2 = \sum(x_i-\bar{x})^2/(n-1)$, the *df* are equal to $\nu = n-1$. The Chi-square distribution is also used for the analysis of frequency (or contingency) tables and as an approximation to the distribution of many complex statistics.

Calculation and moments

The Chi-square distribution, a particular case of the *Gamma* distribution (*see* Mathematical complements), has p.d.f.:

$$p_{\chi^2}(x) = [2^{\nu/2}\Gamma(\nu/2)]^{-1} x^{(\nu-2)/2} e^{-x/2} \quad \{ x \geq 0 \} \ ,$$

where $\Gamma(x)$ is the *Gamma* function and $e \approx 2.7183$. Integration of the χ^2 density depends on whether ν is even or odd. Integrating by parts, we obtain for even ν :

$$P_{\chi^2}(x) = \Pr(X \leq x) = 1 - e^{-y}\left[1 + y + \frac{y^2}{2!} + \dots + \frac{y^{\nu/2-1}}{(\nu/2-1)!}\right] ,$$

and for odd ν:

$$P_{\chi^2}(x) = \Pr(X \le x) = 2\Phi(\sqrt{x}) - 1 - e^{-y}\sqrt{y}\left[\frac{1}{\Gamma(3/2)} + \frac{y}{\Gamma(5/2)} + \cdots + \frac{y^{(\nu-3)/2}}{\Gamma(\nu/2)}\right];$$

in each expression, $y = x/2$. When $\nu = 1$, $\chi^2 = z^2$ by definition, therefore $P_{\chi^2}(x) = 2\Phi(\sqrt{x})-1, \Phi()$ designating the normal d.f.. For $\nu = 2$, the χ^2 variable is the same as a r.v. from the (standard) exponential distribution and $P_{\chi^2}(x) = 1-\exp(-x/2)$; centiles ($C_P$) of this χ^2_2 distribution may be obtained by inversion, *i.e.* $C_P = \chi^2_{2[P]} = -2\ln(1-P)$.

Moments. The expectation, variance and moments for skewness and kurtosis of a χ^2 variable with degrees of freedom ν are:

$$E(x) = \mu = \nu\ ;\ \ \mathrm{var}(x) = \sigma^2 = 2\nu\ ;\ \ \gamma_1 = \sqrt{(8/\nu)}\,;\ \ \gamma_2 = 12/\nu\ .$$

The distribution is positively skew, the more large and right-shifted as ν grows and approaching a normal form. The mode is seated at $\nu-2$ (for $\nu\ge 2$), and the median is approximately equal to $\nu-\frac{2}{3}+\frac{1}{9\nu}$.

Some authors "standardize" the χ^2 variable by dividing it by its parameter ν, *i.e.* $x' = x/\nu$: in that case, $\mu(x') = 1$ and $\mathrm{var}(x') = 2/\nu$. This form facilitates somewhat interpolation of χ^2 for untabled values of ν ; note in that context that $\chi^2/\nu \to 1$ when $\nu \to \infty$.

Three normal approximations to χ^2

The p.d.f. and d.f. of χ^2 can be approximated by the normal distribution through diverse transformations. The simplest one is trivial and uses only the first two moments, *i.e.* $z = (X-\nu)/\sqrt{(2\nu)}, X \sim \chi^2$, and is globally not to be recommended except for large ν such as $\nu > 500$.

Fisher proposes another approximation which compensates for the skewness of X. It reads like:

$$\sqrt{(2X)} - \sqrt{(2\nu-1)} \ \sim N(0,1)\ ;$$

it is the method whose inversion formula is proposed at the foot of the table for extending it to larger ν.

The third method, attributed to Wilson and Hilferty, is quite accurate. Defining $A = 2/(9\nu)$, it may be written:

$$[\ \sqrt[3]{(X/\nu)} - 1 + A\ ]\ /\ \sqrt{A} \sim N(0,1)\ .$$

Its inversion, for the determination of percentage points, is:

$$\chi^2_{\nu[P]} = \nu[z_P\sqrt{A} + 1 - A]^3\ .$$

With the help of a pocket calculator or of a short computer program, this last method can make up for most current applications of χ^2, even when $\nu < 100$.

The distribution of s, the standard deviation (s.d.)

Just as the χ^2 law governs the distribution of variances (s^2) originating from samples of n normal data, with $\nu = n-1$ *df*, the χ ("Chi") law, more precisely $\chi/\sqrt{\nu}$, represents the sampling distribution of s.d.'s (s). Its p.d.f. is:

$$p_\chi(x) = 2(\nu/2)^{\nu/2}[\Gamma(\nu/2)]^{-1}x^{\nu-1}e^{-\nu x^2/2} \quad \{ x > 0 \} .$$

The χ variable being the positive square root of χ^2, its centiles or percentage points may be obtained from it in that way. Thus, centile C_P of the distribution of a s/σ with ν degrees of freedom is given by $\sqrt{[\chi^2_{\nu[P]}/\nu]}$.

Moments. The first two moments of $\chi/\sqrt{\nu}$ are:

$$E(x) = \mu = \sqrt{(2/\nu)}\Gamma[(\nu+1)/2]/\Gamma(\nu/2) \approx 1 - 1/(4\nu)$$

$$\mathrm{var}(x) = \sigma^2 = 1 - \mu^2 \approx (4\nu-1)/(8\nu^2) .$$

The s.d. s being distributed as $\sigma\chi/\sqrt{\nu}$, the expectation above shows that $E(s) < \sigma$, *i.e.* that the sample s.d. underestimates the parameter σ, notwithstanding the fact that $E(s^2) = \sigma^2$. Lastly, the mode of $\chi/\sqrt{\nu}$ equals $\sqrt{(1-1/\nu)}$ and the median is approximated by $1-\frac{1}{3\nu}$.

Generation of pseudo random variates

The schema of a program below allows the production of r.v.'s from χ^2_ν, the Chi-square distribution with ν ($\nu > 2$) *df*, and it requires a function (designated UNIF) which generates serially r.v.'s from the standard uniform $U(0,1)$ distribution. Particular cases, especially those with $\nu = 1$ and 2, are covered in Remark 3.

Preparation: Let $n \equiv \nu$ (the degrees of freedom)

$C_1 = 1 + \sqrt{(2/e)} \approx 1{,}8577638850$; $C_2 = \sqrt{(n/2)}$
$C_3 = (3n^2-2) / [3n(n-2)]$; $C_4 = 4 / (n-2)$
$C_5 = n-2$

Production : **Repeat**

  Repeat $t \leftarrow$ UNIF ; $u \leftarrow t + (1 - C_1 \times \text{UNIF})/C_2$

  Until $0 < u \leq 1$;

 $w \leftarrow C_3 \times t \,/\, u$

 Until $C_4 \times (u-1) + w + 1/w < 2$ **or** $C_4 \times \ln(u) - \ln(w) + w < 1$;

 Return $C_5 \times w \rightarrow x$.

Remarks :

1. Standard temporal cost : 7.8 à 8.7 × t(UNIF)

2. This algorithm, known under the codename "GMK2" (Cheng et Feast 1979, *in* Fishman 1996), performs equally well for any value ν (= n). It uses up from 3 to 3.5 uniform r.v.'s per call.

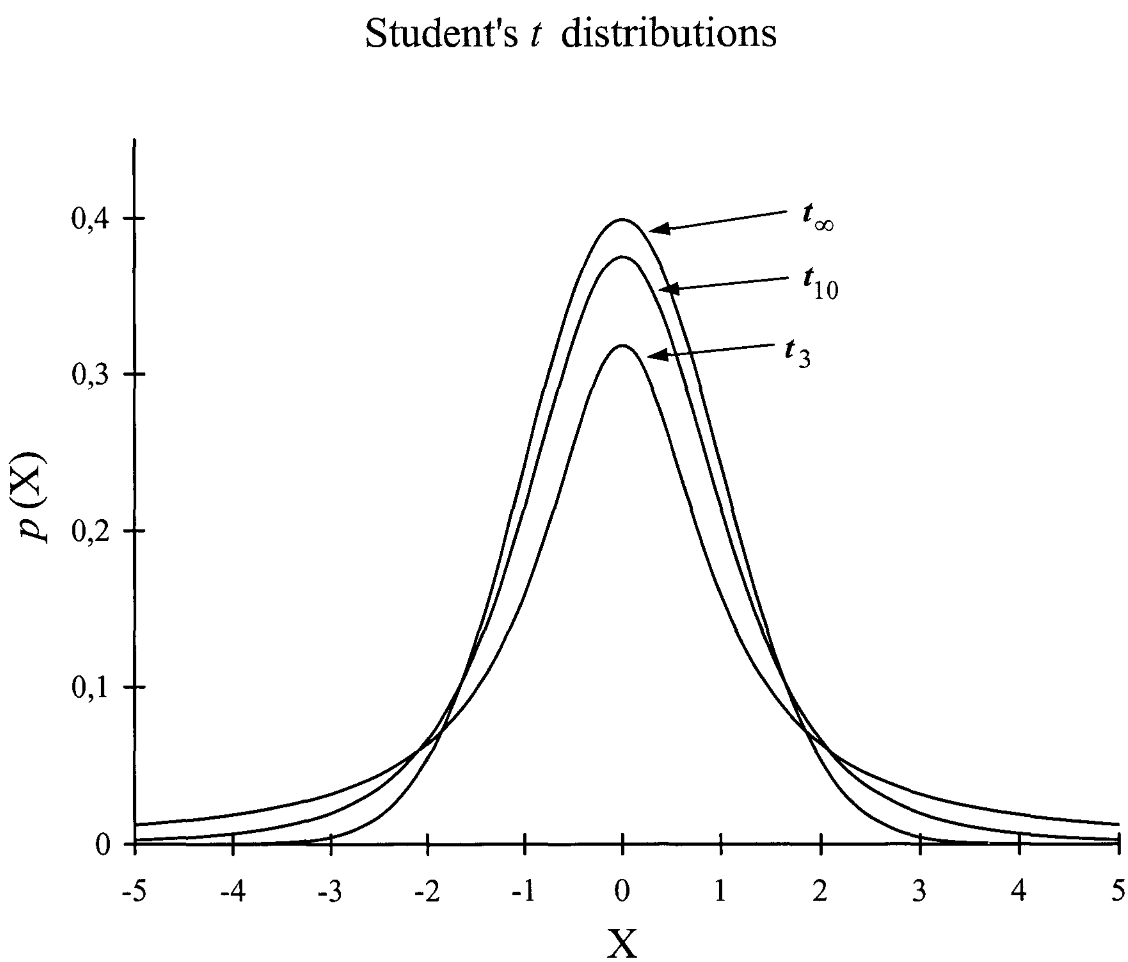
3. There are many other methods, the following being noteworthy. Considering that "$x_{(2)} \leftarrow -2\times\ln(\text{UNIF})$" produces a χ_2^2 r.v. and capitalizing on the additive property of χ^2, we can produce, for instance, a χ_8^2 r.v. with "$x_{(8)} \leftarrow -2\times\ln(\text{UNIF}\times\text{UNIF}\times\text{UNIF}\times\text{UNIF})$". Also from the definition, "$x_{(1)} \leftarrow y^2$" furnishes one r.v. from χ_1^2 using y, a standard $N(0,1)$ normal r.v.. Lastly and for example, we may fabricate a χ_5^2 r.v. through "$x_{(5)} \leftarrow -2\times\ln(\text{UNIF} \times\text{UNIF}) + y^2$" , once more using $y \sim N(0,1)$.

Student's *t* distribution

✓ Graphical representations
✓ Selected percentiles of Student's *t* distribution (table 1)
✓ Critical values of *t* according to Dunn-Šidák's criterion (table 2)
✓ Minimum sample size (n) such that a given correlation coefficient r be significant at various thresholds (table 3)
✓ Reading off the table
✓ Full examples
✓ Mathematical presentation

Calculation and moments

Generation of pseudo random variates


Student's t distributions
0,4
0,3
0,2
0,1
0
p (X)
t∞
t10
t3
-5
-4
-3
-2
-1
0
1
2
3
4
5
X

Selected percentiles of Student's *t* distribution (table 1)

$\nu \setminus P$	.750	.950	.975	.990	.995	$\nu \setminus P$	.750	.950	.975	.990	.995
1	1.000	6.314	12.71	31.82	63.66	51	.679	1.675	2.008	2.402	2.676
2	.816	2.920	4.303	6.965	9.925	52	.679	1.675	2.007	2.400	2.674
3	.765	2.353	3.182	4.541	5.841	53	.679	1.674	2.006	2.399	2.672
4	.741	2.132	2.776	3.747	4.604	54	.679	1.674	2.005	2.397	2.670
5	.727	2.015	2.571	3.365	4.032	55	.679	1.673	2.004	2.396	2.668
6	.718	1.943	2.447	3.143	3.707	56	.679	1.673	2.003	2.395	2.667
7	.711	1.895	2.365	2.998	3.499	57	.679	1.672	2.002	2.394	2.665
8	.706	1.860	2.306	2.896	3.355	58	.679	1.672	2.002	2.392	2.663
9	.703	1.833	2.262	2.821	3.250	59	.679	1.671	2.001	2.391	2.662
10	.700	1.812	2.228	2.764	3.169	60	.679	1.671	2.000	2.390	2.660
11	.697	1.796	2.201	2.718	3.106	61	.679	1.670	2.000	2.389	2.659
12	.695	1.782	2.179	2.681	3.055	62	.678	1.670	1.999	2.388	2.657
13	.694	1.771	2.160	2.650	3.012	63	.678	1.669	1.998	2.387	2.656
14	.692	1.761	2.145	2.624	2.977	64	.678	1.669	1.998	2.386	2.655
15	.691	1.753	2.131	2.602	2.947	65	.678	1.669	1.997	2.385	2.654
16	.690	1.746	2.120	2.583	2.921	66	.678	1.668	1.997	2.384	2.652
17	.689	1.740	2.110	2.567	2.898	67	.678	1.668	1.996	2.383	2.651
18	.688	1.734	2.101	2.552	2.878	68	.678	1.668	1.995	2.382	2.650
19	.688	1.729	2.093	2.539	2.861	69	.678	1.667	1.995	2.382	2.649
20	.687	1.725	2.086	2.528	2.845	70	.678	1.667	1.994	2.381	2.648
21	.686	1.721	2.080	2.518	2.831	71	.678	1.667	1.994	2.380	2.647
22	.686	1.717	2.074	2.508	2.819	72	.678	1.666	1.993	2.379	2.646
23	.685	1.714	2.069	2.500	2.807	73	.678	1.666	1.993	2.379	2.645
24	.685	1.711	2.064	2.492	2.797	74	.678	1.666	1.993	2.378	2.644
25	.684	1.708	2.060	2.485	2.787	75	.678	1.665	1.992	2.377	2.643
26	.684	1.706	2.056	2.479	2.779	76	.678	1.665	1.992	2.376	2.642
27	.684	1.703	2.052	2.473	2.771	77	.678	1.665	1.991	2.376	2.641
28	.683	1.701	2.048	2.467	2.763	78	.678	1.665	1.991	2.375	2.640
29	.683	1.699	2.045	2.462	2.756	79	.678	1.664	1.990	2.374	2.640
30	.683	1.697	2.042	2.457	2.750	80	.678	1.664	1.990	2.374	2.639
31	.682	1.696	2.040	2.453	2.744	81	.678	1.664	1.990	2.373	2.638
32	.682	1.694	2.037	2.449	2.738	82	.677	1.664	1.989	2.373	2.637
33	.682	1.692	2.035	2.445	2.733	83	.677	1.663	1.989	2.372	2.636
34	.682	1.691	2.032	2.441	2.728	84	.677	1.663	1.989	2.372	2.636
35	.682	1.690	2.030	2.438	2.724	85	.677	1.663	1.988	2.371	2.635
36	.681	1.688	2.028	2.434	2.719	86	.677	1.663	1.988	2.370	2.634
37	.681	1.687	2.026	2.431	2.715	87	.677	1.663	1.988	2.370	2.634
38	.681	1.686	2.024	2.429	2.712	88	.677	1.662	1.987	2.369	2.633
39	.681	1.685	2.023	2.426	2.708	89	.677	1.662	1.987	2.369	2.632
40	.681	1.684	2.021	2.423	2.704	90	.677	1.662	1.987	2.368	2.632
41	.681	1.683	2.020	2.421	2.701	91	.677	1.662	1.986	2.368	2.631
42	.680	1.682	2.018	2.418	2.698	92	.677	1.662	1.986	2.368	2.630
43	.680	1.681	2.017	2.416	2.695	93	.677	1.661	1.986	2.367	2.630
44	.680	1.680	2.015	2.414	2.692	94	.677	1.661	1.986	2.367	2.629
45	.680	1.679	2.014	2.412	2.690	95	.677	1.661	1.985	2.366	2.629
46	.680	1.679	2.013	2.410	2.687	96	.677	1.661	1.985	2.366	2.628
47	.680	1.678	2.012	2.408	2.685	97	.677	1.661	1.985	2.365	2.627
48	.680	1.677	2.011	2.407	2.682	98	.677	1.661	1.984	2.365	2.627
49	.680	1.677	2.010	2.405	2.680	99	.677	1.660	1.984	2.365	2.626
50	.679	1.676	2.009	2.403	2.678	100	.677	1.660	1.984	2.364	2.626
						500	.675	1.648	1.965	2.334	2.586
						∞	.674	1.645	1.960	2.326	2.576

Critical values of *t* according to Dunn-Šidák's criterion for one-tailed 5 % tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	4.273	5.292	6.144	6.892	7.566	8.185	8.760	9.300	9.810	10.29	10.76	11.20	11.63	12.04
3	3.166	3.716	4.146	4.506	4.819	5.097	5.349	5.580	5.793	5.993	6.180	6.357	6.525	6.685
4	2.764	3.169	3.474	3.723	3.935	4.121	4.286	4.436	4.574	4.701	4.819	4.929	5.034	5.132
5	2.560	2.897	3.146	3.346	3.514	3.660	3.788	3.904	4.009	4.106	4.195	4.278	4.357	4.430
6	2.438	2.736	2.954	3.127	3.270	3.394	3.503	3.600	3.688	3.769	3.843	3.912	3.976	4.037
7	2.356	2.630	2.828	2.984	3.112	3.223	3.319	3.405	3.482	3.553	3.618	3.678	3.735	3.787
8	2.298	2.555	2.739	2.883	3.002	3.103	3.191	3.269	3.340	3.404	3.463	3.517	3.568	3.615
9	2.254	2.499	2.673	2.809	2.920	3.015	3.097	3.170	3.235	3.295	3.349	3.400	3.447	3.490
10	2.221	2.456	2.623	2.752	2.858	2.947	3.025	3.094	3.156	3.212	3.263	3.310	3.354	3.395
11	2.194	2.422	2.582	2.707	2.808	2.894	2.968	3.034	3.093	3.146	3.195	3.240	3.282	3.320
12	2.172	2.394	2.550	2.670	2.768	2.851	2.922	2.986	3.042	3.093	3.140	3.183	3.223	3.260
13	2.153	2.370	2.523	2.640	2.735	2.815	2.885	2.946	3.000	3.050	3.095	3.136	3.175	3.211
14	2.138	2.351	2.500	2.614	2.707	2.785	2.853	2.912	2.965	3.013	3.057	3.097	3.134	3.169
15	2.125	2.334	2.480	2.592	2.683	2.760	2.826	2.884	2.935	2.982	3.025	3.064	3.100	3.134
16	2.113	2.320	2.463	2.573	2.663	2.738	2.802	2.859	2.910	2.955	2.997	3.035	3.071	3.104
17	2.103	2.307	2.449	2.557	2.645	2.718	2.782	2.838	2.887	2.932	2.973	3.011	3.045	3.077
18	2.094	2.296	2.436	2.543	2.629	2.702	2.764	2.819	2.868	2.912	2.952	2.989	3.023	3.054
19	2.087	2.286	2.424	2.530	2.615	2.687	2.748	2.802	2.850	2.894	2.933	2.969	3.003	3.034
20	2.080	2.277	2.414	2.518	2.603	2.673	2.734	2.788	2.835	2.878	2.917	2.952	2.985	3.016
22	2.068	2.262	2.397	2.499	2.582	2.651	2.710	2.762	2.809	2.850	2.888	2.923	2.955	2.985
24	2.058	2.250	2.382	2.483	2.564	2.632	2.690	2.742	2.787	2.828	2.865	2.899	2.931	2.960
26	2.049	2.239	2.370	2.470	2.550	2.617	2.674	2.724	2.769	2.809	2.846	2.879	2.910	2.939
28	2.042	2.231	2.360	2.458	2.537	2.603	2.660	2.710	2.754	2.793	2.829	2.862	2.893	2.921
30	2.036	2.223	2.351	2.448	2.527	2.592	2.648	2.697	2.741	2.780	2.815	2.848	2.878	2.905
32	2.031	2.216	2.343	2.440	2.517	2.582	2.638	2.686	2.729	2.768	2.803	2.835	2.865	2.892
34	2.026	2.210	2.337	2.432	2.509	2.573	2.628	2.676	2.719	2.757	2.792	2.824	2.853	2.880
36	2.022	2.205	2.331	2.426	2.502	2.566	2.620	2.668	2.710	2.748	2.783	2.814	2.843	2.870
38	2.018	2.201	2.325	2.420	2.496	2.559	2.613	2.660	2.702	2.740	2.774	2.806	2.834	2.861
40	2.015	2.197	2.321	2.415	2.490	2.553	2.607	2.654	2.695	2.733	2.767	2.798	2.826	2.853
42	2.012	2.193	2.316	2.410	2.485	2.547	2.601	2.648	2.689	2.726	2.760	2.791	2.819	2.846
44	2.010	2.190	2.313	2.406	2.480	2.542	2.596	2.642	2.683	2.720	2.754	2.784	2.813	2.839
46	2.007	2.186	2.309	2.402	2.476	2.538	2.591	2.637	2.678	2.715	2.748	2.779	2.807	2.833
48	2.005	2.184	2.306	2.398	2.472	2.534	2.587	2.633	2.673	2.710	2.743	2.773	2.801	2.827
50	2.003	2.181	2.303	2.395	2.469	2.530	2.583	2.628	2.669	2.705	2.738	2.769	2.796	2.822
55	1.998	2.176	2.296	2.388	2.461	2.522	2.574	2.619	2.659	2.696	2.728	2.758	2.786	2.811
60	1.995	2.171	2.291	2.382	2.455	2.515	2.567	2.612	2.652	2.687	2.720	2.749	2.777	2.802
65	1.991	2.167	2.287	2.377	2.449	2.509	2.561	2.605	2.645	2.681	2.713	2.742	2.769	2.794
70	1.989	2.164	2.283	2.373	2.445	2.504	2.555	2.600	2.639	2.675	2.707	2.736	2.763	2.788
75	1.986	2.161	2.279	2.369	2.441	2.500	2.551	2.595	2.634	2.670	2.701	2.731	2.757	2.782
80	1.984	2.158	2.277	2.366	2.437	2.496	2.547	2.591	2.630	2.665	2.697	2.726	2.753	2.777
85	1.983	2.156	2.274	2.363	2.434	2.493	2.544	2.588	2.626	2.661	2.693	2.722	2.748	2.773
90	1.981	2.154	2.272	2.360	2.431	2.490	2.541	2.584	2.623	2.658	2.689	2.718	2.745	2.769
95	1.980	2.152	2.270	2.358	2.429	2.488	2.538	2.581	2.620	2.655	2.686	2.715	2.741	2.766
100	1.978	2.151	2.268	2.356	2.427	2.485	2.535	2.579	2.618	2.652	2.683	2.712	2.738	2.763
125	1.974	2.145	2.261	2.349	2.419	2.477	2.526	2.569	2.607	2.642	2.673	2.701	2.727	2.751
150	1.970	2.141	2.257	2.344	2.413	2.471	2.520	2.563	2.601	2.635	2.665	2.694	2.719	2.743
175	1.968	2.138	2.253	2.340	2.409	2.467	2.516	2.558	2.596	2.630	2.660	2.688	2.714	2.738
200	1.966	2.136	2.251	2.337	2.406	2.464	2.512	2.555	2.592	2.626	2.657	2.684	2.710	2.734
250	1.964	2.133	2.247	2.334	2.402	2.459	2.508	2.550	2.588	2.621	2.651	2.679	2.704	2.728
300	1.962	2.131	2.245	2.331	2.400	2.456	2.505	2.547	2.584	2.618	2.648	2.675	2.701	2.724
350	1.961	2.130	2.244	2.329	2.398	2.454	2.503	2.545	2.582	2.615	2.645	2.673	2.698	2.721
400	1.960	2.129	2.242	2.328	2.396	2.453	2.501	2.543	2.580	2.613	2.643	2.671	2.696	2.719
450	1.960	2.128	2.241	2.327	2.395	2.452	2.500	2.542	2.579	2.612	2.642	2.669	2.694	2.718
500	1.959	2.127	2.241	2.326	2.394	2.451	2.499	2.541	2.578	2.611	2.641	2.668	2.693	2.716
1000	1.957	2.124	2.237	2.322	2.390	2.446	2.494	2.536	2.573	2.606	2.636	2.663	2.688	2.711
∞	1.955	2.121	2.234	2.319	2.386	2.442	2.490	2.531	2.568	2.601	2.630	2.657	2.682	2.705

Critical values of *t* according to Dunn-Šidák's criterion for two-tailed 5% tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	6.164	7.582	8.774	9.823	10.77	11.64	12.45	13.21	13.93	14.61	15.26	15.89	16.49	17.07
3	4.156	4.826	5.355	5.799	6.185	6.529	6.842	7.128	7.394	7.642	7.876	8.096	8.306	8.505
4	3.481	3.941	4.290	4.577	4.822	5.036	5.228	5.402	5.562	5.710	5.848	5.977	6.099	6.214
5	3.152	3.518	3.791	4.012	4.197	4.358	4.501	4.630	4.747	4.855	4.955	5.049	5.136	5.219
6	2.959	3.274	3.505	3.690	3.845	3.978	4.095	4.200	4.296	4.383	4.464	4.539	4.609	4.675
7	2.832	3.115	3.321	3.484	3.620	3.736	3.838	3.929	4.011	4.086	4.156	4.220	4.280	4.336
8	2.743	3.005	3.193	3.342	3.464	3.569	3.661	3.743	3.816	3.883	3.945	4.002	4.055	4.105
9	2.677	2.923	3.099	3.237	3.351	3.448	3.532	3.607	3.675	3.736	3.793	3.845	3.893	3.939
10	2.626	2.860	3.027	3.157	3.264	3.355	3.434	3.505	3.568	3.625	3.677	3.726	3.771	3.813
11	2.586	2.811	2.970	3.094	3.196	3.283	3.358	3.424	3.484	3.538	3.587	3.633	3.675	3.715
12	2.553	2.770	2.924	3.044	3.141	3.224	3.296	3.359	3.416	3.468	3.515	3.558	3.598	3.636
13	2.526	2.737	2.886	3.002	3.096	3.176	3.245	3.306	3.361	3.410	3.455	3.497	3.535	3.571
14	2.503	2.709	2.854	2.967	3.058	3.135	3.202	3.261	3.314	3.362	3.406	3.446	3.483	3.518
15	2.483	2.685	2.827	2.937	3.026	3.101	3.166	3.224	3.275	3.321	3.364	3.402	3.439	3.472
16	2.467	2.665	2.804	2.911	2.998	3.072	3.135	3.191	3.241	3.286	3.327	3.365	3.400	3.433
17	2.452	2.647	2.783	2.889	2.974	3.046	3.108	3.163	3.212	3.256	3.296	3.333	3.367	3.399
18	2.439	2.631	2.766	2.869	2.953	3.024	3.085	3.138	3.186	3.229	3.269	3.305	3.338	3.370
19	2.427	2.617	2.750	2.852	2.934	3.004	3.064	3.116	3.163	3.206	3.245	3.280	3.313	3.343
20	2.417	2.605	2.736	2.836	2.918	2.986	3.045	3.097	3.143	3.185	3.223	3.258	3.290	3.320
22	2.400	2.584	2.712	2.810	2.889	2.956	3.014	3.064	3.109	3.150	3.187	3.220	3.252	3.281
24	2.385	2.566	2.692	2.788	2.866	2.931	2.988	3.037	3.081	3.121	3.157	3.190	3.220	3.249
26	2.373	2.551	2.675	2.770	2.847	2.911	2.966	3.014	3.058	3.096	3.132	3.164	3.194	3.222
28	2.363	2.539	2.661	2.755	2.830	2.893	2.948	2.995	3.038	3.076	3.111	3.142	3.172	3.199
30	2.354	2.528	2.649	2.742	2.816	2.878	2.932	2.979	3.021	3.058	3.092	3.124	3.153	3.180
32	2.346	2.519	2.639	2.730	2.804	2.865	2.918	2.965	3.006	3.043	3.077	3.108	3.136	3.163
34	2.340	2.511	2.630	2.720	2.793	2.854	2.906	2.952	2.993	3.030	3.063	3.094	3.122	3.148
36	2.334	2.504	2.622	2.711	2.784	2.844	2.896	2.941	2.982	3.018	3.051	3.081	3.109	3.135
38	2.328	2.498	2.614	2.703	2.775	2.835	2.887	2.932	2.972	3.008	3.040	3.070	3.098	3.123
40	2.323	2.492	2.608	2.696	2.768	2.827	2.878	2.923	2.963	2.998	3.031	3.060	3.088	3.113
42	2.319	2.487	2.602	2.690	2.761	2.820	2.871	2.915	2.954	2.990	3.022	3.051	3.079	3.104
44	2.315	2.482	2.597	2.684	2.755	2.813	2.864	2.908	2.947	2.982	3.014	3.043	3.070	3.095
46	2.312	2.478	2.592	2.679	2.749	2.807	2.858	2.901	2.940	2.975	3.007	3.036	3.063	3.088
48	2.309	2.474	2.588	2.674	2.744	2.802	2.852	2.895	2.934	2.969	3.001	3.029	3.056	3.081
50	2.306	2.470	2.584	2.670	2.739	2.797	2.847	2.890	2.929	2.963	2.995	3.023	3.050	3.074
55	2.299	2.463	2.575	2.660	2.729	2.786	2.835	2.878	2.916	2.951	2.982	3.010	3.036	3.060
60	2.294	2.456	2.568	2.653	2.721	2.777	2.826	2.869	2.906	2.940	2.971	2.999	3.025	3.049
65	2.289	2.451	2.562	2.646	2.714	2.770	2.818	2.860	2.898	2.931	2.962	2.990	3.016	3.039
70	2.285	2.446	2.557	2.640	2.707	2.764	2.812	2.853	2.891	2.924	2.954	2.982	3.008	3.031
75	2.282	2.442	2.552	2.635	2.702	2.758	2.806	2.847	2.884	2.918	2.948	2.975	3.001	3.024
80	2.279	2.439	2.548	2.631	2.698	2.753	2.801	2.842	2.879	2.912	2.942	2.969	2.995	3.018
85	2.277	2.436	2.545	2.627	2.694	2.749	2.796	2.838	2.874	2.907	2.937	2.964	2.989	3.012
90	2.274	2.433	2.542	2.624	2.690	2.745	2.792	2.833	2.870	2.903	2.932	2.959	2.984	3.008
95	2.272	2.431	2.539	2.621	2.687	2.742	2.789	2.830	2.866	2.899	2.928	2.955	2.980	3.003
100	2.271	2.428	2.537	2.618	2.684	2.739	2.786	2.827	2.863	2.895	2.925	2.952	2.976	2.999
125	2.264	2.420	2.527	2.608	2.673	2.727	2.774	2.814	2.850	2.882	2.911	2.938	2.962	2.985
150	2.259	2.415	2.521	2.602	2.666	2.720	2.766	2.806	2.841	2.873	2.902	2.928	2.953	2.975
175	2.256	2.411	2.517	2.597	2.661	2.715	2.760	2.800	2.835	2.867	2.896	2.922	2.946	2.968
200	2.253	2.408	2.514	2.593	2.657	2.711	2.756	2.796	2.831	2.862	2.891	2.917	2.941	2.963
250	2.250	2.404	2.509	2.588	2.652	2.705	2.750	2.790	2.825	2.856	2.884	2.910	2.934	2.956
300	2.248	2.401	2.506	2.585	2.649	2.701	2.746	2.786	2.820	2.852	2.880	2.906	2.929	2.951
350	2.246	2.399	2.504	2.583	2.646	2.699	2.744	2.783	2.817	2.849	2.877	2.902	2.926	2.948
400	2.245	2.398	2.502	2.581	2.644	2.697	2.741	2.781	2.815	2.846	2.874	2.900	2.924	2.945
450	2.244	2.397	2.501	2.580	2.643	2.695	2.740	2.779	2.813	2.844	2.872	2.898	2.922	2.943
500	2.243	2.396	2.500	2.579	2.641	2.694	2.739	2.778	2.812	2.843	2.871	2.897	2.920	2.942
1000	2.240	2.392	2.495	2.574	2.636	2.688	2.733	2.772	2.806	2.837	2.864	2.890	2.913	2.935
∞	2.236	2.388	2.491	2.569	2.631	2.683	2.727	2.766	2.800	2.830	2.858	2.883	2.906	2.928

Critical values of *t* according to Dunn-Šidák's criterion for one-tailed 1 % tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	9.912	12.17	14.06	15.73	17.24	18.63	19.92	21.13	22.28	23.37	24.41	25.41	26.37	27.29
3	5.836	6.733	7.444	8.041	8.563	9.028	9.451	9.839	10.20	10.54	10.85	11.15	11.44	11.71
4	4.601	5.162	5.592	5.945	6.247	6.513	6.750	6.966	7.165	7.348	7.520	7.680	7.832	7.975
5	4.030	4.452	4.769	5.026	5.242	5.431	5.599	5.750	5.887	6.014	6.132	6.242	6.345	6.443
6	3.705	4.055	4.313	4.520	4.694	4.844	4.976	5.095	5.203	5.302	5.393	5.478	5.558	5.633
7	3.498	3.803	4.026	4.204	4.352	4.479	4.591	4.691	4.781	4.864	4.940	5.011	5.077	5.139
8	3.354	3.630	3.830	3.988	4.119	4.232	4.330	4.418	4.497	4.570	4.636	4.698	4.755	4.809
9	3.248	3.503	3.687	3.832	3.951	4.054	4.143	4.222	4.294	4.359	4.419	4.474	4.526	4.574
10	3.168	3.407	3.579	3.714	3.825	3.919	4.002	4.075	4.141	4.201	4.256	4.306	4.354	4.398
11	3.104	3.332	3.494	3.621	3.726	3.815	3.892	3.960	4.022	4.078	4.129	4.177	4.220	4.262
12	3.053	3.271	3.426	3.547	3.647	3.731	3.804	3.869	3.927	3.980	4.028	4.073	4.114	4.153
13	3.011	3.221	3.371	3.487	3.582	3.662	3.732	3.794	3.850	3.900	3.946	3.988	4.028	4.064
14	2.976	3.179	3.324	3.436	3.527	3.605	3.672	3.732	3.785	3.833	3.877	3.918	3.956	3.991
15	2.945	3.144	3.284	3.393	3.482	3.557	3.622	3.679	3.731	3.777	3.820	3.859	3.895	3.929
16	2.920	3.113	3.250	3.356	3.442	3.515	3.578	3.634	3.684	3.729	3.770	3.808	3.843	3.876
17	2.897	3.087	3.221	3.324	3.408	3.479	3.541	3.595	3.644	3.688	3.728	3.764	3.799	3.830
18	2.877	3.064	3.195	3.296	3.378	3.448	3.508	3.561	3.608	3.651	3.690	3.726	3.759	3.790
19	2.860	3.043	3.172	3.271	3.352	3.420	3.479	3.531	3.577	3.619	3.657	3.693	3.725	3.755
20	2.844	3.025	3.152	3.249	3.329	3.396	3.454	3.504	3.550	3.591	3.628	3.663	3.695	3.724
22	2.818	2.994	3.117	3.212	3.289	3.354	3.410	3.459	3.503	3.543	3.579	3.612	3.643	3.671
24	2.796	2.968	3.089	3.182	3.257	3.320	3.374	3.422	3.465	3.503	3.539	3.571	3.601	3.628
26	2.778	2.947	3.065	3.156	3.230	3.291	3.345	3.391	3.433	3.471	3.505	3.537	3.566	3.593
28	2.762	2.929	3.045	3.135	3.207	3.267	3.320	3.366	3.406	3.443	3.477	3.508	3.536	3.562
30	2.749	2.914	3.028	3.116	3.187	3.247	3.298	3.343	3.383	3.420	3.453	3.483	3.511	3.537
32	2.737	2.900	3.013	3.100	3.170	3.229	3.280	3.324	3.364	3.399	3.432	3.461	3.489	3.514
34	2.727	2.889	3.000	3.086	3.155	3.213	3.263	3.307	3.346	3.381	3.413	3.443	3.470	3.495
36	2.718	2.878	2.989	3.074	3.142	3.200	3.249	3.292	3.331	3.366	3.397	3.426	3.453	3.478
38	2.711	2.869	2.979	3.063	3.131	3.188	3.236	3.279	3.317	3.352	3.383	3.412	3.438	3.463
40	2.703	2.861	2.970	3.053	3.120	3.177	3.225	3.267	3.305	3.339	3.370	3.399	3.425	3.449
42	2.697	2.853	2.962	3.044	3.111	3.167	3.215	3.257	3.294	3.328	3.359	3.387	3.413	3.437
44	2.691	2.847	2.954	3.036	3.102	3.158	3.206	3.247	3.284	3.318	3.348	3.376	3.402	3.426
46	2.686	2.840	2.947	3.029	3.095	3.150	3.197	3.239	3.276	3.309	3.339	3.366	3.392	3.416
48	2.681	2.835	2.941	3.022	3.088	3.142	3.190	3.231	3.267	3.300	3.330	3.358	3.383	3.406
50	2.677	2.830	2.936	3.016	3.081	3.136	3.183	3.223	3.260	3.293	3.322	3.350	3.375	3.398
55	2.667	2.819	2.923	3.003	3.067	3.121	3.167	3.208	3.244	3.276	3.305	3.332	3.357	3.380
60	2.659	2.809	2.913	2.992	3.056	3.109	3.155	3.195	3.230	3.262	3.291	3.318	3.342	3.365
65	2.653	2.802	2.905	2.983	3.046	3.099	3.144	3.184	3.219	3.251	3.279	3.306	3.330	3.352
70	2.647	2.795	2.897	2.975	3.038	3.090	3.135	3.174	3.209	3.241	3.269	3.295	3.319	3.342
75	2.642	2.789	2.891	2.968	3.031	3.083	3.127	3.166	3.201	3.232	3.260	3.286	3.310	3.332
80	2.638	2.785	2.886	2.963	3.024	3.076	3.121	3.159	3.194	3.225	3.253	3.279	3.302	3.324
85	2.634	2.780	2.881	2.957	3.019	3.070	3.115	3.153	3.187	3.218	3.246	3.272	3.295	3.317
90	2.631	2.776	2.877	2.953	3.014	3.065	3.109	3.148	3.182	3.212	3.240	3.266	3.289	3.311
95	2.628	2.773	2.873	2.949	3.010	3.061	3.105	3.143	3.177	3.207	3.235	3.260	3.284	3.305
100	2.625	2.770	2.869	2.945	3.006	3.057	3.100	3.138	3.172	3.203	3.230	3.255	3.279	3.300
125	2.615	2.758	2.856	2.931	2.991	3.041	3.084	3.122	3.155	3.185	3.212	3.237	3.260	3.281
150	2.608	2.750	2.848	2.922	2.982	3.031	3.074	3.111	3.144	3.174	3.201	3.225	3.248	3.269
175	2.603	2.745	2.842	2.915	2.975	3.024	3.066	3.103	3.136	3.165	3.192	3.217	3.239	3.260
200	2.600	2.741	2.837	2.911	2.970	3.019	3.061	3.097	3.130	3.159	3.186	3.210	3.233	3.253
250	2.595	2.735	2.831	2.904	2.962	3.011	3.053	3.089	3.122	3.151	3.177	3.201	3.223	3.244
300	2.591	2.731	2.827	2.899	2.958	3.006	3.048	3.084	3.116	3.145	3.171	3.195	3.217	3.238
350	2.589	2.728	2.824	2.896	2.954	3.003	3.044	3.080	3.112	3.141	3.167	3.191	3.213	3.233
400	2.587	2.726	2.821	2.894	2.952	3.000	3.041	3.077	3.109	3.138	3.164	3.188	3.210	3.230
450	2.586	2.725	2.820	2.892	2.950	2.998	3.039	3.075	3.107	3.136	3.162	3.185	3.207	3.228
500	2.585	2.723	2.818	2.890	2.948	2.996	3.037	3.073	3.105	3.134	3.160	3.184	3.205	3.226
1000	2.580	2.718	2.812	2.884	2.941	2.989	3.030	3.065	3.097	3.125	3.151	3.175	3.196	3.216
∞	2.575	2.712	2.806	2.877	2.934	2.981	3.022	3.057	3.089	3.117	3.143	3.166	3.187	3.207

Critical values of *t* according to Dunn-Šidák's criterion for two-tailed 1 % tests (table 2)

ν	nc=2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	14.07	17.25	19.92	22.28	24.41	26.37	28.20	29.91	31.53	33.07	34.54	35.95	37.31	38.62
3	7.447	8.565	9.453	10.20	10.85	11.44	11.97	12.45	12.90	13.33	13.72	14.10	14.46	14.80
4	5.594	6.248	6.751	7.166	7.520	7.832	8.112	8.367	8.600	8.817	9.019	9.208	9.387	9.556
5	4.771	5.243	5.599	5.888	6.133	6.346	6.535	6.706	6.862	7.006	7.139	7.264	7.381	7.491
6	4.315	4.695	4.977	5.203	5.394	5.559	5.704	5.835	5.954	6.063	6.164	6.258	6.345	6.428
7	4.027	4.353	4.591	4.782	4.941	5.078	5.198	5.306	5.404	5.493	5.576	5.652	5.724	5.791
8	3.831	4.120	4.331	4.498	4.637	4.756	4.860	4.953	5.038	5.115	5.185	5.251	5.312	5.370
9	3.688	3.952	4.143	4.294	4.419	4.526	4.619	4.703	4.778	4.846	4.909	4.967	5.021	5.072
10	3.580	3.825	4.002	4.141	4.256	4.354	4.439	4.515	4.584	4.646	4.703	4.756	4.806	4.852
11	3.495	3.726	3.892	4.022	4.129	4.221	4.300	4.371	4.434	4.492	4.545	4.594	4.639	4.682
12	3.427	3.647	3.804	3.927	4.029	4.114	4.189	4.256	4.315	4.369	4.419	4.465	4.507	4.547
13	3.371	3.582	3.733	3.850	3.946	4.028	4.099	4.162	4.218	4.270	4.317	4.360	4.400	4.438
14	3.324	3.528	3.673	3.785	3.878	3.956	4.024	4.084	4.138	4.187	4.232	4.273	4.311	4.347
15	3.285	3.482	3.622	3.731	3.820	3.895	3.961	4.019	4.071	4.118	4.160	4.200	4.237	4.271
16	3.251	3.443	3.579	3.684	3.771	3.844	3.907	3.963	4.013	4.058	4.100	4.138	4.173	4.206
17	3.221	3.409	3.541	3.644	3.728	3.799	3.860	3.914	3.963	4.007	4.047	4.084	4.118	4.150
18	3.195	3.379	3.508	3.609	3.691	3.760	3.820	3.872	3.920	3.962	4.001	4.037	4.071	4.102
19	3.173	3.353	3.479	3.578	3.658	3.725	3.784	3.835	3.881	3.923	3.961	3.996	4.029	4.059
20	3.152	3.329	3.454	3.550	3.629	3.695	3.752	3.802	3.848	3.888	3.926	3.960	3.992	4.021
22	3.118	3.289	3.410	3.503	3.579	3.643	3.698	3.747	3.790	3.830	3.865	3.898	3.929	3.957
24	3.089	3.257	3.375	3.465	3.539	3.601	3.654	3.702	3.744	3.782	3.816	3.848	3.878	3.905
26	3.066	3.230	3.345	3.433	3.505	3.566	3.618	3.664	3.705	3.742	3.776	3.807	3.835	3.862
28	3.046	3.207	3.320	3.407	3.477	3.536	3.587	3.632	3.672	3.708	3.741	3.771	3.799	3.825
30	3.029	3.188	3.298	3.384	3.453	3.511	3.561	3.605	3.644	3.680	3.712	3.742	3.769	3.794
32	3.014	3.171	3.280	3.364	3.432	3.489	3.538	3.582	3.620	3.655	3.687	3.716	3.743	3.768
34	3.001	3.156	3.264	3.346	3.414	3.470	3.518	3.561	3.599	3.633	3.665	3.693	3.720	3.744
36	2.990	3.143	3.249	3.331	3.397	3.453	3.501	3.543	3.581	3.614	3.645	3.673	3.699	3.724
38	2.979	3.131	3.237	3.318	3.383	3.438	3.485	3.527	3.564	3.597	3.628	3.656	3.681	3.705
40	2.970	3.121	3.225	3.305	3.370	3.425	3.472	3.513	3.549	3.582	3.612	3.640	3.665	3.689
42	2.962	3.111	3.215	3.295	3.359	3.413	3.459	3.500	3.536	3.569	3.599	3.626	3.651	3.674
44	2.955	3.103	3.206	3.285	3.348	3.402	3.448	3.488	3.524	3.557	3.586	3.613	3.638	3.661
46	2.948	3.095	3.197	3.276	3.339	3.392	3.438	3.478	3.513	3.546	3.575	3.601	3.626	3.649
48	2.942	3.088	3.190	3.268	3.330	3.383	3.428	3.468	3.504	3.535	3.564	3.591	3.615	3.638
50	2.936	3.082	3.183	3.260	3.322	3.375	3.420	3.459	3.495	3.526	3.555	3.581	3.606	3.628
55	2.924	3.068	3.167	3.244	3.305	3.357	3.401	3.440	3.475	3.506	3.534	3.560	3.584	3.606
60	2.914	3.056	3.155	3.230	3.291	3.342	3.386	3.425	3.459	3.489	3.517	3.543	3.567	3.589
65	2.905	3.046	3.144	3.219	3.279	3.330	3.373	3.411	3.445	3.476	3.503	3.529	3.552	3.574
70	2.898	3.038	3.135	3.209	3.269	3.319	3.362	3.400	3.434	3.464	3.491	3.516	3.539	3.561
75	2.892	3.031	3.128	3.201	3.261	3.310	3.353	3.390	3.424	3.454	3.481	3.506	3.528	3.550
80	2.886	3.025	3.121	3.194	3.253	3.302	3.345	3.382	3.415	3.445	3.472	3.496	3.519	3.540
85	2.881	3.019	3.115	3.188	3.246	3.295	3.338	3.374	3.407	3.437	3.464	3.488	3.511	3.532
90	2.877	3.014	3.110	3.182	3.240	3.289	3.331	3.368	3.401	3.430	3.457	3.481	3.503	3.524
95	2.873	3.010	3.105	3.177	3.235	3.284	3.326	3.362	3.395	3.424	3.450	3.475	3.497	3.518
100	2.870	3.006	3.101	3.172	3.230	3.279	3.320	3.357	3.389	3.418	3.445	3.469	3.491	3.512
125	2.857	2.992	3.085	3.155	3.213	3.260	3.301	3.337	3.369	3.397	3.423	3.447	3.469	3.489
150	2.848	2.982	3.074	3.144	3.201	3.248	3.288	3.324	3.355	3.383	3.409	3.433	3.454	3.474
175	2.842	2.975	3.067	3.136	3.192	3.239	3.279	3.314	3.346	3.374	3.399	3.422	3.444	3.464
200	2.838	2.970	3.061	3.130	3.186	3.233	3.273	3.307	3.339	3.366	3.392	3.415	3.436	3.456
250	2.831	2.963	3.053	3.122	3.177	3.224	3.263	3.298	3.329	3.356	3.381	3.404	3.425	3.445
300	2.827	2.958	3.048	3.116	3.171	3.217	3.257	3.291	3.322	3.349	3.374	3.397	3.418	3.438
350	2.824	2.954	3.044	3.112	3.167	3.213	3.252	3.287	3.317	3.345	3.369	3.392	3.413	3.432
400	2.822	2.952	3.041	3.110	3.164	3.210	3.249	3.283	3.314	3.341	3.366	3.388	3.409	3.429
450	2.820	2.950	3.039	3.107	3.162	3.207	3.247	3.281	3.311	3.338	3.363	3.385	3.406	3.426
500	2.819	2.948	3.038	3.105	3.160	3.205	3.244	3.279	3.309	3.336	3.361	3.383	3.404	3.423
000	2.812	2.941	3.030	3.097	3.151	3.196	3.235	3.269	3.299	3.326	3.350	3.373	3.393	3.412
∞	2.806	2.934	3.022	3.089	3.143	3.188	3.226	3.260	3.289	3.316	3.340	3.362	3.383	3.402

Minimum sample size (n) such that a given correlation coefficient de correlation r be significant (*i.e.* significantly different from zero) at various thresholds (table 3)

r	P = 0.95	0.975	0.99	0.995	r	P = 0.95	0.975	0.99	0.995
		minimum n					minimum n		
.99	3	4	4	5	.49	13	17	23	27
.98	4	4	5	5	.48	13	18	24	28
.97	4	4	5	5	.47	14	18	25	30
.96	4	4	5	5	.46	14	19	26	31
.95	4	5	5	6	.45	15	20	27	32
.94	4	5	5	6	.44	16	21	28	34
.93	4	5	6	6	.43	16	22	29	35
.92	4	5	6	6	.42	17	23	31	37
.91	4	5	6	7	.41	18	24	32	39
.90	5	5	6	7	.40	19	25	34	41
.89	5	5	6	7	.39	19	26	36	43
.88	5	5	7	7	.38	20	28	38	46
.87	5	6	7	8	.37	21	29	40	48
.86	5	6	7	8	.36	22	31	42	51
.85	5	6	7	8	.35	24	32	44	54
.84	5	6	7	8	.34	25	34	47	57
.83	5	6	8	9	.33	26	36	50	61
.82	5	6	8	9	.32	28	39	53	64
.81	5	7	8	9	.31	30	41	57	69
.80	6	7	8	9	.30	32	44	60	73
.79	6	7	8	10	.29	34	47	65	79
.78	6	7	9	10	.28	36	50	69	84
.77	6	7	9	10	.27	39	54	74	91
.76	6	7	9	11	.26	42	58	80	98
.75	6	8	9	11	.25	45	63	87	106
.74	6	8	10	11	.24	49	68	94	115
.73	6	8	10	12	.23	53	74	103	125
.72	7	8	10	12	.22	58	80	112	137
.71	7	8	11	12	.21	63	88	123	150
.70	7	9	11	13	.20	69	97	136	166
.69	7	9	11	13	.19	77	107	150	183
.68	7	9	12	14	.18	85	120	167	204
.67	7	9	12	14	.17	95	134	188	229
.66	8	10	12	15	.16	107	151	212	259
.65	8	10	13	15	.15	122	172	241	295
.64	8	10	13	16	.14	140	197	276	338
.63	8	11	14	16	.13	162	228	321	392
.62	9	11	14	17	.12	190	268	376	460
.61	9	11	15	17	.11	225	319	448	548
.60	9	12	15	18	.10	272	385	541	663
.59	9	12	16	18	.09	336	475	668	819
.58	10	12	16	19	.08	424	601	846	1036
.57	10	13	17	20	.07	554	785	1105	1354
.56	10	13	17	21	.06	753	1068	1504	1843
.55	10	14	18	21	.05	1084	1538	2165	2654
.54	11	14	19	22	.04	1693	2402	3383	4146
.53	11	15	19	23	.03	3008	4269	6014	7372
.52	12	15	20	24	.02	6766	9605	13530	16587
.51	12	16	21	25	.01	27057	38416	54119	66349
.50	12	16	22	26					

Reading off the tables

Table 1 furnishes values of Student's t statistic corresponding to probability integrals $P = 0.75$, 0.95, 0.975, 0.99 and 0.995 for $df\,(\nu)$ from 1 to 99, plus some others. For $P < 0.50$, one may use relation: $t_{[P]} = -t_{[1-P]}$ due to the symmetry of the distribution about zero.

Table 2 also gives t values for $P = 0.95$, 0.975, 0.99 and 0.995 following Dunn-Šidák protection criterion: the other parameters are degrees of freedom (ν, or df) and number of planned comparisons (nc).

Table 3 is a significance table for the correlation coefficient r. For each significance level α ($P = 1-\alpha = 0.95$, 0.975, 0.99, 0.995), it indicates the minimum sample size n for which any r value departs significantly from zero. This table is based on a logical inversion of the t test appropriate for this situation.

Illustration 1. Find $t_{14[.95]}$. In table 1, at line $\nu = 14$ and under $P = 0.95$, we read $t = 1.761$. Likewise, for $t_{9[.05]}$, we use $-t_{9[.95]} = -1.833$.

Illutration 2. With 72 *df*, what are the (two) critical values of Student's t for a two-tailed 1 % test? The total extreme probability of 1 %, assigned equally to the right and left, leads to probability grades 0.005 (on the left) and 0.995 (on the right). In table 1, under $\nu = 72$ and $P = 0.995$, we obtain 2.646, so the searched-for values are -2.646 and 2.646.

Illustration 3. Find $t_{DS}(20[.95];4)$, *i.e.* the appropriate t value under the Dunn-Šidák criterion, for $\nu = 20$ *df*, a one-tailed 5 % threshold and $nc = 4$ comparisons. In table 2, on the page prepared for one-tailed 5 % tests, line $\nu = 20$ and column $nc = 4$ point to $t_{DS} = 2.414$.

Illustration 4. Find $n(r{=}\pm 0.30)$ for $P = 0.975$ (or, equivalently, for a two-tailed 5 % test). In table 3, at line $r = 0.30$ and under $P = 0.975$, we read $n = 44$. Hence, any coefficient $r \geq 0.30$ or $r \leq -0.30$ differs significantly from zero (*i.e.* is significant) at the 5 % (bilateral) level if $n \geq 44$.

Full examples

Example 1. With a sample of 40 children in first grade of primary school, we measured an average weight ($\bar{x}$) of 19.89 kg, with a s.d. of 4.74. Using a confidence coefficient

$c = 0.95$, find the interval inside which the true mean weight (μ_X) of children in this category should lie. *Solution*: The confidence interval for a mean, or average, is given by:

$$\Pr\{ \bar{x}+t_{n-1[(1-c)/2]}s/\sqrt{n} < \mu_X < \bar{x}+t_{n-1[(1+c)/2]}s/\sqrt{n} \} = c \ .$$

Here, $c = 0.95$, $n = 40$, $\bar{x} = 19.89$, $s = 4.74$. In table 1, we find $t_{39[.975]} = 2.023$, hence $t_{39[.025]} = -2.023$ by symmetry. These data allow us to calculate and state that:

$$\Pr\{ 18{,}37 < \mu_X < 21{,}41 \} \approx 0.95.$$

Example 2. Thirty-eight patients suffering a phobia to dogs were randomly assigned either to therapy 1 or 2. After four individual sessions of one hour, a questionnaire measuring the respondent's "bravery toward dogs" was administered to each patient. A summary of results follows:

Therapy 1	$n_1 = 20$	$\bar{x}_1 = 3.49$	$s_1 = 0.96$;
Therapy 2	$n_2 = 18$	$\bar{x}_2 = 2.04$	$s_2 = 0.93$.

May we conclude that one therapy is more effective than the other? *Solution*: This simple research paradigm and the associated *t*-test procedure are among the most currently encountered in statistical practice. Under appropriate conditions (*i.e.* independent samples from a unique normally distributed population, homogeneity of variance estimates), the *t* test for the difference of means between two samples is defined by:

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sqrt{\dfrac{(n_1-1)s_1^2 + (n_2-1)s_2^2}{n_1+n_2-2}\left(\dfrac{1}{n_1}+\dfrac{1}{n_2}\right)}} \ ,$$

and is distributed as Student's *t* with $n_1 + n_2 - 2$ *df*. Selecting a 5 % significance level with a two-tailed test (whence $P = 0.025$ and 0.975) and with $df = 20 + 18 - 2 = 36$, table 1 furnishes $t = \pm 2.028$. Calculating the above formula with our data, we get $t \approx 1.45 / \sqrt{(0.8948 \times 0.10556)} \approx 1.45 / 0.30733 \approx 4.718$. As the calculated *t* test exceeds the critical value, we reject the hypothesis of the equality of population means. Therefore, therapy 1 appears more effective, at least in the short term, than therapy 2.

Example 3. A researcher in experimental biology allocates 50 laboratory rats into 5 groups of 10, in a random manner. In his research protocol, he compares three experimental diets: E1, E2 and E3, and two control diets, C1 and C2. The body weight – the dependent variable in this study – is measured in grams (g) at the end of the experiment. Here is a summary of results:

Diet group	n	$\bar{x}$	s
E1	10	352.7	32.14
E2	10	318.2	29.56
E3	10	336.4	33.37
C1	10	308.5	28.61
C2	10	273.4	30.19

The researcher wishes to determine, using a 1 % significance threshold, whether each experimental diet has benefitted the rats' weight in contrast with each control diet. *Solution*: The researcher's questioning design corresponds to a set of 6 planned comparisons (and 6 one-tailed tests), each of type H_1: $\mu(E_j) > \mu(C_r)$, $j = 1,2,3$ and $r = 1,2$; the criterion and method of Dunn-Šidák are appropriate here. The t test for this situation is:

$$t = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{\hat{\sigma}^2\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}} .$$

The (error) variance estimate, $\hat{\sigma}^2$, is given here by the Within-group mean square (MS_{Within}) of a one-way analysis of variance, here $MS_{Within} = 950.06$ (calculted as a weighted average of each group's s^2, with weights equal to $n_j - 1$), with $\nu = 5 \times 9 = 45$ *df.* Table 2 furnishes, for one-tailed tests, $\alpha = 0.01$ and $nc = 6$ comparisons, the two values $t_{DS}(44[0.99];6) = 3.102$ and $t_{DS}(46[0.99];6) = 3.095$, whence, by interpolation for $\nu = 45$, $t(45) \approx 3.099$. To be significant, the t test must exceed 3.099; therefore, each difference $\bar{x}(E_j) - \bar{x}(C_r)$ must exceed $2.099 \times \sqrt{[950.06 \times (1/10 + 1/10)]} \approx 42.72$. Computations show that only differences (E1 − C1), (E1 − C2) and (E3 − C2) are significant at the prescribed level.

Example 4. The (linear) correlation between the height of 20 women and that of their consort has been calculated at $r = 0.54$. With this value, can we infer that there exists some relation between the statures of partners in human couples in the whole population, a non-zero correlation (using a 5 %, two-tailed test)? *Solution*: In table 3, under column $P = 0.975$, we find $n(0.54) = 14$, this being the minimum sample size for our correlation to be significant. As our sample is of size 20, we conclude that the obtained correlation is statistically significant, hence that the members of human couples tend moderately to be akin in stature.

Mathematical presentation

Student's t variable is basically the quotient of a random normal $N(\mu,\sigma^2)$ variable divided by an estimate s of σ having ν *df*, that estimate itself emanating from the same statistical population[1]. As the following equality shows, the t formula compensates for our ignorance of the parametric value of σ^2 :

$$t_\nu = \frac{x-\mu}{s_\nu} = \frac{(x-\mu)/\sigma}{\sqrt{\dfrac{\nu \cdot s_\nu^2}{\sigma^2}/\nu}} = \frac{z}{\sqrt{\chi_\nu^2/\nu}} .$$

We see, in addition, that the t variable can be manufactured, or generated, by dividing a standard normal r.v. (z) with the square root of a χ_ν^2 r.v., itself divided by its parameter ν.

The most popular applications of Student's t distribution concern the sampling distribution of the mean $\bar{x}$, the distribution of a difference $(\bar{x}_1-\bar{x}_2)$ of two independent or two paired means, and the significance of a correlation coefficient. The distribution was discovered by W. S. Gosset (alias *Student*), and R. A. Fisher established it mathematically. Fisher also established the kinship of Student's t with correlation coefficient r in a bivariate normal population where $\rho = 0$, and with the χ^2 and F distributions. Fisher promoted the use of t tests and other small-sample tests in fundamental and applied research.

Calculation and moments

The p.d.f. of t having ν *df* is:

$$p(t_\nu) = K_\nu(1+t^2/\nu)^{-(\nu+1)/2}$$

where:

$$K_\nu = \Gamma[(\nu+1)/2]/\{\sqrt{(\nu\pi)}\Gamma(\nu/2)\} .$$

[1] This last clause is not mandatory. It is stated here as it evokes the habitual context in which most t tests are applied, a context wherein some sample mean is divided by the square root of the same sample's variance estimate. The reader is reminded that, in samples from a normal distribution, the sample estimates $\bar{X}$ and s^2 are independently distributed.

The distribution has one mode, is symmetrical about zero and is somewhat wider than the normal density, tending progressively to it as $\nu \to \infty$. $\Gamma(x)$ indicates the *Gamma* function.

Integration of the above p.d.f. is not simple, that being the case for most statistical distributions. The binomial expansion of $(1 + t^2/\nu)^{-r}$, followed by term-by-term integration on t, results in the converging power series:

$$P(t_\nu) = \tfrac{1}{2} + K_\nu t \sum_{i=0}^{\infty} \frac{(-1)^i}{i!\,(2i+1)} \left(\frac{\nu+1}{2}\right)^{(i)} \left(\frac{t^2}{\nu}\right)^i ,$$

where $x^{(i)}$ denotes the ascending factorial function. Gosset, using the change of variables $\theta = \tan^{-1}(t/\sqrt{\nu})$, developped the following series, one for even ν:

$$P(t_\nu) = \frac{1}{2} + \frac{\sin\theta}{2}\left\{1 + \sum_{i=1}^{\frac{1}{2}(\nu-2)} (\cos\theta)^{2i} \prod_{j=1}^{i} \frac{2j-1}{2j}\right\},$$

the other for odd ν:

$$P(t_\nu) = \frac{1}{2} + \frac{1}{\pi}\left\{\theta + \sin\theta\cos\theta\left[1 + \sum_{i=1}^{\frac{1}{2}(\nu-3)} (\cos\theta)^{2i} \prod_{j=1}^{i} \frac{2j}{2j+1}\right]\right\};$$

both series become cluttered for large ν values. For such cases, the preceding binomial expansion seems preferable, or else, the approximation proposed by Fisher:

$$P(t_\nu) \approx \Phi(t) + \varphi(t)\cdot t\,[A/(4\nu) + B/(96\nu^2) + C/(384\nu^3) + D/(92160\nu^4) + \cdots],$$

where Φ and φ are respectively the d.f. and the p.d.f. of the standard normal distribution, and $A = t^2+1$, $B = 3t^6 - 7t^4 - 5t^2 - 3$, $C = t^{10} - 11t^8 + 14t^6 + 6t^4 - 3t^2 - 15$, $D = 15t^{14} - 375t^{12} + 2225t^{10} - 2141t^8 - 939t^6 - 213t^4 + 915t^2 + 945$. Many other computational algorithms are available.

The t variable with $\nu = 1$ has p.d.f. $1/[\pi(1+t^2)]$ and coincides with a standard Cauchy variable. This variable, which also corresponds to the quotient of two standard normal r.v.'s ($= z_1/z_2$), has no expectation, no variance, nor moments of any order. The d.f. is simply $P(t_1) = \tfrac{1}{2} + (\tan^{-1}t)/\pi$, whence, by inversion, we may obtain the percentage point $t_1[P] = \tan\{\pi(P-\tfrac{1}{2})\}$.

For $\nu = 2$, through Gosset's trigonometric series, we have $P(t_2) = \tfrac{1}{2} + \tfrac{1}{2}t/\sqrt{(t^2+2)}$, and $t_2[P] = (2P-1)/\sqrt{[2P(1-P)]}$.

Note, finally, the kinship of t_ν and $F_{1,\nu}$, from which we have, for instance, the correspondance:

$$t_{\nu[P]} = +\sqrt{F_{1,\nu[2P-1]}} \quad \{ P > ½ \} ,$$

and also $t_\nu \rightarrow z \sim N(0,1)$ as $\nu \rightarrow \infty$, reflecting the trend of t toward the normal law. These diverse relations may help to interpolate in a table of percentage points of t, when necessary.

Moments. Thanks to the symmetry of its p.d.f. centered at zero, the expectation, median and mode of t_ν are all zero, as is the skewness index, γ_1. The variance (σ^2) equals $\nu/(\nu-2)$, and the kurtosis index (γ_2) is $6/(\nu-4)$, reflecting a leptokurtic, somewhat narrow, center. Note that the moment of order r ($r = 1$ for μ, 2 for σ^2, etc.) is defined only if $\nu > r$.

With only the first two moments of t_ν, we can approximate a t centile using the rough equivalence: $t_{\nu[P]} \approx z_{[P]}\sigma_\nu$, $z_{[P]}$ being the standard normal P-centile, and $\sigma_\nu = \sqrt{[\nu/(\nu-2)]}$. The literature offers a better approximation:

$$t_{\nu[P]} \approx z_{[P]}\left(1 + \frac{z_{[P]}^2 + 1}{4\nu - 4}\right) ,$$

which renders centile values with a ±0.01 precision as soon as $\nu \geq 7$ for $P \leq 0.975$, and $\nu \geq 21$ for $P \leq 0.995$.

The Dunn-Šidák criterion (see table 2)

The t distribution is also useful in the realm of analysis of variance (ANOVA), for doing multiple comparisons of means associated with a factor or dimension of ANOVA. In some cases, a few comparisons of type "Mean *vs.* Mean" are planned (before obtaining the data or independently of it). A standard statistical practice consists in establishing a global error rate α for all nc comparisons or tests, and in performing each test using an effective $\alpha_C = \alpha/nc$ significance level: this is the so-called Bonferroni criterion. Now, according to a theorem by Z. Šidák:

$$\alpha_{(\text{global})} \leq 1 - (1-\alpha_C)^{nc} ,$$

whether the nc comparisons be statistically independent or not. O. J. Dunn then proposes, for each comparison, to calculate a t test and apply, for the significance criterion, an individual significance level α_C obtained with:

$$\alpha_C = 1 - \sqrt[nc]{(1-\alpha)} ,$$

this being slightly more powerful than Bonferroni's (α/nc) value. Table 2 was built to meet the requirements of such a procedure, with parameters ν, α and nc: the values given are simply $t_{\nu[1-\alpha_c]}$ or $t_{\nu[1-\alpha_c/2]}$, for one-tailed or two-tailed tests respectively.

The significance or r (see table 3)

The distribution of the correlation coefficient r for samples from a bivariate normal population having correlation $\rho = 0$ has been discovered by Fisher, who linked it with Student's t through the transformation:

$$\frac{r}{\sqrt{\dfrac{1-r^2}{n-2}}} \rightarrow t_{n-2}$$

This direct correspondance between Student's t and correlation coefficient r allowed us to set up an exhaustive significance table of r, in table 3. The table indicates the sample size n (= ν+2) sufficient for a given r value or a larger one to be significant (*i.e.* significantly different from zero) at the prescribed level. Note the approximate relation: $n_{[P]} \approx t^2_{n[P]}/r^2$; linear interpolation on r and $1/\sqrt{n}$ can also be used to determine the sufficient size n for intermediate r values.

Generation of pseudo random variates

The following program outline permits the generation of r.v.'s from Student's $t(\nu)$ distribution using a function "UNIF" that produces serially independent uniform $U(0,1)$ r.v.'s. Some particular cases are given in Remark 2.

Preparation : Let $n \equiv \nu$ (the degrees of freedom, *df*)

$C = -2/n$

Production :

Repeat $t \leftarrow 2\times\text{UNIF}-1$; $u \leftarrow 2\times\text{UNIF}-1$; $r \leftarrow t^2 + u^2$
Until $r < 1$;
Return $t \times \sqrt{[n \times (r^C - 1)/r]} \rightarrow x$.

Remarks :

1. Standard temporal cost : 9.9 × t(UNIF)

2. The above method is attributed to R. W. Bailey (1994, in Gentle 1998) and is valid for any ν (= n) value. The generation of one x variate requires an expected number of $8/\pi \approx 2.55$ calls to the UNIF generator. When $\nu = 1$ or 2, one may also exploit the simple inversion formulae given earlier, while substituting $u \sim$ UNIF instead of the required P value.

F distribution

✓ Graphical representations

✓ Percentiles 95, 97.5, 99 and 99.5 of the *F* distribution

✓ Reading off the table

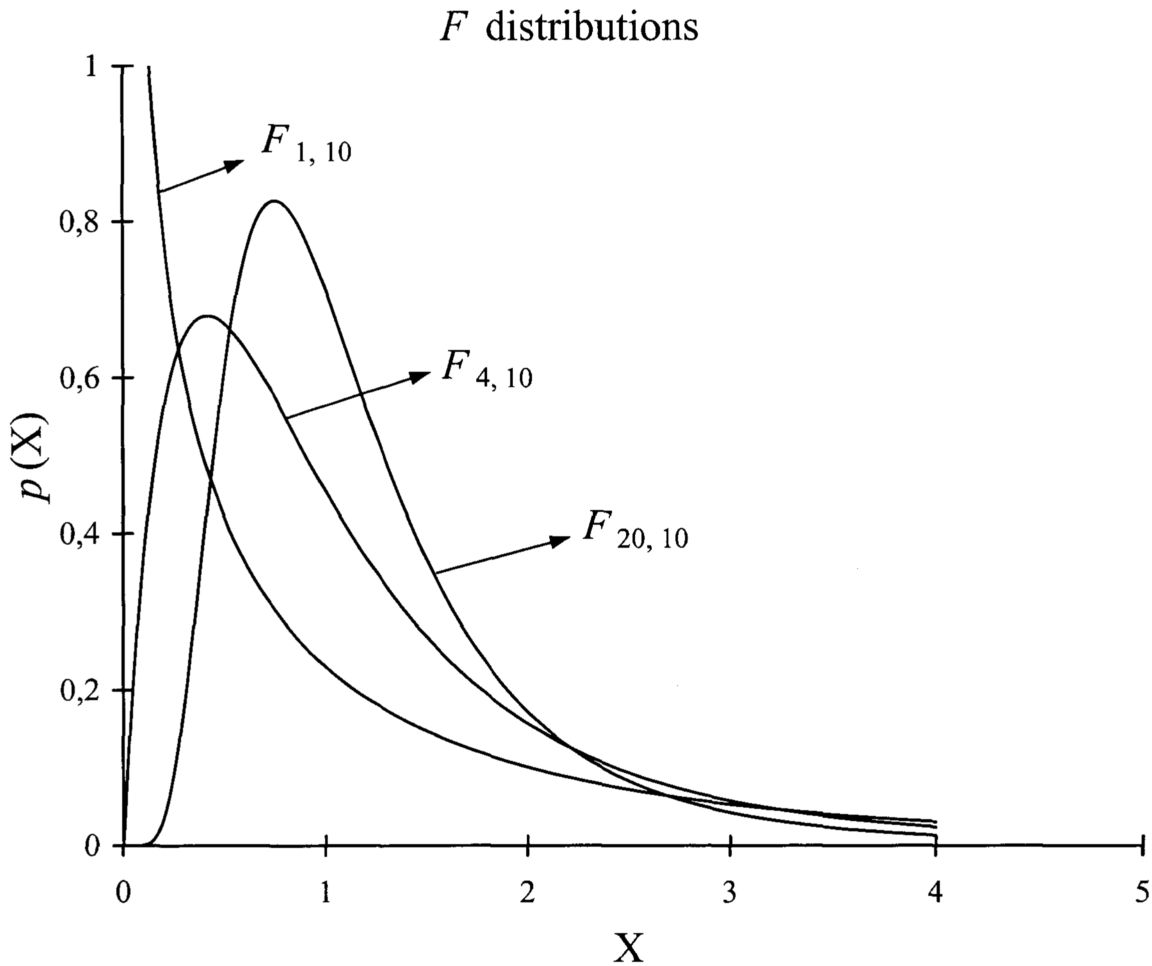
✓ Full examples

✓ Mathematical presentation

Calculation and moments

Relationship between *F* and binomial distributions

Generation of pseudo random variates


F distributions
1
0,8
0,6
0,4
0,2
0
p (X)
F 1, 10
F 4, 10
F 20, 10
0
1
2
3
4
5
X

Percentile 95 of the *F* distribution

$\nu_2 \setminus \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.0	243.9
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.40	19.41
3	10.13	9.552	9.277	9.117	9.013	8.941	8.887	8.845	8.812	8.786	8.763	8.745
4	7.709	6.944	6.591	6.388	6.256	6.163	6.094	6.041	5.999	5.964	5.936	5.912
5	6.608	5.786	5.409	5.192	5.050	4.950	4.876	4.818	4.772	4.735	4.704	4.678
6	5.987	5.143	4.757	4.534	4.387	4.284	4.207	4.147	4.099	4.060	4.027	4.000
7	5.591	4.737	4.347	4.120	3.972	3.866	3.787	3.726	3.677	3.637	3.603	3.575
8	5.318	4.459	4.066	3.838	3.687	3.581	3.500	3.438	3.388	3.347	3.313	3.284
9	5.117	4.256	3.863	3.633	3.482	3.374	3.293	3.230	3.179	3.137	3.102	3.073
10	4.965	4.103	3.708	3.478	3.326	3.217	3.135	3.072	3.020	2.978	2.943	2.913
11	4.844	3.982	3.587	3.357	3.204	3.095	3.012	2.948	2.896	2.854	2.818	2.788
12	4.747	3.885	3.490	3.259	3.106	2.996	2.913	2.849	2.796	2.753	2.717	2.687
13	4.667	3.806	3.411	3.179	3.026	2.915	2.832	2.767	2.714	2.671	2.635	2.604
14	4.600	3.739	3.344	3.112	2.958	2.848	2.764	2.699	2.646	2.602	2.565	2.534
15	4.543	3.682	3.287	3.056	2.901	2.790	2.707	2.641	2.588	2.544	2.507	2.475
16	4.494	3.634	3.239	3.007	2.852	2.741	2.657	2.591	2.538	2.494	2.456	2.425
17	4.451	3.592	3.197	2.965	2.810	2.699	2.614	2.548	2.494	2.450	2.413	2.381
18	4.414	3.555	3.160	2.928	2.773	2.661	2.577	2.510	2.456	2.412	2.374	2.342
19	4.381	3.522	3.127	2.895	2.740	2.628	2.544	2.477	2.423	2.378	2.340	2.308
20	4.351	3.493	3.098	2.866	2.711	2.599	2.514	2.447	2.393	2.348	2.310	2.278
22	4.301	3.443	3.049	2.817	2.661	2.549	2.464	2.397	2.342	2.297	2.259	2.226
24	4.260	3.403	3.009	2.776	2.621	2.508	2.423	2.355	2.300	2.255	2.216	2.183
26	4.225	3.369	2.975	2.743	2.587	2.474	2.388	2.321	2.265	2.220	2.181	2.148
28	4.196	3.340	2.947	2.714	2.558	2.445	2.359	2.291	2.236	2.190	2.151	2.118
30	4.171	3.316	2.922	2.690	2.534	2.421	2.334	2.266	2.211	2.165	2.126	2.092
32	4.149	3.295	2.901	2.669	2.512	2.399	2.313	2.244	2.189	2.142	2.103	2.070
34	4.130	3.276	2.883	2.650	2.494	2.380	2.294	2.225	2.170	2.123	2.084	2.050
36	4.113	3.259	2.866	2.634	2.477	2.364	2.277	2.209	2.153	2.106	2.067	2.033
38	4.098	3.245	2.852	2.619	2.463	2.349	2.262	2.194	2.138	2.091	2.051	2.017
40	4.085	3.232	2.839	2.606	2.449	2.336	2.249	2.180	2.124	2.077	2.038	2.003
42	4.073	3.220	2.827	2.594	2.438	2.324	2.237	2.168	2.112	2.065	2.025	1.991
44	4.062	3.209	2.816	2.584	2.427	2.313	2.226	2.157	2.101	2.054	2.014	1.980
46	4.052	3.200	2.807	2.574	2.417	2.304	2.216	2.147	2.091	2.044	2.004	1.970
48	4.043	3.191	2.798	2.565	2.409	2.295	2.207	2.138	2.082	2.035	1.995	1.960
50	4.034	3.183	2.790	2.557	2.400	2.286	2.199	2.130	2.073	2.026	1.986	1.952
55	4.016	3.165	2.773	2.540	2.383	2.269	2.181	2.112	2.055	2.008	1.968	1.933
60	4.001	3.150	2.758	2.525	2.368	2.254	2.167	2.097	2.040	1.993	1.952	1.917
65	3.989	3.138	2.746	2.513	2.356	2.242	2.154	2.084	2.027	1.980	1.939	1.904
70	3.978	3.128	2.736	2.503	2.346	2.231	2.143	2.074	2.017	1.969	1.928	1.893
75	3.968	3.119	2.727	2.494	2.337	2.222	2.134	2.064	2.007	1.959	1.919	1.884
80	3.960	3.111	2.719	2.486	2.329	2.214	2.126	2.056	1.999	1.951	1.910	1.875
85	3.953	3.104	2.712	2.479	2.322	2.207	2.119	2.049	1.992	1.944	1.903	1.868
90	3.947	3.098	2.706	2.473	2.316	2.201	2.113	2.043	1.986	1.938	1.897	1.861
95	3.941	3.092	2.700	2.468	2.310	2.196	2.107	2.037	1.980	1.932	1.891	1.856
100	3.936	3.087	2.696	2.463	2.305	2.191	2.103	2.032	1.975	1.927	1.886	1.850
125	3.917	3.069	2.677	2.444	2.287	2.172	2.084	2.013	1.956	1.907	1.866	1.830
150	3.904	3.056	2.665	2.432	2.274	2.160	2.071	2.001	1.943	1.894	1.853	1.817
175	3.895	3.048	2.656	2.423	2.266	2.151	2.062	1.992	1.934	1.885	1.844	1.808
200	3.888	3.041	2.650	2.417	2.259	2.144	2.056	1.985	1.927	1.878	1.837	1.801
250	3.879	3.032	2.641	2.408	2.250	2.135	2.046	1.976	1.917	1.869	1.827	1.791
500	3.860	3.014	2.623	2.390	2.232	2.117	2.028	1.957	1.899	1.850	1.808	1.772
∞	3.841	2.996	2.605	2.372	2.214	2.099	2.010	1.938	1.880	1.831	1.789	1.752

Percentile 95 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	244.7	245.4	245.9	246.5	247.3	248.0	249.3	250.1	251.1	251.8	253.0	254.3
2	19.42	19.42	19.43	19.43	19.44	19.45	19.46	19.46	19.47	19.48	19.49	19.50
3	8.729	8.715	8.703	8.694	8.675	8.660	8.634	8.617	8.594	8.581	8.554	8.526
4	5.891	5.873	5.858	5.844	5.821	5.803	5.769	5.746	5.717	5.699	5.664	5.628
5	4.655	4.636	4.619	4.604	4.579	4.558	4.521	4.496	4.464	4.444	4.405	4.365
6	3.976	3.956	3.938	3.922	3.896	3.874	3.835	3.808	3.774	3.754	3.712	3.669
7	3.550	3.529	3.511	3.494	3.467	3.445	3.404	3.376	3.340	3.319	3.275	3.230
8	3.259	3.237	3.218	3.202	3.173	3.150	3.108	3.079	3.043	3.021	2.975	2.928
9	3.048	3.025	3.006	2.989	2.960	2.936	2.893	2.864	2.826	2.803	2.756	2.707
10	2.887	2.865	2.845	2.828	2.798	2.774	2.730	2.700	2.661	2.637	2.588	2.538
11	2.761	2.739	2.719	2.701	2.671	2.646	2.601	2.570	2.531	2.507	2.457	2.404
12	2.660	2.637	2.617	2.599	2.568	2.544	2.498	2.466	2.426	2.401	2.350	2.296
13	2.577	2.554	2.533	2.515	2.484	2.459	2.412	2.380	2.339	2.314	2.261	2.206
14	2.507	2.484	2.463	2.445	2.413	2.388	2.341	2.308	2.266	2.241	2.187	2.131
15	2.448	2.424	2.403	2.385	2.353	2.328	2.280	2.247	2.204	2.178	2.123	2.066
16	2.397	2.373	2.352	2.333	2.302	2.276	2.227	2.194	2.151	2.124	2.068	2.010
17	2.353	2.329	2.308	2.289	2.257	2.230	2.181	2.148	2.104	2.077	2.020	1.960
18	2.314	2.290	2.269	2.250	2.217	2.191	2.141	2.107	2.063	2.035	1.978	1.917
19	2.280	2.256	2.234	2.215	2.182	2.155	2.106	2.071	2.026	1.999	1.940	1.878
20	2.250	2.225	2.203	2.184	2.151	2.124	2.074	2.039	1.994	1.966	1.907	1.843
22	2.198	2.173	2.151	2.131	2.098	2.071	2.020	1.984	1.938	1.909	1.849	1.783
24	2.155	2.130	2.108	2.088	2.054	2.027	1.975	1.939	1.892	1.863	1.800	1.733
26	2.119	2.094	2.072	2.052	2.018	1.990	1.938	1.901	1.853	1.823	1.760	1.691
28	2.089	2.064	2.041	2.021	1.987	1.959	1.906	1.869	1.820	1.790	1.725	1.654
30	2.063	2.037	2.015	1.995	1.960	1.932	1.878	1.841	1.792	1.761	1.695	1.622
32	2.040	2.015	1.992	1.972	1.937	1.908	1.854	1.817	1.767	1.736	1.669	1.594
34	2.021	1.995	1.972	1.952	1.917	1.888	1.833	1.795	1.745	1.713	1.645	1.569
36	2.003	1.977	1.954	1.934	1.899	1.870	1.815	1.776	1.726	1.694	1.625	1.547
38	1.988	1.962	1.939	1.918	1.883	1.853	1.798	1.760	1.708	1.676	1.606	1.527
40	1.974	1.948	1.924	1.904	1.868	1.839	1.783	1.744	1.693	1.660	1.589	1.509
42	1.961	1.935	1.912	1.891	1.855	1.826	1.770	1.731	1.679	1.646	1.574	1.492
44	1.950	1.924	1.900	1.879	1.844	1.814	1.758	1.718	1.666	1.633	1.560	1.477
46	1.940	1.913	1.890	1.869	1.833	1.803	1.747	1.707	1.654	1.621	1.547	1.463
48	1.930	1.904	1.880	1.859	1.823	1.793	1.737	1.697	1.644	1.610	1.536	1.450
50	1.921	1.895	1.871	1.850	1.814	1.784	1.727	1.687	1.634	1.599	1.525	1.438
55	1.903	1.876	1.852	1.831	1.795	1.764	1.707	1.666	1.612	1.577	1.501	1.412
60	1.887	1.860	1.836	1.815	1.778	1.748	1.690	1.649	1.594	1.559	1.481	1.389
65	1.874	1.847	1.823	1.802	1.765	1.734	1.676	1.635	1.579	1.543	1.464	1.370
70	1.863	1.836	1.812	1.790	1.753	1.722	1.664	1.622	1.566	1.530	1.450	1.353
75	1.853	1.826	1.802	1.780	1.743	1.712	1.653	1.611	1.555	1.518	1.437	1.338
80	1.845	1.817	1.793	1.772	1.734	1.703	1.644	1.602	1.545	1.508	1.426	1.325
85	1.837	1.810	1.786	1.764	1.726	1.695	1.636	1.593	1.536	1.499	1.416	1.313
90	1.830	1.803	1.779	1.757	1.720	1.688	1.629	1.586	1.528	1.491	1.407	1.302
95	1.825	1.797	1.773	1.751	1.713	1.682	1.622	1.579	1.521	1.484	1.399	1.292
100	1.819	1.792	1.768	1.746	1.708	1.676	1.616	1.573	1.515	1.477	1.392	1.283
125	1.799	1.772	1.747	1.725	1.687	1.655	1.594	1.551	1.491	1.452	1.364	1.248
150	1.786	1.758	1.734	1.711	1.673	1.641	1.580	1.535	1.475	1.436	1.345	1.223
175	1.776	1.749	1.724	1.702	1.663	1.631	1.569	1.525	1.464	1.424	1.331	1.204
200	1.769	1.742	1.717	1.694	1.656	1.623	1.561	1.516	1.455	1.415	1.321	1.189
250	1.759	1.732	1.707	1.684	1.645	1.613	1.550	1.505	1.443	1.402	1.306	1.166
500	1.740	1.712	1.686	1.664	1.625	1.592	1.528	1.482	1.419	1.376	1.275	1.113
∞	1.720	1.692	1.666	1.644	1.604	1.571	1.506	1.459	1.394	1.350	1.243	1.000

Percentile 97.5 of the *F* distribution

$\nu_2 \setminus \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	647.8	799.5	864.2	899.6	921.8	937.1	948.2	956.7	963.3	968.6	973.0	976.7
2	38.51	39.00	39.17	39.25	39.30	39.33	39.36	39.37	39.39	39.40	39.41	39.41
3	17.44	16.04	15.44	15.10	14.88	14.73	14.62	14.54	14.47	14.42	14.37	14.34
4	12.22	10.65	9.979	9.605	9.364	9.197	9.074	8.980	8.905	8.844	8.794	8.751
5	10.01	8.434	7.764	7.388	7.146	6.978	6.853	6.757	6.681	6.619	6.568	6.525
6	8.813	7.260	6.599	6.227	5.988	5.820	5.695	5.600	5.523	5.461	5.410	5.366
7	8.073	6.542	5.890	5.523	5.285	5.119	4.995	4.899	4.823	4.761	4.709	4.666
8	7.571	6.059	5.416	5.053	4.817	4.652	4.529	4.433	4.357	4.295	4.243	4.200
9	7.209	5.715	5.078	4.718	4.484	4.320	4.197	4.102	4.026	3.964	3.912	3.868
10	6.937	5.456	4.826	4.468	4.236	4.072	3.950	3.855	3.779	3.717	3.665	3.621
11	6.724	5.256	4.630	4.275	4.044	3.881	3.759	3.664	3.588	3.526	3.474	3.430
12	6.554	5.096	4.474	4.121	3.891	3.728	3.607	3.512	3.436	3.374	3.321	3.277
13	6.414	4.965	4.347	3.996	3.767	3.604	3.483	3.388	3.312	3.250	3.197	3.153
14	6.298	4.857	4.242	3.892	3.663	3.501	3.380	3.285	3.209	3.147	3.095	3.050
15	6.200	4.765	4.153	3.804	3.577	3.415	3.293	3.199	3.123	3.060	3.008	2.963
16	6.115	4.687	4.077	3.729	3.502	3.341	3.219	3.125	3.049	2.986	2.934	2.889
17	6.042	4.619	4.011	3.665	3.438	3.277	3.156	3.061	2.985	2.922	2.870	2.825
18	5.978	4.560	3.954	3.608	3.382	3.221	3.100	3.005	2.929	2.866	2.814	2.769
19	5.922	4.508	3.903	3.559	3.333	3.172	3.051	2.956	2.880	2.817	2.765	2.720
20	5.871	4.461	3.859	3.515	3.289	3.128	3.007	2.913	2.837	2.774	2.721	2.676
22	5.786	4.383	3.783	3.440	3.215	3.055	2.934	2.839	2.763	2.700	2.647	2.602
24	5.717	4.319	3.721	3.379	3.155	2.995	2.874	2.779	2.703	2.640	2.586	2.541
26	5.659	4.265	3.670	3.329	3.105	2.945	2.824	2.729	2.653	2.590	2.536	2.491
28	5.610	4.221	3.626	3.286	3.063	2.903	2.782	2.687	2.611	2.547	2.494	2.448
30	5.568	4.182	3.589	3.250	3.026	2.867	2.746	2.651	2.575	2.511	2.458	2.412
32	5.531	4.149	3.557	3.218	2.995	2.836	2.715	2.620	2.543	2.480	2.426	2.381
34	5.499	4.120	3.529	3.191	2.968	2.808	2.688	2.593	2.516	2.453	2.399	2.353
36	5.471	4.094	3.505	3.167	2.944	2.785	2.664	2.569	2.492	2.429	2.375	2.329
38	5.446	4.071	3.483	3.145	2.923	2.763	2.643	2.548	2.471	2.407	2.353	2.307
40	5.424	4.051	3.463	3.126	2.904	2.744	2.624	2.529	2.452	2.388	2.334	2.288
42	5.404	4.033	3.446	3.109	2.887	2.727	2.607	2.512	2.435	2.371	2.317	2.271
44	5.386	4.016	3.430	3.093	2.871	2.712	2.591	2.496	2.419	2.355	2.302	2.255
46	5.369	4.001	3.415	3.079	2.857	2.698	2.577	2.482	2.405	2.341	2.287	2.241
48	5.354	3.987	3.402	3.066	2.844	2.685	2.565	2.470	2.393	2.329	2.274	2.228
50	5.340	3.975	3.390	3.054	2.833	2.674	2.553	2.458	2.381	2.317	2.263	2.216
55	5.310	3.948	3.364	3.029	2.807	2.648	2.528	2.433	2.355	2.291	2.237	2.190
60	5.286	3.925	3.343	3.008	2.786	2.627	2.507	2.412	2.334	2.270	2.216	2.169
65	5.265	3.906	3.324	2.990	2.769	2.610	2.489	2.394	2.317	2.252	2.198	2.151
70	5.247	3.890	3.309	2.975	2.754	2.595	2.474	2.379	2.302	2.237	2.183	2.136
75	5.232	3.876	3.296	2.962	2.741	2.582	2.461	2.366	2.289	2.224	2.170	2.123
80	5.218	3.864	3.284	2.950	2.730	2.571	2.450	2.355	2.277	2.213	2.158	2.111
85	5.207	3.854	3.274	2.940	2.720	2.561	2.440	2.345	2.268	2.203	2.148	2.101
90	5.196	3.844	3.265	2.932	2.711	2.552	2.432	2.336	2.259	2.194	2.140	2.092
95	5.187	3.836	3.257	2.924	2.703	2.544	2.424	2.328	2.251	2.186	2.132	2.084
100	5.179	3.828	3.250	2.917	2.696	2.537	2.417	2.321	2.244	2.179	2.124	2.077
125	5.147	3.800	3.222	2.890	2.670	2.511	2.390	2.295	2.217	2.153	2.098	2.050
150	5.126	3.781	3.204	2.872	2.652	2.494	2.373	2.278	2.200	2.135	2.080	2.032
175	5.111	3.768	3.192	2.860	2.640	2.481	2.361	2.265	2.187	2.122	2.067	2.020
200	5.100	3.758	3.182	2.850	2.630	2.472	2.351	2.256	2.178	2.113	2.058	2.010
250	5.085	3.744	3.169	2.837	2.618	2.459	2.338	2.243	2.165	2.100	2.045	1.997
500	5.054	3.716	3.142	2.811	2.592	2.434	2.313	2.217	2.139	2.074	2.019	1.971
∞	5.024	3.689	3.116	2.786	2.567	2.408	2.288	2.192	2.114	2.048	1.993	1.945

Percentile 97.5 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	979.8	982.5	984.9	986.9	990.4	993.1	998.1	1001	1006	1008	1013	1018
2	39.42	39.43	39.43	39.44	39.44	39.45	39.46	39.46	39.47	39.48	39.49	39.50
3	14.30	14.28	14.25	14.23	14.20	14.17	14.12	14.08	14.04	14.01	13.96	13.90
4	8.715	8.684	8.657	8.633	8.592	8.560	8.501	8.461	8.411	8.381	8.319	8.257
5	6.488	6.456	6.428	6.403	6.362	6.329	6.268	6.227	6.175	6.144	6.080	6.015
6	5.329	5.297	5.269	5.244	5.202	5.168	5.107	5.065	5.012	4.980	4.915	4.849
7	4.628	4.596	4.568	4.543	4.501	4.467	4.405	4.362	4.309	4.276	4.210	4.142
8	4.162	4.130	4.101	4.076	4.034	3.999	3.937	3.894	3.840	3.807	3.739	3.670
9	3.831	3.798	3.769	3.744	3.701	3.667	3.604	3.560	3.505	3.472	3.403	3.333
10	3.583	3.550	3.522	3.496	3.453	3.419	3.355	3.311	3.255	3.221	3.152	3.080
11	3.392	3.359	3.330	3.304	3.261	3.226	3.162	3.118	3.061	3.027	2.956	2.883
12	3.239	3.206	3.177	3.152	3.108	3.073	3.008	2.963	2.906	2.871	2.800	2.725
13	3.115	3.082	3.053	3.027	2.983	2.948	2.882	2.837	2.780	2.744	2.671	2.595
14	3.012	2.979	2.949	2.923	2.879	2.844	2.778	2.732	2.674	2.638	2.565	2.487
15	2.925	2.891	2.862	2.836	2.792	2.756	2.689	2.644	2.585	2.549	2.474	2.395
16	2.851	2.817	2.788	2.761	2.717	2.681	2.614	2.568	2.509	2.472	2.396	2.316
17	2.786	2.753	2.723	2.697	2.652	2.616	2.548	2.502	2.442	2.405	2.329	2.247
18	2.730	2.696	2.667	2.640	2.596	2.559	2.491	2.445	2.384	2.347	2.269	2.187
19	2.681	2.647	2.617	2.591	2.546	2.509	2.441	2.394	2.333	2.295	2.217	2.133
20	2.637	2.603	2.573	2.547	2.501	2.464	2.396	2.349	2.287	2.249	2.170	2.085
22	2.563	2.528	2.498	2.472	2.426	2.389	2.320	2.272	2.210	2.171	2.090	2.003
24	2.502	2.468	2.437	2.411	2.365	2.327	2.257	2.209	2.146	2.107	2.024	1.935
26	2.451	2.417	2.387	2.360	2.314	2.276	2.205	2.157	2.093	2.053	1.969	1.878
28	2.409	2.374	2.344	2.317	2.270	2.232	2.161	2.112	2.048	2.007	1.922	1.829
30	2.372	2.338	2.307	2.280	2.233	2.195	2.124	2.074	2.009	1.968	1.882	1.787
32	2.341	2.306	2.275	2.248	2.201	2.163	2.091	2.041	1.975	1.934	1.846	1.750
34	2.313	2.278	2.248	2.220	2.173	2.135	2.062	2.012	1.946	1.904	1.815	1.717
36	2.289	2.254	2.223	2.196	2.148	2.110	2.037	1.986	1.919	1.877	1.787	1.687
38	2.267	2.232	2.201	2.174	2.126	2.088	2.015	1.963	1.896	1.854	1.763	1.661
40	2.248	2.213	2.182	2.154	2.107	2.068	1.994	1.943	1.875	1.832	1.741	1.637
42	2.231	2.196	2.164	2.137	2.089	2.050	1.976	1.924	1.856	1.813	1.720	1.615
44	2.215	2.180	2.149	2.121	2.073	2.034	1.960	1.908	1.839	1.796	1.702	1.596
46	2.201	2.165	2.134	2.106	2.058	2.019	1.945	1.893	1.824	1.780	1.685	1.577
48	2.188	2.152	2.121	2.093	2.045	2.006	1.931	1.879	1.809	1.765	1.670	1.561
50	2.176	2.140	2.109	2.081	2.033	1.993	1.919	1.866	1.796	1.752	1.656	1.545
55	2.150	2.114	2.083	2.055	2.006	1.967	1.891	1.838	1.768	1.723	1.625	1.511
60	2.129	2.093	2.061	2.033	1.985	1.944	1.869	1.815	1.744	1.699	1.599	1.482
65	2.111	2.075	2.043	2.015	1.966	1.926	1.850	1.796	1.724	1.678	1.577	1.457
70	2.095	2.059	2.028	1.999	1.950	1.910	1.833	1.779	1.707	1.660	1.558	1.436
75	2.082	2.046	2.014	1.986	1.937	1.896	1.819	1.765	1.692	1.645	1.542	1.417
80	2.071	2.035	2.003	1.974	1.925	1.884	1.807	1.752	1.679	1.632	1.527	1.400
85	2.060	2.024	1.992	1.964	1.915	1.874	1.796	1.741	1.668	1.620	1.514	1.385
90	2.051	2.015	1.983	1.955	1.905	1.864	1.787	1.731	1.657	1.610	1.503	1.371
95	2.043	2.007	1.975	1.946	1.897	1.856	1.778	1.723	1.648	1.600	1.493	1.359
100	2.036	2.000	1.968	1.939	1.890	1.849	1.770	1.715	1.640	1.592	1.483	1.347
125	2.009	1.973	1.940	1.911	1.862	1.820	1.741	1.685	1.609	1.559	1.448	1.303
150	1.991	1.955	1.922	1.893	1.843	1.801	1.722	1.665	1.588	1.538	1.423	1.271
175	1.978	1.942	1.909	1.880	1.830	1.788	1.708	1.651	1.573	1.522	1.406	1.248
200	1.969	1.932	1.900	1.870	1.820	1.778	1.698	1.640	1.562	1.511	1.393	1.229
250	1.955	1.919	1.886	1.857	1.806	1.764	1.683	1.625	1.546	1.495	1.374	1.201
500	1.929	1.892	1.859	1.830	1.779	1.736	1.655	1.596	1.515	1.462	1.336	1.137
∞	1.903	1.866	1.833	1.803	1.751	1.708	1.626	1.566	1.484	1.428	1.296	1.000

Percentile 99 of the *F* distribution

$\nu_2 \backslash \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	4052	5000	5403	5625	5764	5859	5928	5981	6022	6056	6083	6106
2	98.50	99.00	99.17	99.25	99.30	99.33	99.36	99.37	99.39	99.40	99.41	99.42
3	34.12	30.82	29.46	28.71	28.24	27.91	27.67	27.49	27.35	27.23	27.13	27.05
4	21.20	18.00	16.69	15.98	15.52	15.21	14.98	14.80	14.66	14.55	14.45	14.37
5	16.26	13.27	12.06	11.39	10.97	10.67	10.46	10.29	10.16	10.05	9.963	9.888
6	13.75	10.92	9.780	9.148	8.746	8.466	8.260	8.102	7.976	7.874	7.790	7.718
7	12.25	9.547	8.451	7.847	7.460	7.191	6.993	6.840	6.719	6.620	6.538	6.469
8	11.26	8.649	7.591	7.006	6.632	6.371	6.178	6.029	5.911	5.814	5.734	5.667
9	10.56	8.022	6.992	6.422	6.057	5.802	5.613	5.467	5.351	5.257	5.178	5.111
10	10.04	7.559	6.552	5.994	5.636	5.386	5.200	5.057	4.942	4.849	4.772	4.706
11	9.646	7.206	6.217	5.668	5.316	5.069	4.886	4.744	4.632	4.539	4.462	4.397
12	9.330	6.927	5.953	5.412	5.064	4.821	4.640	4.499	4.388	4.296	4.220	4.155
13	9.074	6.701	5.739	5.205	4.862	4.620	4.441	4.302	4.191	4.100	4.025	3.960
14	8.862	6.515	5.564	5.035	4.695	4.456	4.278	4.140	4.030	3.939	3.864	3.800
15	8.683	6.359	5.417	4.893	4.556	4.318	4.142	4.004	3.895	3.805	3.730	3.666
16	8.531	6.226	5.292	4.773	4.437	4.202	4.026	3.890	3.780	3.691	3.616	3.553
17	8.400	6.112	5.185	4.669	4.336	4.102	3.927	3.791	3.682	3.593	3.519	3.455
18	8.285	6.013	5.092	4.579	4.248	4.015	3.841	3.705	3.597	3.508	3.434	3.371
19	8.185	5.926	5.010	4.500	4.171	3.939	3.765	3.631	3.523	3.434	3.360	3.297
20	8.096	5.849	4.938	4.431	4.103	3.871	3.699	3.564	3.457	3.368	3.294	3.231
22	7.945	5.719	4.817	4.313	3.988	3.758	3.587	3.453	3.346	3.258	3.184	3.121
24	7.823	5.614	4.718	4.218	3.895	3.667	3.496	3.363	3.256	3.168	3.094	3.032
26	7.721	5.526	4.637	4.140	3.818	3.591	3.421	3.288	3.182	3.094	3.020	2.958
28	7.636	5.453	4.568	4.074	3.754	3.528	3.358	3.226	3.120	3.032	2.959	2.896
30	7.562	5.390	4.510	4.018	3.699	3.473	3.304	3.173	3.067	2.979	2.906	2.843
32	7.499	5.336	4.459	3.969	3.652	3.427	3.258	3.127	3.021	2.934	2.860	2.798
34	7.444	5.289	4.416	3.927	3.611	3.386	3.218	3.087	2.981	2.894	2.821	2.758
36	7.396	5.248	4.377	3.890	3.574	3.351	3.183	3.052	2.946	2.859	2.786	2.723
38	7.353	5.211	4.343	3.858	3.542	3.319	3.152	3.021	2.915	2.828	2.755	2.692
40	7.314	5.179	4.313	3.828	3.514	3.291	3.124	2.993	2.888	2.801	2.727	2.665
42	7.280	5.149	4.285	3.802	3.488	3.266	3.099	2.968	2.863	2.776	2.703	2.640
44	7.248	5.123	4.261	3.778	3.465	3.243	3.076	2.946	2.841	2.754	2.680	2.618
46	7.220	5.099	4.238	3.757	3.444	3.222	3.056	2.925	2.820	2.733	2.660	2.598
48	7.194	5.077	4.218	3.737	3.425	3.204	3.037	2.907	2.802	2.715	2.642	2.579
50	7.171	5.057	4.199	3.720	3.408	3.186	3.020	2.890	2.785	2.698	2.625	2.562
55	7.119	5.013	4.159	3.681	3.370	3.149	2.983	2.853	2.748	2.662	2.589	2.526
60	7.077	4.977	4.126	3.649	3.339	3.119	2.953	2.823	2.718	2.632	2.559	2.496
65	7.041	4.947	4.098	3.622	3.313	3.093	2.928	2.798	2.693	2.607	2.534	2.471
70	7.011	4.922	4.074	3.600	3.291	3.071	2.906	2.777	2.672	2.585	2.512	2.450
75	6.985	4.900	4.054	3.580	3.272	3.052	2.887	2.758	2.653	2.567	2.494	2.431
80	6.963	4.881	4.036	3.563	3.255	3.036	2.871	2.742	2.637	2.551	2.478	2.415
85	6.943	4.864	4.021	3.548	3.241	3.022	2.857	2.728	2.623	2.537	2.464	2.401
90	6.925	4.849	4.007	3.535	3.228	3.009	2.845	2.715	2.611	2.524	2.451	2.389
95	6.910	4.836	3.995	3.523	3.216	2.998	2.833	2.704	2.600	2.513	2.440	2.378
100	6.895	4.824	3.984	3.513	3.206	2.988	2.823	2.694	2.590	2.503	2.430	2.368
125	6.842	4.779	3.942	3.473	3.167	2.950	2.785	2.657	2.552	2.466	2.393	2.330
150	6.807	4.749	3.915	3.447	3.142	2.924	2.761	2.632	2.528	2.441	2.368	2.305
175	6.782	4.728	3.895	3.428	3.123	2.906	2.743	2.614	2.510	2.424	2.351	2.288
200	6.763	4.713	3.881	3.414	3.110	2.893	2.730	2.601	2.497	2.411	2.337	2.275
250	6.737	4.691	3.861	3.395	3.091	2.875	2.711	2.583	2.479	2.393	2.319	2.257
500	6.686	4.648	3.821	3.357	3.054	2.838	2.675	2.547	2.443	2.356	2.283	2.220
∞	6.635	4.605	3.782	3.319	3.017	2.802	2.639	2.511	2.407	2.321	2.248	2.185

Percentile 99 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	6126	6143	6157	6170	6192	6209	6240	6261	6287	6303	6334	6366
2	99.42	99.43	99.43	99.44	99.44	99.45	99.46	99.47	99.47	99.48	99.49	99.50
3	26.98	26.92	26.87	26.83	26.75	26.69	26.58	26.50	26.41	26.35	26.24	26.13
4	14.31	14.25	14.20	14.15	14.08	14.02	13.91	13.84	13.75	13.69	13.58	13.46
5	9.825	9.770	9.722	9.680	9.610	9.553	9.449	9.379	9.291	9.238	9.130	9.020
6	7.657	7.605	7.559	7.519	7.451	7.396	7.296	7.229	7.143	7.091	6.987	6.880
7	6.410	6.359	6.314	6.275	6.209	6.155	6.058	5.992	5.908	5.858	5.755	5.650
8	5.609	5.559	5.515	5.477	5.412	5.359	5.263	5.198	5.116	5.065	4.963	4.859
9	5.055	5.005	4.962	4.924	4.860	4.808	4.713	4.649	4.567	4.517	4.415	4.311
10	4.650	4.601	4.558	4.520	4.457	4.405	4.311	4.247	4.165	4.115	4.014	3.909
11	4.342	4.293	4.251	4.213	4.150	4.099	4.005	3.941	3.860	3.810	3.708	3.602
12	4.100	4.052	4.010	3.972	3.909	3.858	3.765	3.701	3.619	3.569	3.467	3.361
13	3.905	3.857	3.815	3.778	3.716	3.665	3.571	3.507	3.425	3.375	3.272	3.165
14	3.745	3.698	3.656	3.619	3.556	3.505	3.412	3.348	3.266	3.215	3.112	3.004
15	3.612	3.564	3.522	3.485	3.423	3.372	3.278	3.214	3.132	3.081	2.977	2.868
16	3.498	3.451	3.409	3.372	3.310	3.259	3.165	3.101	3.018	2.967	2.863	2.753
17	3.401	3.353	3.312	3.275	3.212	3.162	3.068	3.003	2.920	2.869	2.764	2.653
18	3.316	3.269	3.227	3.190	3.128	3.077	2.983	2.919	2.835	2.784	2.678	2.566
19	3.242	3.195	3.153	3.117	3.054	3.003	2.909	2.844	2.761	2.709	2.602	2.489
20	3.177	3.130	3.088	3.051	2.989	2.938	2.843	2.778	2.695	2.643	2.535	2.421
22	3.067	3.019	2.978	2.941	2.879	2.827	2.733	2.667	2.583	2.531	2.422	2.305
24	2.977	2.930	2.889	2.852	2.789	2.738	2.643	2.577	2.492	2.440	2.329	2.211
26	2.904	2.857	2.815	2.778	2.715	2.664	2.569	2.503	2.417	2.364	2.252	2.131
28	2.842	2.795	2.753	2.716	2.653	2.602	2.506	2.440	2.353	2.300	2.187	2.064
30	2.789	2.742	2.700	2.663	2.600	2.549	2.453	2.386	2.299	2.245	2.131	2.006
32	2.744	2.696	2.655	2.618	2.555	2.503	2.406	2.340	2.252	2.198	2.082	1.956
34	2.704	2.657	2.615	2.578	2.515	2.463	2.366	2.299	2.211	2.156	2.040	1.911
36	2.669	2.622	2.580	2.543	2.480	2.428	2.331	2.263	2.175	2.120	2.002	1.872
38	2.638	2.591	2.549	2.512	2.449	2.397	2.299	2.232	2.143	2.087	1.968	1.837
40	2.611	2.563	2.522	2.484	2.421	2.369	2.271	2.203	2.114	2.058	1.938	1.805
42	2.586	2.539	2.497	2.460	2.396	2.344	2.246	2.178	2.088	2.032	1.911	1.776
44	2.564	2.516	2.475	2.437	2.374	2.321	2.223	2.155	2.065	2.008	1.887	1.750
46	2.544	2.496	2.454	2.417	2.353	2.301	2.203	2.134	2.044	1.987	1.864	1.726
48	2.525	2.478	2.436	2.399	2.335	2.282	2.184	2.115	2.024	1.967	1.844	1.703
50	2.508	2.461	2.419	2.382	2.318	2.265	2.167	2.098	2.007	1.949	1.825	1.683
55	2.472	2.424	2.382	2.345	2.281	2.228	2.129	2.060	1.968	1.910	1.784	1.638
60	2.442	2.394	2.352	2.315	2.251	2.198	2.098	2.028	1.936	1.877	1.749	1.601
65	2.417	2.369	2.327	2.289	2.225	2.172	2.072	2.002	1.909	1.850	1.720	1.568
70	2.395	2.348	2.306	2.268	2.204	2.150	2.050	1.980	1.886	1.826	1.695	1.541
75	2.377	2.329	2.287	2.249	2.185	2.132	2.031	1.960	1.866	1.806	1.674	1.516
80	2.361	2.313	2.271	2.233	2.169	2.115	2.015	1.944	1.849	1.788	1.655	1.495
85	2.347	2.299	2.257	2.219	2.154	2.101	2.000	1.929	1.834	1.773	1.638	1.475
90	2.334	2.286	2.244	2.206	2.142	2.088	1.987	1.916	1.820	1.759	1.623	1.458
95	2.323	2.275	2.233	2.195	2.130	2.077	1.976	1.904	1.808	1.746	1.610	1.442
100	2.313	2.265	2.223	2.185	2.120	2.067	1.965	1.893	1.797	1.735	1.598	1.428
125	2.276	2.228	2.185	2.147	2.082	2.028	1.926	1.853	1.756	1.693	1.551	1.371
150	2.251	2.203	2.160	2.122	2.057	2.003	1.900	1.827	1.729	1.665	1.520	1.331
175	2.233	2.185	2.143	2.105	2.039	1.985	1.882	1.808	1.709	1.645	1.498	1.302
200	2.220	2.172	2.129	2.091	2.026	1.971	1.868	1.794	1.694	1.629	1.481	1.279
250	2.202	2.154	2.111	2.073	2.007	1.953	1.849	1.774	1.674	1.608	1.457	1.244
500	2.166	2.118	2.075	2.036	1.970	1.915	1.810	1.735	1.633	1.566	1.408	1.164
∞	2.130	2.082	2.039	2.000	1.934	1.878	1.773	1.696	1.592	1.523	1.358	1.000

Percentile 99.5 of the *F* distribution

$\nu_2 \setminus \nu_1$	1	2	3	4	5	6	7	8	9	10	11	12
1	16211	20000	21615	22500	23056	23437	23715	23925	24091	24225	24334	24426
2	198.5	199.0	199.2	199.2	199.3	199.3	199.4	199.4	199.4	199.4	199.4	199.4
3	55.55	49.80	47.47	46.19	45.39	44.84	44.43	44.13	43.88	43.69	43.52	43.39
4	31.33	26.28	24.26	23.15	22.46	21.97	21.62	21.35	21.14	20.97	20.82	20.70
5	22.78	18.31	16.53	15.56	14.94	14.51	14.20	13.96	13.77	13.62	13.49	13.38
6	18.64	14.54	12.92	12.03	11.46	11.07	10.79	10.57	10.39	10.25	10.13	10.03
7	16.24	12.40	10.88	10.05	9.522	9.155	8.885	8.678	8.514	8.380	8.270	8.176
8	14.69	11.04	9.596	8.805	8.302	7.952	7.694	7.496	7.339	7.211	7.104	7.015
9	13.61	10.11	8.717	7.956	7.471	7.134	6.885	6.693	6.541	6.417	6.314	6.227
10	12.83	9.427	8.081	7.343	6.872	6.545	6.302	6.116	5.968	5.847	5.746	5.661
11	12.23	8.912	7.600	6.881	6.422	6.102	5.865	5.682	5.537	5.418	5.320	5.236
12	11.75	8.510	7.226	6.521	6.071	5.757	5.525	5.345	5.202	5.085	4.988	4.906
13	11.37	8.186	6.926	6.233	5.791	5.482	5.253	5.076	4.935	4.820	4.724	4.643
14	11.06	7.922	6.680	5.998	5.562	5.257	5.031	4.857	4.717	4.603	4.508	4.428
15	10.80	7.701	6.476	5.803	5.372	5.071	4.847	4.674	4.536	4.424	4.329	4.250
16	10.58	7.514	6.303	5.638	5.212	4.913	4.692	4.521	4.384	4.272	4.179	4.099
17	10.38	7.354	6.156	5.497	5.075	4.779	4.559	4.389	4.254	4.142	4.050	3.971
18	10.22	7.215	6.028	5.375	4.956	4.663	4.445	4.276	4.141	4.030	3.938	3.860
19	10.07	7.093	5.916	5.268	4.853	4.561	4.345	4.177	4.043	3.933	3.841	3.763
20	9.944	6.986	5.818	5.174	4.762	4.472	4.257	4.090	3.956	3.847	3.756	3.678
22	9.727	6.806	5.652	5.017	4.609	4.322	4.109	3.944	3.812	3.703	3.612	3.535
24	9.551	6.661	5.519	4.890	4.486	4.202	3.991	3.826	3.695	3.587	3.497	3.420
26	9.406	6.541	5.409	4.785	4.384	4.103	3.893	3.730	3.599	3.492	3.402	3.325
28	9.284	6.440	5.317	4.698	4.300	4.020	3.811	3.649	3.519	3.412	3.322	3.246
30	9.180	6.355	5.239	4.623	4.228	3.949	3.742	3.580	3.450	3.344	3.255	3.179
32	9.090	6.281	5.171	4.559	4.166	3.889	3.682	3.521	3.392	3.286	3.197	3.121
34	9.012	6.217	5.113	4.504	4.112	3.836	3.630	3.470	3.341	3.235	3.146	3.071
36	8.943	6.161	5.062	4.455	4.065	3.790	3.585	3.425	3.296	3.191	3.102	3.027
38	8.882	6.111	5.016	4.412	4.023	3.749	3.545	3.385	3.257	3.152	3.063	2.988
40	8.828	6.066	4.976	4.374	3.986	3.713	3.509	3.350	3.222	3.117	3.028	2.953
42	8.779	6.027	4.940	4.339	3.953	3.680	3.477	3.318	3.191	3.086	2.997	2.922
44	8.735	5.991	4.907	4.308	3.923	3.651	3.448	3.290	3.162	3.057	2.969	2.894
46	8.695	5.958	4.877	4.280	3.896	3.625	3.422	3.264	3.137	3.032	2.944	2.869
48	8.659	5.929	4.850	4.255	3.871	3.601	3.398	3.240	3.113	3.009	2.921	2.846
50	8.626	5.902	4.826	4.232	3.849	3.579	3.376	3.219	3.092	2.988	2.900	2.825
55	8.554	5.843	4.773	4.181	3.800	3.531	3.330	3.173	3.046	2.942	2.854	2.779
60	8.494	5.795	4.729	4.140	3.760	3.492	3.291	3.134	3.008	2.904	2.817	2.742
65	8.445	5.755	4.692	4.105	3.726	3.459	3.259	3.103	2.977	2.873	2.785	2.711
70	8.402	5.720	4.661	4.076	3.698	3.431	3.232	3.075	2.950	2.846	2.759	2.684
75	8.366	5.691	4.635	4.050	3.674	3.407	3.208	3.052	2.927	2.823	2.736	2.661
80	8.335	5.665	4.612	4.029	3.652	3.387	3.188	3.032	2.907	2.803	2.716	2.641
85	8.307	5.643	4.591	4.009	3.634	3.368	3.170	3.014	2.889	2.786	2.698	2.624
90	8.282	5.623	4.573	3.992	3.617	3.352	3.154	2.999	2.873	2.770	2.683	2.608
95	8.260	5.605	4.556	3.977	3.603	3.338	3.140	2.985	2.860	2.756	2.669	2.595
100	8.241	5.589	4.542	3.963	3.589	3.325	3.127	2.972	2.847	2.744	2.657	2.583
125	8.166	5.530	4.488	3.912	3.540	3.277	3.079	2.925	2.801	2.698	2.611	2.536
150	8.118	5.490	4.453	3.878	3.508	3.245	3.048	2.894	2.770	2.667	2.580	2.506
175	8.083	5.462	4.427	3.855	3.485	3.223	3.026	2.872	2.748	2.645	2.559	2.484
200	8.057	5.441	4.408	3.837	3.468	3.206	3.010	2.856	2.732	2.629	2.543	2.468
250	8.022	5.412	4.382	3.812	3.443	3.182	2.987	2.833	2.710	2.607	2.520	2.446
500	7.948	5.355	4.330	3.764	3.397	3.137	2.941	2.789	2.665	2.563	2.476	2.402
∞	7.880	5.299	4.279	3.715	3.350	3.091	2.897	2.744	2.621	2.519	2.432	2.358

Percentile 99.5 of the F distribution (cont.)

$\nu_2 \setminus \nu_1$	13	14	15	16	18	20	25	30	40	50	100	∞
1	24505	24572	24630	24682	24767	24836	24960	25044	25148	25211	25337	25465
2	199.4	199.4	199.4	199.4	199.4	199.4	199.5	199.5	199.5	199.5	199.5	199.5
3	43.27	43.17	43.08	43.01	42.88	42.78	42.59	42.47	42.31	42.21	42.02	41.83
4	20.60	20.51	20.44	20.37	20.26	20.17	20.00	19.89	19.75	19.67	19.50	19.32
5	13.29	13.21	13.15	13.09	12.98	12.90	12.76	12.66	12.53	12.45	12.30	12.14
6	9.950	9.877	9.814	9.758	9.664	9.589	9.451	9.358	9.241	9.170	9.026	8.879
7	8.097	8.028	7.968	7.915	7.826	7.754	7.623	7.534	7.422	7.354	7.217	7.076
8	6.938	6.872	6.814	6.763	6.678	6.608	6.482	6.396	6.288	6.222	6.088	5.951
9	6.153	6.089	6.032	5.983	5.899	5.832	5.708	5.625	5.519	5.454	5.322	5.188
10	5.589	5.526	5.471	5.422	5.340	5.274	5.153	5.071	4.966	4.902	4.772	4.639
11	5.165	5.103	5.049	5.001	4.921	4.855	4.736	4.654	4.551	4.488	4.359	4.226
12	4.836	4.775	4.721	4.674	4.595	4.530	4.412	4.331	4.228	4.165	4.037	3.904
13	4.573	4.513	4.460	4.413	4.334	4.270	4.153	4.073	3.970	3.908	3.780	3.647
14	4.359	4.299	4.247	4.200	4.122	4.059	3.942	3.862	3.760	3.698	3.569	3.436
15	4.181	4.122	4.070	4.024	3.946	3.883	3.766	3.687	3.585	3.523	3.394	3.260
16	4.031	3.972	3.920	3.875	3.797	3.734	3.618	3.539	3.437	3.375	3.246	3.112
17	3.903	3.844	3.793	3.747	3.670	3.607	3.492	3.412	3.311	3.248	3.119	2.984
18	3.793	3.734	3.683	3.637	3.560	3.498	3.382	3.303	3.201	3.139	3.009	2.873
19	3.696	3.638	3.587	3.541	3.465	3.402	3.287	3.208	3.106	3.043	2.913	2.776
20	3.611	3.553	3.502	3.457	3.380	3.318	3.203	3.123	3.022	2.959	2.828	2.690
22	3.469	3.411	3.360	3.315	3.239	3.176	3.061	2.982	2.880	2.817	2.685	2.545
24	3.354	3.296	3.246	3.201	3.125	3.062	2.947	2.868	2.765	2.702	2.569	2.428
26	3.259	3.202	3.151	3.107	3.031	2.969	2.853	2.774	2.671	2.607	2.473	2.330
28	3.180	3.123	3.073	3.028	2.952	2.890	2.775	2.695	2.592	2.527	2.392	2.247
30	3.113	3.056	3.006	2.961	2.885	2.823	2.708	2.628	2.524	2.459	2.323	2.176
32	3.056	2.998	2.948	2.904	2.828	2.766	2.650	2.570	2.466	2.401	2.264	2.114
34	3.005	2.948	2.898	2.854	2.778	2.716	2.600	2.520	2.415	2.350	2.212	2.060
36	2.961	2.905	2.854	2.810	2.734	2.672	2.556	2.475	2.371	2.305	2.166	2.013
38	2.923	2.866	2.816	2.771	2.695	2.633	2.517	2.436	2.331	2.265	2.125	1.970
40	2.888	2.831	2.781	2.737	2.661	2.598	2.482	2.401	2.296	2.230	2.088	1.932
42	2.857	2.800	2.750	2.706	2.630	2.567	2.451	2.370	2.264	2.198	2.056	1.897
44	2.829	2.772	2.722	2.678	2.602	2.540	2.423	2.342	2.236	2.169	2.026	1.866
46	2.804	2.747	2.697	2.653	2.577	2.514	2.398	2.316	2.210	2.143	1.999	1.837
48	2.781	2.724	2.674	2.630	2.554	2.491	2.375	2.293	2.186	2.119	1.974	1.811
50	2.760	2.703	2.653	2.609	2.533	2.470	2.353	2.272	2.164	2.097	1.951	1.786
55	2.714	2.658	2.608	2.563	2.487	2.425	2.308	2.226	2.118	2.049	1.902	1.733
60	2.677	2.620	2.570	2.526	2.450	2.387	2.270	2.187	2.079	2.010	1.861	1.689
65	2.646	2.589	2.539	2.495	2.419	2.356	2.238	2.155	2.046	1.977	1.826	1.650
70	2.619	2.563	2.513	2.468	2.392	2.329	2.211	2.128	2.019	1.949	1.796	1.619
75	2.597	2.540	2.490	2.445	2.369	2.306	2.188	2.105	1.995	1.925	1.771	1.590
80	2.577	2.520	2.470	2.425	2.349	2.286	2.168	2.084	1.974	1.903	1.749	1.564
85	2.559	2.503	2.453	2.408	2.332	2.269	2.150	2.067	1.956	1.885	1.729	1.541
90	2.544	2.487	2.437	2.393	2.316	2.253	2.134	2.051	1.939	1.868	1.711	1.521
95	2.530	2.474	2.424	2.379	2.303	2.239	2.120	2.037	1.925	1.853	1.695	1.503
100	2.518	2.461	2.411	2.367	2.290	2.227	2.108	2.024	1.912	1.840	1.681	1.486
125	2.472	2.415	2.365	2.320	2.244	2.180	2.061	1.976	1.863	1.790	1.627	1.420
150	2.441	2.385	2.335	2.290	2.213	2.150	2.030	1.944	1.830	1.756	1.590	1.375
175	2.420	2.363	2.313	2.268	2.191	2.128	2.007	1.922	1.807	1.733	1.564	1.340
200	2.404	2.347	2.297	2.252	2.175	2.112	1.991	1.905	1.790	1.715	1.544	1.314
250	2.381	2.325	2.275	2.230	2.153	2.089	1.968	1.882	1.765	1.690	1.516	1.275
500	2.337	2.281	2.230	2.185	2.108	2.044	1.922	1.835	1.717	1.640	1.460	1.184
∞	2.294	2.237	2.187	2.142	2.064	2.000	1.877	1.789	1.669	1.590	1.402	1.000

Reading off the table

The preceding table gives the critical values of $F(\nu_1,\nu_2)$ at percentage points $P = 0.95$, 0.975, 0.99 and 0.995, for pairs of degrees of freedom (ν_1,ν_2), ν_1 usually referring to the numerator, ν_2 to the denominator of F.

Illustration 1. Find $F_{3,8[.95]}$, *i.e.* the 95th percentage point of F with 3 and 8 *df.* Taking the first page for percentile 95, at the junction of column $\nu_1 = 3$ and line $\nu_2 = 8$, we read off 4.066. In the same manner, we find $F_{10,12[.99]} = 4.296$ and $F_{20,5[.975]} = 6.329$.

Illustration 2. Determine $F_{12,3[.05]}$. The table furnishes only percentage points over the 50th. For lower percentage points, we must resort to the inversion formula for F, *i.e.*

$$F_{\nu_1,\nu_2[P]} = 1 \,/\, F_{\nu_2,\nu_1[1-P]} \ .$$

Finding first $F_{3,12[.95]} = 3.490$ (note the interversion of the two *df*'s and the transposition of percentiles .05 → .95), we obtain $F_{12,3[.05]} = 1 \,/\, F_{3,12[.95]} = 1 \,/\, 3.490 \approx 0.287$.

Illustration 3. Determine $F_{4,160[.99]}$. At page 1 for percentile 99, we find only $F_{4,150} =$ 3.447 and $F_{4,175} = 3.428$; consequently, we must interpolate. Linear interpolation seems adequate for estimating F for $\nu_2 = 160$ between values at $\nu_2 = 150$ and 175. Simplifying the notation, we calculate:

$$F_\nu \approx F_{\nu'} + (F_{\nu''} - F_{\nu'}) \times (\nu - \nu')/(\nu'' - \nu') \ .$$

Here, $F_{160} \approx F_{150} + (F_{175} - F_{150}) \times (160-150)/(175-150) = 3.439$, a value precise to 3 decimal places.

Illustration 4. Determine if a calculated F value corresponding to $F_{22,140}$ is significant at the 5 % (one-tailed) significance level, *i.e.* at $P = 0.95$. The two pages for percentage point 95 show neither $\nu_1 = 22$, nor $\nu_2 = 140$. However, many manners of solution are possible:

a) For $P = .95$, the table furnishes $F_{20,125} = 1.655$ and $F_{25,150} = 1.580$. Therefore, if our test value equals, say, 2.19, we can declare it significant (because it goes beyond the more demanding $F_{20,125}$), whereas if it equals 1.50, it is certainly not significant.

b) If it is required, we can determine approximately $F_{22,140}$ via a double interpolation scheme: first, on ν_1 for each value of ν_2, then on ν_2. Harmonic interpolation is generally superior; we must resort to it anyway with our table if $\nu_1 > 100$ or $\nu_2 > 500$. However, if the interpolation gap is narrow, simple linear interpolation may suffice. For this case, on page 2 for $P = .95$, we find

$F_{20,125} = 1.655$; $F_{20,150} = 1.641$; $F_{25,125} = 1.594$; $F_{25,150} = 1.580$. Using harmonic interpolation, let us first calculate $F_{22,125} \approx 1.6272$ [$= 1.655 + (1.594 - 1.655) \times (22^{-1} - 20^{-1})/(25^{-1} - 20^{-1})$] and $F_{22,150} \approx 1.6133$; then, combining these two interpolated values, one more harmonic interpolation renders $F_{22,140} \approx 1.618$. Linear interpolation would have produced 1.622, the exact value being 1.61885.

c) Lastly, there exists a remarkable normal approximation to F (traced back to Wilson and Hilferty), whose formula is:

$$z = \frac{(1-\mathrm{B})F^{1/3} - (1-\mathrm{A})}{\sqrt{\mathrm{B}F^{2/3} + \mathrm{A}}} \ ,$$

where A = $2/(9\nu_1)$ and B = $2/(9\nu_2)$. The obtained z value is then compared to percentile $z_{[P]}$ in the appropriate table of the standard $N(0,1)$ distribution. For example, in that table, we may read $z_{[.95]} = 1.6449$, whence any value $z \geq 1.6449$ would be declared significant at 5 %. The interpolated value $F_{22,140[.95]} \approx 1.618$, with A = $2/(9\times22)$ and $B = 2/(9\times140)$, $z \approx 1.644$, would (correctly) be deemed non-significant, and $F = 1.619$, giving $z = 1.646$, would be declared significant.

Full examples

Example 1. A researcher wants to compare the effects of three alimentary diets by evaluating weight loss (in kg). He recruits 18 subjects, randomly assigning 6 of them to each group. Here are his summary results.

Diet	n	$\bar{x}$	s^2
1	6	4.40	1.300
2	6	6.00	1.333
3	6	4.00	1.200

Using the 0.05 significance level, can he confirm that the diets have truly different effects? *Solution*: This example illustrates the method of "analysis of variance", or ANOVA, conceived by R. A. Fisher and for which he also derived the so-called F distribution. In the research design presented here, the null hypothesis at stake, $\mathrm{H}_0 : \mu_1 = \mu_2 = \cdots = \mu_k$, postulates that all data come from a common normal population (or, equivalently, from normal populations with identical parameter values). Under that hypothesis, the quotient $F = \mathrm{MS}_{\text{Between-groups}}/\mathrm{MS}_{\text{Within-groups}}$

is distributed as $F_{k-1,\mathrm{N}-k}$, $\mathrm{N} = \sum n_j = 18$; hypothesis H_0 can be rejected, or discredited, if $F > F_{k-1,\mathrm{N}-k[1-\alpha]} = F_{2,15[.95]} = 3.682$. Formulae for calculating the two MS's are:

$$\mathrm{MS_{Between}} = n_\mathrm{h} s^2(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_k)$$

$$\mathrm{MS_{Within}} = \sum(n_j - 1)s_j^2 \;/\; (\mathrm{N} - k)\;.$$

Quantity n_h in $\mathrm{MS_{Between}}$ represents the *harmonic mean* of n_j's, obtained with $n_\mathrm{h} = k/(1/n_1+1/n_2+\cdots+1/n_k)$; when all n_j are equal to n, then $n_\mathrm{h} = n$.

Calculations for the above data are: $n_\mathrm{h} = 6$; $s^2(\bar{x}_1,\bar{x}_2,\ldots) = s^2(4.40;\ 6.00;\ 4.00) \approx 1.120$; $\mathrm{MS_{Between}} = 6\times 1.120 = 6.720$; $\mathrm{MS_{Within}} = [5\times 1.3 + 5\times 1.333 + 5\times 1.2]/[18-3] \approx 1.278$ and, lastly, $F = \mathrm{MS_{Between}}/\mathrm{MS_{Within}} = 6.720/1.278 \approx 5.258$. This value exceeds $F_{2,15[.95]} = 3.682$, and the researcher may reject H_0 and state that the various diets entail different amounts of weight loss.

There exist a very great number of research designs and a corresponding variety of procedures for ANOVA (factorial designs, with or without repeated measures, with nested factors, by latin squares, with confounded interactions, etc.). Also, different calculating formulae are possible [see B. J. Winer, D. R. Brown, & K. M. Michels, "Statistical principles in experimental design" (3rd ed.), 1991, New York, McGraw-Hill; R. E. Kirk, "Experimental design: Procedures for the behavioral sciences" (2nd ed.), 1994, Belmont (CA), Brooks/Cole].

***Example** 2*. In order to investigate a theory on the relationship between gender and abilities, a university psychologist puts forward the idea that mechanical ability is higher in boys than in girls: a test of ability is suitably chosen. With random samples composed of 25 girls and 31 boys, all aged 12 years, she measures each subject and obtains, in brief:

Girls :	$n = 25$	$\bar{x} = 18.56$	$s^2 = 86.816$;
Boys :	$n = 31$	$\bar{x} = 26.43$	$s^2 = 19.431$.

Now, beyond the fact that the boys' mean ability is higher as predicted, our researcher notes that the scores' variance for girls is higher, as if ability levels were sprinkled in a more heterogeneous manner among girls. Is there a difference in scatter of ability between boys and girls? Recall that, anyhow, the t test appropriate for deciding on the difference between means $(\bar{x}_1 - \bar{x}_2)$ subsumes the condition called "homogeneity of variance", which brings us back to a comparison of variances. *Solution*: The null hypothesis to be tested is that s_1^2 and s_2^2 are two sample estimates of the same parametric variance σ^2. Hence, according to hypothesis H_0: $\sigma_1^2 = \sigma_2^2 = \sigma^2$, the quotient s_1^2/s_2^2 is distributed as $F(\nu_1,\nu_2)$, $\nu_1 = n_1 - 1$, $\nu_2 = n_2 - 1$. The appropriate test, the test of homogeneity for two variance estimates, calls for a bilateral criterion; the resulting F quotient will be deemed significant if $F \le F_{\nu_1,\nu_2[1-\alpha/2]}$ or $F \ge F_{\nu_1,\nu_2[\alpha/2]}$. Let us here choose $\alpha = 0{,}10$ and put the greater variance in the numerator (to simplify the location of the critical value). We obtain $86.816/19.431 \approx 4.468$, vs. $F_{24,30[.95]} \approx 1.887$ [approximated through harmonic interpolation, between $F_{20,30} = 1.932$ and $F_{25,30} = 1.878$]. The girls' subpopulation thus appears more heterogeneous than the boys' as regards to mechanical ability: there are very talented girls and other much less talented, the range of variation being broader for girls than for boys. On the other hand, for the purpose of testing her first hypothesis (on $\bar{x}_1 - \bar{x}_2$), the researcher should resort to a different test procedure other than the usual t test, for instance to Welch's t^* procedure (see Winer 1991).

To test the homogeneity hypothesis for $k \ge 2$ variance estimates, other techniques are available, in particular Hartley's $F_{\max}$ test and Cochran's C (see appropriate sections of this book).

Mathematical presentation

The quotient of two variances originating from two independant normal samples is distributed according to the F distribution or, more precisely :

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{\nu_1,\nu_2} ,$$

where ν_1 and ν_2 are the *df*'s associated with variances s_1^2 and s_2^2 respectively, and σ_1^2 and σ_2^2 are the (unknown) true variances of the corresponding normal populations. In fact, the F r.v. corresponds to the quotient of two independent χ^2 (Chi-square) r.v.'s, each one divided by its own *df* :

$$F_{\nu_1,\nu_2} = \frac{\chi_1^2/\nu_1}{\chi_2^2/\nu_2} .$$

When, by hypothesis, we postulate that $\sigma_1^2 = \sigma_2^2$, the quotient s_1^2/s_2^2 allows us to test the homogeneity of variance condition as stipulated for ANOVA. In accordance with its definition, the F variable may be *inverted*, concurrently with a swap of the two *df*'s. Through that inversion, one obtains the lower percentage points of F (e.g. $P = 0.05$) from the higher ones (e.g. $P = 0.95$), using:

$$F_{\nu_1,\nu_2[P]} = 1 \,/\, F_{\nu_2,\nu_1[1-P]} ,$$

where the quantity within brackets designates the centile rank (reduced to range 0-1) of the given *F* value.

Calculation and moments

The p.d.f. of the *F* distribution, for given ν_1 and ν_2, is :

$$p(F) = K_{\nu_1,\nu_2} F^{\nu_1/2-1} \left(1 + \frac{\nu_1}{\nu_2} F\right)^{-\frac{1}{2}(\nu_1+\nu_2)} ,$$

where:

$$K_{\nu_1,\nu_2} = \frac{(\nu_1/\nu_2)^{\nu_1/2}}{B(\nu_1/2,\nu_2/2)} = \frac{(\nu_1/\nu_2)^{\nu_1/2}\,\Gamma([\nu_1+\nu_2]/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} .$$

According to the values of ν_1 and ν_2, this function can be integrated either analytically (by integration by parts) or numerically (with the generalized Simpson's rule, for instance). We thus obtain the d.f. $P(F) = \int_0^F p(x)\,dx = \Pr\{x \le F \mid \nu_1,\nu_2\}$, enabling us to determine percentage points or critical values of *F*. In the formulae for constant $K(\nu_1,\nu_2)$, $B(a,b)$ denotes the *Beta* function and $\Gamma(x)$, the *Gamma* function.

Apart from the possible swapping of *df*'s ν_1 and ν_2, a few special cases of *F* permit an easy computing of the d.f. and critical values. In order to find $F(\nu_1,\nu_2[P])$, we can use:

$\nu_1=1,\ \nu_2=1$: $\tan^2(\frac{1}{2}\pi P)$

$\nu_1=1,\ \nu_2\ge 1$: $t^2_{\nu_2[(1+P)/2]}$

$\nu_1,\ \ \nu_2=\infty$: $\chi^2_{\nu_1[P]}/\nu_1$

$\nu_1=2,\ \nu_2\ge 1$: $\frac{\nu_2}{2}\left[\frac{1}{\sqrt[\nu_2]{(1-P)^2}} - 1\right]$.

In these expressions, t_ν is a Student's *t* variable with ν *df.* Kendall and Stuart (1977) note one more interesting case,

$$\frac{\sqrt{\nu}}{2}\left[\sqrt{F_{\nu,\nu\,[P]}} - \sqrt{1/F_{\nu,\nu\,[P]}}\right] \sim t_{\nu\,[(1+P)/2]} .$$

Finally, except for the relation of F's d.f. with the binomial distribution (see below), exact integration of the F distribution is laborious[1], and we resort readily to numerical methods.

Relationship between *F* and binomial distributions

There exists a working relationship between F's d.f. and the binomial sum of probabilities, through an underlying *Beta* distribution. The respective parameters are, for F : the *df* ν_1, ν_2, and for the binomial : n, x, π ; the binomial distribution pertains to the probability of obtaining x (= 0, 1, ···, n) "successes" out of n independent "trials", the probability of a success being π at each trial (see section on binomial distribution). We illustrate the F - binomial relationship with three categories of applications.

– Case 1 : Is the number of "successes" (x) significant?

Let us suppose a binomial process for n = 10 trials and yielding x = 7 successes. The hypothesized probability of success is π = 0.3. Does the observed rate of success of 7/10 contradict the hypothetical π = 0.3, at the significance level α = 0.01?

To answer this question, we may convert the binomial data (n, x, p) into F data (ν_1, ν_2, F), with:

$$2(n-x+1) \rightarrow \nu_1 \,;\; 2x \rightarrow \nu_2 \,;\; \nu_2(1-\pi)/(\nu_1\pi) \rightarrow F,$$

obtaining F = 4.0833, ν_1 = 8 and ν_2 = 14. The critical value $F_{8,14[.99]}$ is 4.140; thus, our F = 4.0833 is not significant at the 0.01 level, and the proposed probability π = 0.3 remains plausible.

Note that the computed binomial sum for our example is $\Pr\{x \geq 7 \mid 10;\, 0.3\} \approx 0.010592$, a non-significant result. In (binomial) table 3a, we find π^*, the maximum value of π such that $\Pr\{r \geq x \mid n,\pi\} \leq \alpha$. Here, for n = 10, x = 7 and α = 0.01, table 3a gives π^* = 0.2971, our π = 0.3 being too high to obtain significance.

– Case 2 : What is the maximum value π^* such that a number x of successes be sgnificant?

With n trials and x observed successes, the success rate (x/n) will be judged significant only if the individual probability of success (π) is small. In fact, π must lie in the interval (0, π^*), within which $\Pr\{r \geq x \mid n,\pi\} \leq \alpha$; value $\pi = \pi^*$ brings about equality with α. Take an example with n = 12, x = 3 and α = 0.05. Under what value π^* of π does the success rate of 3 out of 12 trials will be found significant at 5 %?

[1] For example, for $\nu_1 = \nu_2 = 3$, $P(F) = \Pr(x \leq F) = K_{3,3}[(F+1)^2\tan^{-1}\sqrt{F} + \sqrt{F}(F-1)]\,/\,(F+1)^2$, where $K_{3,3} \approx$ 2.54648.

We can exploit again the $F(\nu_1,\nu_2)$ distribution, by determining as above $\nu_1 = 2(12-3+1) = 20$ and $\nu_2 = 2\times 3 = 6$. Percentile 95 of $F_{20,6}$ is about 3.874, and $\pi^* = \nu_2/(\nu_2+\nu_1 F) \approx 0.071874$, as can be found also in (binomial) table 3a. Thus, for any probability $\pi \le 0.07187$, the occurence of 3 or more successes in 12 trials is a rare, unusual event, according to the 5 % significance level.

– Case 3 : Is a given *F* value significant?

The binomial-*F* relationship can also be serve to determine the significance of an observed *F* value, with the restriction that both *F*'s ν_1 and ν_2 be even. For instance, let $\nu_1 = 6$, $\nu_2 = 4$ and $F = 6.30$. At the (one-tailed) upper 5 % significance level, does this *F* value reach significance?

In order to answer that question, we first compute:

$$n = \tfrac{1}{2}[\nu_1 + \nu_2 - 2]\,;\; x = \nu_2/2\,;\; \pi = \nu_2/[\nu_2 + \nu_1 F]\ ,$$

obtaining $n = 4$, $x = 2$, $\pi \approx 0.0957$. Then, by direct computation of the binomial sum, $\Pr\{r \ge 2 \mid 4, 0.0957\} \approx 0.0482$, a significant result at $\alpha = 0.05$. Besides, in (binomial) table 3a with $(n,x,\alpha) = (4,2,0.05)$, we read an upper limit $\pi^* = 0.0976$; our $\pi = 0.0957$ falls under this upper limit and, therefore, carries significance.

Normal approximation to the F distribution. Any function for approximating the *F* distribution must take into account its usually marked positive skewness. Based upon the very good Wilson-Hilferty normal approximation to χ^2, the following:

$$z = \frac{(1-\mathrm{B})F^{1/3} - (1-\mathrm{A})}{\sqrt{\mathrm{B}F^{2/3} + \mathrm{A}}}\ ,$$

with $\mathrm{A} = 2/(9\nu_1)$ and $\mathrm{B} = 2/(9\nu_2)$, is approximately distributed as a standard $N(0,1)$ r.v..

Moments of F. The moments and shape coefficients of the *F* distribution depend upon its two parameters, ν_1 and ν_2. They are given below, simpler expressions being obtained when both parameters are equal ($\nu_1 = \nu_2$).

$$\mathrm{E}(F) = \mu = \frac{\nu_2}{\nu_2 - 2}\ ;$$

$$\mathrm{var}(F) = \frac{2\nu_2^2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-2)^2(\nu_2-4)}$$

$$= \frac{4\nu(\nu-1)}{(\nu-2)^2(\nu-4)} \qquad \text{(when } \nu_1 = \nu_2 = \nu\text{)}\ ;$$

$$\gamma_1 = \frac{2\nu_1+\nu_2-2}{\nu_2-6}\cdot\sqrt{\frac{8(\nu_2-4)}{\nu_1(\nu_1+\nu_2-2)}}$$

$$= \frac{2(3\nu-2)}{\nu-6}\cdot\sqrt{\frac{\nu-4}{\nu(\nu-1)}}\ ; \qquad (\text{when } \nu_1 = \nu_2 = \nu)\ ;$$

$$\gamma_2 = \frac{12\left[(\nu_2-2)^2(\nu_2-4)+\nu_1(\nu_1+\nu_2-2)(5\nu_2-22)\right]}{\nu_1(\nu_2-6)(\nu_2-8)(\nu_1+\nu_2-2)}$$

$$= \frac{6(11\nu^3-62\nu^2+64\nu-16)}{\nu(\nu-1)(\nu-6)(\nu-8)}\ . \qquad (\text{when } \nu_1 = \nu_2 = \nu)\ .$$

These moments are determinate when ν_2 (or ν) is sufficient; for instance, var(F) exists if $\nu_2 \geq 5$.

Furthermore, the mode of F, Mo(F), equals $\nu_2(\nu_1-2)/[\nu_1(\nu_2+2)]$; it is used notably in the numerical integration of the p.d.f. and it represents also a lower limit for the median $F_{[0.5]}$. Note that the median satisfies inequalities: Mo(F) < $F_{[0.5]}$ < $\mu(F)$. Moreover, $F_{[0.5]} < 1$ if $\nu_1 < \nu_2$, $F_{[0.5]} = 1$ if $\nu_1 = \nu_2$, and $F_{[0.5]} > 1$ if $\nu_1 > \nu_2$.

For the inquisitive reader, the quotient of two *correlated* variances is not distributed as F. If both variances originate from a normal bivariate population with parametric correlation ρ, the (parametric) correlation coefficient between sample variances, *i.e.* $\rho(s_1^2,s_2^2)$, equals ρ^2. Also, postulating $E(s_1^2) = E(s_2^2)$ as for F, then the quotient s_1^2/s_2^2 has expectation $(\nu-2\rho^2)/(\nu-2)$ and variance $4[\nu(\nu-1)-\rho^2(5\nu-8)](1-\rho^2)/[(\nu-2)^2(\nu-4)]$, where ν is the common parameter of s_1^2 and s_2^2. The p.d.f. is also known. More information, some of it about the comparative validity of tests on the equality of two correlated variances (*see* Supplementary examples, n° 8), is to be found in Laurencelle (2000).

Generation of pseudo random variates

The schema of a program below allows the production of r.v.'s from the $F(\nu_1,\nu_2)$ distribution; it requires a function (designated UNIF) which generates serially r.v.'s from the standard uniform $U(0,1)$ distribution.

Preparation : Let ν_1 and ν_2 (the degrees of freedom)

$a = \nu_1/2$; $b = \nu_2/2$; $s = a + b$;
If min(a,b) ≤ 1 **then** $\beta \leftarrow 1 / \min(a,b)$
else $\beta \leftarrow \sqrt{[(s-2)/(2\times a\times b-s)]}$;
$g \leftarrow a + 1/\beta$; C = ln(4) ≈ 1,3862943611 .

Production : **Repeat** $t \leftarrow$ UNIF ; $u \leftarrow \beta \times \ln[t/(1-t)]$; $w \leftarrow a \times \exp(u)$
Until $s \times \ln[s/(b+w)] + g \times u - C \geq \ln(t^2 \times \text{UNIF})$;
Return $w / a \rightarrow x$.

Remarks :

1. Standard temporal cost : 8.7 à 16.0 × *t*(UNIF), depending on values of ν_1 et ν_2.

2. This method is due to R. C. H. Cheng (1978, *in* Bratley, Fox and Schrage 1987) and it is suitable for any (ν_1, ν_2) combination. There are adaptations of this method, and other methods, that are more efficient for particular values of (ν_1, ν_2), like those in which $\max(\nu_1, \nu_2) < 2$ or $\nu_1 = \nu_2$ (Devroye 1986). The inversion formulae given earlier (for cases where $\nu_1 = \nu_2 = 1$ or $\nu_1 = 2$, $\nu_2 \geq 2$) allow the generation of r.v.'s from the *F* distribution through the simple replacement of *P* by one r.v. from the standard $U(0,1)$ distribution, *i.e.* $u \leftarrow$ UNIF.

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Studentized range (q) distribution

✓ Percentiles 95 and 99 of the Studentized range (q)
✓ Reading off the table
✓ Full examples
✓ Mathematical presentation
Calculation and moments

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Percentile 95 of the Studentized range (q)

$\nu \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	6.085	8.331	9.798	10.88	11.73	12.43	13.03	13.54	13.99	14.39	14.75	15.08	15.37	15.65
3	4.501	5.910	6.824	7.502	8.037	8.478	8.853	9.176	9.462	9.716	9.946	10.15	10.35	10.52
4	3.926	5.040	5.757	6.287	6.706	7.052	7.346	7.601	7.826	8.027	8.208	8.373	8.524	8.664
5	3.635	4.602	5.218	5.673	6.033	6.330	6.582	6.801	6.995	7.167	7.323	7.465	7.596	7.716
6	3.460	4.339	4.896	5.305	5.628	5.895	6.122	6.319	6.493	6.649	6.789	6.917	7.034	7.143
7	3.344	4.165	4.681	5.060	5.359	5.606	5.815	5.997	6.158	6.302	6.431	6.550	6.658	6.759
8	3.261	4.041	4.529	4.886	5.167	5.399	5.596	5.767	5.918	6.053	6.175	6.286	6.389	6.483
9	3.199	3.949	4.415	4.755	5.024	5.244	5.432	5.595	5.738	5.867	5.983	6.089	6.186	6.276
10	3.151	3.877	4.327	4.654	4.912	5.124	5.304	5.460	5.598	5.722	5.833	5.935	6.028	6.114
11	3.113	3.820	4.256	4.574	4.823	5.028	5.202	5.353	5.486	5.605	5.713	5.811	5.901	5.984
12	3.081	3.773	4.199	4.508	4.750	4.950	5.119	5.265	5.395	5.510	5.615	5.710	5.797	5.878
13	3.055	3.734	4.151	4.453	4.690	4.884	5.049	5.192	5.318	5.431	5.533	5.625	5.710	5.789
14	3.033	3.701	4.111	4.407	4.639	4.829	4.990	5.130	5.253	5.364	5.463	5.554	5.637	5.714
15	3.014	3.673	4.076	4.367	4.595	4.782	4.940	5.077	5.198	5.306	5.403	5.492	5.574	5.649
16	2.998	3.649	4.046	4.333	4.557	4.741	4.896	5.031	5.150	5.256	5.352	5.439	5.519	5.593
17	2.984	3.628	4.020	4.303	4.524	4.705	4.858	4.991	5.108	5.212	5.306	5.392	5.471	5.544
18	2.971	3.609	3.997	4.276	4.494	4.673	4.824	4.955	5.071	5.173	5.266	5.351	5.429	5.501
19	2.960	3.593	3.977	4.253	4.468	4.645	4.794	4.924	5.037	5.139	5.231	5.314	5.391	5.462
20	2.950	3.578	3.958	4.232	4.445	4.620	4.768	4.895	5.008	5.108	5.199	5.282	5.357	5.427
22	2.933	3.553	3.927	4.196	4.405	4.577	4.722	4.847	4.957	5.056	5.144	5.225	5.299	5.368
24	2.919	3.532	3.901	4.166	4.373	4.541	4.684	4.807	4.915	5.012	5.099	5.179	5.251	5.319
26	2.907	3.514	3.880	4.141	4.345	4.511	4.652	4.773	4.880	4.975	5.061	5.139	5.211	5.277
28	2.897	3.499	3.861	4.120	4.322	4.486	4.625	4.745	4.850	4.944	5.029	5.106	5.177	5.242
30	2.888	3.486	3.845	4.102	4.301	4.464	4.601	4.720	4.824	4.917	5.001	5.077	5.147	5.211
32	2.881	3.475	3.832	4.086	4.284	4.445	4.581	4.698	4.802	4.894	4.976	5.052	5.121	5.185
34	2.874	3.465	3.820	4.072	4.268	4.428	4.563	4.680	4.782	4.873	4.955	5.030	5.098	5.161
36	2.868	3.457	3.809	4.060	4.255	4.414	4.547	4.663	4.764	4.855	4.936	5.010	5.078	5.141
38	2.863	3.449	3.799	4.049	4.243	4.400	4.533	4.648	4.749	4.838	4.919	4.993	5.060	5.122
40	2.858	3.442	3.791	4.039	4.232	4.388	4.521	4.634	4.735	4.824	4.904	4.977	5.044	5.106
42	2.854	3.436	3.783	4.030	4.222	4.378	4.509	4.622	4.722	4.810	4.890	4.963	5.029	5.091
44	2.850	3.430	3.776	4.022	4.213	4.368	4.499	4.611	4.710	4.798	4.878	4.950	5.016	5.077
46	2.847	3.425	3.770	4.015	4.205	4.359	4.489	4.601	4.700	4.787	4.866	4.938	5.004	5.065
48	2.843	3.420	3.764	4.008	4.197	4.351	4.481	4.592	4.690	4.777	4.856	4.927	4.993	5.053
50	2.841	3.416	3.758	4.002	4.190	4.344	4.473	4.584	4.681	4.768	4.847	4.918	4.983	5.043
55	2.834	3.407	3.747	3.989	4.176	4.328	4.455	4.566	4.662	4.748	4.826	4.896	4.961	5.020
60	2.829	3.399	3.737	3.977	4.163	4.314	4.441	4.550	4.646	4.732	4.808	4.878	4.942	5.001
65	2.824	3.392	3.729	3.968	4.153	4.303	4.429	4.538	4.633	4.718	4.794	4.863	4.927	4.985
70	2.821	3.386	3.722	3.960	4.144	4.293	4.419	4.527	4.621	4.706	4.781	4.850	4.913	4.971
75	2.817	3.382	3.716	3.953	4.136	4.285	4.410	4.517	4.611	4.695	4.771	4.839	4.902	4.960
80	2.814	3.377	3.711	3.947	4.129	4.278	4.402	4.509	4.603	4.686	4.761	4.829	4.892	4.949
85	2.812	3.374	3.706	3.942	4.124	4.271	4.395	4.502	4.595	4.678	4.753	4.821	4.883	4.940
90	2.810	3.370	3.702	3.937	4.118	4.265	4.389	4.495	4.588	4.671	4.746	4.813	4.875	4.932
95	2.808	3.367	3.698	3.933	4.114	4.260	4.383	4.489	4.582	4.665	4.739	4.806	4.868	4.925
100	2.806	3.365	3.695	3.929	4.109	4.256	4.379	4.484	4.577	4.659	4.733	4.800	4.862	4.918
125	2.799	3.354	3.683	3.915	4.093	4.238	4.360	4.465	4.556	4.637	4.711	4.777	4.838	4.894
150	2.794	3.348	3.674	3.905	4.083	4.227	4.348	4.451	4.542	4.623	4.696	4.762	4.822	4.877
175	2.791	3.343	3.668	3.898	4.075	4.219	4.339	4.442	4.533	4.613	4.685	4.751	4.811	4.866
200	2.789	3.339	3.664	3.893	4.069	4.212	4.332	4.435	4.525	4.605	4.677	4.742	4.802	4.857
250	2.785	3.334	3.658	3.886	4.062	4.204	4.323	4.425	4.515	4.595	4.666	4.731	4.790	4.845
∞	2.772	3.315	3.633	3.858	4.030	4.170	4.286	4.387	4.474	4.552	4.622	4.685	4.743	4.796

Percentile 99 of the Studentized range (q)

$\nu\backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	14.04	19.02	22.30	24.72	26.63	28.20	29.53	30.68	31.69	32.59	33.40	34.13	34.81	35.43
3	8.260	10.62	12.17	13.32	14.24	15.00	15.64	16.20	16.69	17.13	17.52	17.88	18.22	18.52
4	6.511	8.119	9.172	9.958	10.58	11.10	11.54	11.92	12.26	12.57	12.84	13.09	13.32	13.53
5	5.702	6.976	7.804	8.421	8.913	9.321	9.669	9.971	10.24	10.48	10.70	10.89	11.08	11.24
6	5.243	6.331	7.034	7.556	7.973	8.318	8.613	8.870	9.097	9.301	9.485	9.654	9.808	9.951
7	4.949	5.919	6.543	7.005	7.373	7.679	7.939	8.166	8.368	8.548	8.711	8.860	8.998	9.125
8	4.745	5.635	6.204	6.625	6.959	7.237	7.474	7.680	7.863	8.027	8.175	8.311	8.436	8.551
9	4.596	5.428	5.957	6.347	6.658	6.915	7.134	7.325	7.495	7.647	7.784	7.910	8.025	8.133
10	4.482	5.270	5.769	6.136	6.428	6.669	6.875	7.054	7.213	7.356	7.485	7.603	7.712	7.812
11	4.392	5.146	5.620	5.970	6.246	6.476	6.671	6.841	6.992	7.127	7.249	7.361	7.464	7.559
12	4.320	5.046	5.502	5.837	6.102	6.321	6.508	6.670	6.814	6.943	7.060	7.167	7.265	7.356
13	4.260	4.964	5.404	5.726	5.981	6.192	6.372	6.528	6.667	6.791	6.903	7.006	7.100	7.188
14	4.210	4.895	5.322	5.634	5.881	6.085	6.258	6.410	6.543	6.663	6.772	6.871	6.962	7.047
15	4.167	4.836	5.252	5.556	5.796	5.994	6.162	6.309	6.439	6.555	6.660	6.756	6.845	6.927
16	4.131	4.786	5.192	5.489	5.722	5.915	6.079	6.222	6.348	6.462	6.564	6.658	6.744	6.823
17	4.099	4.742	5.140	5.430	5.659	5.847	6.007	6.147	6.270	6.381	6.480	6.572	6.656	6.734
18	4.071	4.703	5.094	5.379	5.603	5.787	5.944	6.081	6.201	6.309	6.407	6.496	6.579	6.655
19	4.046	4.669	5.054	5.334	5.553	5.735	5.889	6.022	6.141	6.247	6.342	6.430	6.510	6.585
20	4.024	4.639	5.018	5.293	5.510	5.688	5.839	5.970	6.087	6.191	6.285	6.371	6.450	6.523
22	3.986	4.588	4.957	5.225	5.435	5.608	5.754	5.882	5.994	6.095	6.186	6.269	6.346	6.417
24	3.955	4.546	4.907	5.168	5.373	5.542	5.685	5.809	5.919	6.017	6.106	6.186	6.261	6.330
26	3.930	4.510	4.865	5.121	5.322	5.487	5.627	5.749	5.856	5.951	6.038	6.117	6.190	6.257
28	3.908	4.481	4.830	5.082	5.279	5.441	5.578	5.697	5.802	5.896	5.981	6.058	6.130	6.195
30	3.889	4.455	4.799	5.048	5.242	5.401	5.536	5.653	5.756	5.849	5.932	6.008	6.078	6.142
32	3.873	4.433	4.773	5.018	5.210	5.367	5.500	5.615	5.716	5.807	5.889	5.964	6.033	6.096
34	3.859	4.413	4.750	4.992	5.181	5.336	5.468	5.581	5.682	5.771	5.852	5.926	5.994	6.056
36	3.846	4.396	4.729	4.969	5.156	5.310	5.440	5.552	5.651	5.739	5.819	5.892	5.959	6.021
38	3.835	4.381	4.711	4.949	5.134	5.286	5.414	5.526	5.623	5.711	5.790	5.862	5.928	5.989
40	3.825	4.367	4.695	4.931	5.114	5.265	5.392	5.502	5.599	5.686	5.764	5.835	5.900	5.961
42	3.816	4.355	4.681	4.914	5.097	5.246	5.372	5.481	5.577	5.663	5.740	5.811	5.875	5.935
44	3.807	4.344	4.667	4.900	5.080	5.228	5.354	5.462	5.557	5.642	5.719	5.789	5.853	5.912
46	3.800	4.334	4.655	4.886	5.066	5.213	5.337	5.444	5.539	5.623	5.699	5.769	5.832	5.891
48	3.793	4.324	4.644	4.874	5.052	5.198	5.322	5.428	5.522	5.606	5.682	5.750	5.814	5.872
50	3.787	4.316	4.634	4.863	5.040	5.185	5.308	5.414	5.507	5.590	5.665	5.734	5.796	5.854
55	3.773	4.297	4.612	4.838	5.013	5.157	5.278	5.382	5.474	5.556	5.630	5.697	5.759	5.816
60	3.762	4.282	4.594	4.818	4.991	5.133	5.253	5.356	5.447	5.528	5.601	5.667	5.728	5.785
65	3.753	4.269	4.579	4.801	4.973	5.113	5.231	5.334	5.424	5.504	5.576	5.642	5.702	5.758
70	3.745	4.258	4.566	4.786	4.957	5.096	5.214	5.315	5.404	5.483	5.555	5.620	5.680	5.735
75	3.738	4.249	4.555	4.774	4.943	5.081	5.198	5.299	5.387	5.466	5.537	5.602	5.661	5.716
80	3.732	4.241	4.545	4.763	4.931	5.069	5.185	5.284	5.372	5.451	5.521	5.585	5.644	5.699
85	3.726	4.233	4.537	4.753	4.921	5.057	5.173	5.272	5.359	5.437	5.507	5.571	5.630	5.683
90	3.722	4.227	4.529	4.745	4.911	5.047	5.162	5.261	5.348	5.425	5.495	5.558	5.616	5.670
95	3.717	4.221	4.522	4.737	4.903	5.039	5.153	5.251	5.338	5.415	5.484	5.547	5.605	5.658
100	3.714	4.216	4.516	4.730	4.896	5.031	5.144	5.242	5.328	5.405	5.474	5.537	5.594	5.648
125	3.699	4.197	4.493	4.704	4.868	5.000	5.112	5.209	5.293	5.369	5.437	5.498	5.555	5.607
150	3.690	4.184	4.478	4.687	4.849	4.980	5.091	5.187	5.270	5.345	5.412	5.473	5.529	5.580
175	3.683	4.175	4.467	4.675	4.836	4.966	5.076	5.171	5.254	5.328	5.394	5.455	5.510	5.561
200	3.678	4.168	4.459	4.666	4.826	4.956	5.065	5.159	5.242	5.315	5.381	5.441	5.496	5.547
250	3.671	4.158	4.448	4.653	4.812	4.941	5.049	5.143	5.224	5.297	5.363	5.422	5.477	5.527
∞	3.643	4.120	4.403	4.603	4.757	4.882	4.987	5.077	5.157	5.227	5.290	5.348	5.400	5.449

Reading off the table

The table furnishes the 95th and 99th percentiles of q, the Studentized range, for sets of (k =) 2 to 15 data to be compared and degrees of freedom (df) ν from 2 to infinity. The Studentized range test for k means admits only right-tailed rejection. The quantity $q_{k,\nu}$ is a quotient obtained by dividing the range of k standard normal r.v.'s by an independent estimate of standard deviation having ν df. It is used in some multiple comparison procedures, concurrent to or following analysis of variance (ANOVA).

Illustration 1. What is the value of $q_{4,20[.95]}$? Looking at the page for percentile 95, column $k = 4$ and line $\nu = 20$, we read off $q = 3.958$.

Illustration 2. Find $q_{10,96[.99]}$ and $q_{6,400[.95]}$. For the first case, with $P = 0.99$ and $k = 10$, $df = 96$ lies between $\nu = 95$ and $\nu = 100$: we can interpolate on $q_{10,95[.99]} = 5.338$ and $q_{10,100[.99]} = 5.328$. The 95-100 interval being narrow, linear interpolation is sufficient; for $\nu = 96$, $q \approx 5.338 + (5.328-5.338)\times(96-95)/(100-95) \approx 5.336$. For the second case, $\nu = 400$ sits between 250 and infinity, which commands harmonic interpolation. Using $q_{6,250[.95]}$=4.062 and $q_{6,\infty[.99]}$=4.030, we compute $q \approx 4.062+(4.030-4.062)\times(400^{-1}-250^{-1})/(1/\infty-250^{-1}) \approx 4.050$.

Full examples

Example 1 *- Tukey's HSD (Honestly Significant Difference) method.* An experimentation using $k = 5$ groups of randomly assigned participants, $n = 10$ per group, produced the following results:

Group	Mean	S.d.
1	22.63	3.17
2	18.90	2.86
3	26.14	2.59
4	23.35	3.06
5	19.31	2.77

Using the HSD method, are there any differences between means, at the $\alpha = .05$ significance threshold? Which means differ one from the other? *Solution*: The HSD method makes use of a unique critical value, *i.e.* percentile $q_{k,\nu[P]}$, to assess all possible differences between two means, with the test statistic:

$$q = \frac{\bar{x}_i - \bar{x}_j}{\sqrt{\frac{\text{MS}_{\text{error}}}{n}}} \ .$$

Quantity MS_{error} under the square root designates the residual or *error* Mean square against which the set of means are to be assessed, and n is the common sample size. In cases where the k groups are of varying sizes n_j, authors suggest the computation and use of the harmonic mean, n_h. In our example, MS_{error} is the common within-groups variance estimate, as obtained from each group's standard deviation. We thus obtain $\text{MS}_{\text{error}} = [3.17^2+2.86^2+2.59^2+3.06^2+2.77^2]/5 \approx 8.395$. The test statistic is: $q = (\bar{x}_i - \bar{x}_j)/\sqrt{[8.395/10]} \approx (\bar{x}_i - \bar{x}_j)/0.9162$. At the $\alpha = 0.05$ threshold, the Tukey HSD criterion equals $q_{5,45[.95]} \approx 4.019$ (after interpolation between 4.022 and 4.015), so that any quotient $q \geq 4.019$ would be judged significant. For instance, the difference between groups 1 and 2 gives $q = (22.63 - 18.90)/0.9162 \approx 4.071$, exceeding the agreed-upon critical value. In fact, every mean-to-mean difference higher than $0.9162 \times 4.019 \approx 3.682$ (in absolute value) would be declared significant, which occurs for pairs of means: 1-2, 2-3, 2-4, 3-5 and 4-5.

Example 2 *- Newman-Keuls' method.* Statistical literature reports another popular procedure for multiple comparisons of means, Newman-Keuls's method, where the criterion for judging an observed difference $(\bar{x}_i - \bar{x}_j)$ changes according to the *ordinal distance* between the compared means. Thus, once the means are arrayed in order of decreasing values, if one has rank r_i and another, rank r_j, distance d is established as $d_{i,j} = |r_i - r_j| + 1$. Distance d runs from 2 to k, this last case pertaining to the difference between the two most extreme means, *i.e.* the range of the k means. In this setting, Newman-Keuls' procedure uses the same formulae as HSD's, but it exploits a modified criterion for each comparison, *i.e.* critical value $q_{d,\nu[P]}$.

Let us illustrate the Newman-Keuls' method with the same data as in example 1. In order of decreasing values, the $k = 5$ means are { 26.4, 23.35, 22.63, 19.31, 18.90 }, in concordance with group numbers { 3, 4, 1, 5, 2 }. The different critical values applicable here for 0.05 α-level are $q_{2,45[.95]} \approx 2.849$ (via linear interpolation), $q_{3,45[.95]} \approx 3.428$, $q_{4,45[.95]} \approx 3.773$ and $q_{5,45[.95]} \approx 4.019$. For group pair 3-4, $q = (26.14-23.35)/0.9162$ [see example 1] must be judged against $q_{2,45[.95]} = 2.849$ and is thus non-significant. The systematic application of the method uncovers the following significant differences: $3\text{-}4_{(d=2)}$, $1\text{-}5_{(d=2)}$, $3\text{-}1_{(d=3)}$, $4\text{-}5_{(d=3)}$, $1\text{-}2_{(d=3)}$, $3\text{-}5_{(d=4)}$, $4\text{-}2_{(d=4)}$ et $3\text{-}2_{(d=5)}$.

Mathematical presentation

The Studentized range distribution, with variable q, is used by procedures for deciding if elements from a set of k variables (or means of variables) originate or not from a common normal population: such procedures complement or, occasionally, substitute for ANOVA.

Consider a sample $\{x_1, x_2, \cdots, x_k\}$ from a normal $N(\mu,\sigma^2)$ population, and s^2, an independent estimate of σ^2 with ν *df.* In this context:

$$q = \frac{\max(x_i) - \min(x_i)}{s}$$

is a Studentized (or standardized) range q, with parameters k and ν. In ANOVA, the x_i usually represent means, each one being based on n individual observations, and the variance estimate is a mean square (MS), e.g. the "error" (MS_{error}) or "within-groups" ($MS_{w\text{-}g}$) mean square, pertaining to a one-way ANOVA design with k independent groups. The test statistic for this typical setup would be:

$$q = \frac{\max(\bar{x}_i) - \min(\bar{x}_i)}{\sqrt{\dfrac{MS_{error}}{n}}} \ ;$$

the obtained values should be compared to percentile $100(1-\alpha)$ of q, *i.e.* $q_{k,\nu[1-\alpha]}$. In cases where the two compared means have unequal sizes n_j and $n_{j'}$, the "n" in the above formula can be replaced with the harmonic mean, $n_h = 2/(n_j^{-1}+n_{j'}^{-1})$, lending a handy approximation.

Calculation and moments

The range $\max(x_i)-\min(x_i)$ and s.d. s are statistically independent[1] one from the other, so that the p.d.f. for their quotient q can be obtained by convoluting their respective p.d.f.'s, *i.e.*:

$$p(q) = \int_0^{\infty} f(u) \left[\int_{-\infty}^{\infty} g_u(r)\, dr \right] du \ .$$

[1] This independence arises from their separate origins. For instance, the "x_i" could represent each one of the k sample averages, while "s^2" be a pooled within-sample variance estimate.

The external density $f(u)$ portrays the variation of the s.d.'s estimate, with parameter ν:

$$f(u) = \frac{2(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)} u^{\nu-1} e^{-\nu \cdot u^2/2} ,$$

and $g(r)$, the inside density conditioned on u, portrays the variation of the range of k standard normal r.v.'s:

$$g_u(r) = k\{\Phi(r) - \Phi(r-qu)\}^{k-1} \varphi(r) .$$

In these expressions, Φ designates the standard normal d.f., φ its corresponding p.d.f., and $\Gamma(x)$ is the *Gamma* function.

Taking the normal d.f. as a simple function, density $p(q)$ appears as a double integral, whose shape changes seriously across different combinations of the $\{k,\nu\}$ parameters. Let us point out, for instance, that p.d.f. $f(s)$, given above, is non-modal for $\nu = 1$, much spread out and unimodal for intermediate values of ν, the mode being located at $s = \sqrt{(1-1/\nu)}$, then more pointed for higher values of ν. Percentiles 95 and 99 have been determined on the basis of a double integration using Simpson's rule. For $k = 2$, the equivalence with Student's t, $q_{2,\nu[P]} = \sqrt{2}\, t_{\nu[½(1+P)]}$, has been used.

Moments. Moments of the Studentized range variable q are generally not known. It is theoretically possible to compute their value with the double or triple integral mentioned above. However, q being the product of two statistically independent r.v.'s, *i.e.* $q = R_k \times 1/s_\nu$, we can obtain its moments through its factors's moments, specifically:

$$\mu(X \cdot Y) = \mu_X \cdot \mu_Y$$

and :

$$\text{var}(X \cdot Y) = \mu_Y^2 \cdot \text{var}(X) + \mu_X^2 \cdot \text{var}(Y) + \text{var}(X) \cdot \text{var}(Y) .$$

The distribution of R_k, the range of k standard $N(0,1)$ variables, is known and well documented; such is not the case for $1/s_\nu$, the reciprocal of a normal s.d. s. Nonetheless, the p.d.f. of $y = 1/s_\nu$ can easily be obtained from that of the s.d., $f(s)$[2]. For illustrative purposes, we present, on the next page, some calculated values of the expectation and variance of q. Note that the variance of $1/s$ for $\nu = 2$ is not determinate, hence its absence for q.

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[2] Let $y = 1/s$, where s is a sample s.d. based on $\nu+1$ standard normal data and having ν *df.* The p.d.f. of y is then $K_\nu y^{-\nu-1}\exp(-½\nu/y^2)$, K_ν being the same multiplicative factor as for p.d.f. $f(s)$, and y ranging from infinity to zero (as s ranges from zero to infinity).

Expectations (normal characters) and variances (in italics) of q, the Studentized range statistic, for some combinations of parameters k (number of groups) and ν (degrees of freedom)

$\nu \setminus k$	2	3	5	10	20
2	2.000	3.000	4.122	5.454	6.619
3	1.559	2.339	3.215	4.253	5.162
	3.570	*5.494*	*8.144*	*12.241*	*16.815*
5	1.342	2.013	2.767	3.660	4.442
	1.532	*2.037*	*2.608*	*3.446*	*4.400*
10	1.223	1.834	2.521	3.335	4.048
	1.005	*1.203*	*1.342*	*1.509*	*1.717*
20	1.173	1.759	2.418	3.199	3.883
	0.846	*0.964*	*0.994*	*0.994*	*1.014*

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Dunnett's *t* distribution

- ✓ Percentiles 90, 95 and 99 of the one-tailed Dunnett's *t* distribution (table 1)
- ✓ Percentiles 90, 95 and 99 of the two-tailed Dunnett's *t* distribution (table 2)
- ✓ Reading off the table
- ✓ Full example
- ✓ Mathematical presentation

  Calculation and moments

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Percentile 90 of the one-tailed Dunnett's t distribution (table 1)

$\nu \backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	1.886	2.538	2.924	3.195	3.403	3.572	3.712	3.833	3.938	4.032	4.115	4.191	4.260	4.324	4.383
3	1.638	2.130	2.411	2.607	2.755	2.875	2.975	3.060	3.134	3.200	3.260	3.313	3.362	3.407	3.449
4	1.533	1.963	2.204	2.371	2.497	2.598	2.682	2.754	2.817	2.873	2.922	2.967	3.008	3.046	3.081
5	1.476	1.873	2.094	2.245	2.359	2.451	2.527	2.592	2.649	2.699	2.744	2.784	2.821	2.855	2.887
6	1.440	1.817	2.025	2.167	2.274	2.360	2.431	2.492	2.545	2.592	2.633	2.671	2.706	2.737	2.767
7	1.415	1.779	1.978	2.114	2.217	2.298	2.366	2.424	2.474	2.519	2.559	2.595	2.627	2.658	2.685
8	1.397	1.751	1.944	2.076	2.175	2.254	2.319	2.375	2.424	2.466	2.505	2.539	2.571	2.600	2.627
9	1.383	1.730	1.919	2.047	2.143	2.220	2.284	2.338	2.385	2.427	2.464	2.498	2.528	2.556	2.582
10	1.372	1.713	1.899	2.024	2.119	2.194	2.256	2.309	2.355	2.396	2.432	2.465	2.495	2.522	2.548
11	1.363	1.700	1.882	2.006	2.099	2.173	2.234	2.286	2.331	2.371	2.406	2.439	2.468	2.495	2.520
12	1.356	1.689	1.869	1.991	2.082	2.155	2.215	2.267	2.311	2.350	2.385	2.417	2.446	2.472	2.497
13	1.350	1.680	1.858	1.978	2.069	2.141	2.200	2.251	2.294	2.333	2.368	2.399	2.427	2.454	2.478
14	1.345	1.672	1.849	1.968	2.057	2.128	2.187	2.237	2.280	2.319	2.353	2.384	2.412	2.438	2.461
15	1.341	1.666	1.841	1.959	2.047	2.118	2.176	2.225	2.268	2.306	2.340	2.371	2.398	2.424	2.448
16	1.337	1.660	1.833	1.951	2.039	2.108	2.166	2.215	2.258	2.295	2.329	2.359	2.387	2.412	2.435
17	1.333	1.655	1.827	1.944	2.031	2.100	2.158	2.206	2.249	2.286	2.319	2.349	2.377	2.402	2.425
18	1.330	1.650	1.822	1.938	2.024	2.093	2.150	2.198	2.240	2.277	2.311	2.340	2.368	2.392	2.415
19	1.328	1.646	1.817	1.932	2.018	2.087	2.143	2.191	2.233	2.270	2.303	2.333	2.360	2.384	2.407
20	1.325	1.643	1.813	1.927	2.013	2.081	2.137	2.185	2.227	2.263	2.296	2.325	2.352	2.377	2.400
22	1.321	1.636	1.805	1.919	2.004	2.071	2.127	2.175	2.216	2.252	2.284	2.313	2.340	2.364	2.387
24	1.318	1.631	1.799	1.912	1.996	2.063	2.119	2.166	2.206	2.242	2.274	2.303	2.330	2.354	2.376
26	1.315	1.627	1.794	1.906	1.990	2.057	2.112	2.158	2.199	2.234	2.266	2.295	2.321	2.345	2.367
28	1.313	1.623	1.789	1.901	1.985	2.051	2.105	2.152	2.192	2.228	2.259	2.288	2.314	2.338	2.360
30	1.310	1.620	1.786	1.897	1.980	2.046	2.100	2.146	2.186	2.222	2.253	2.282	2.308	2.331	2.353
32	1.309	1.617	1.782	1.893	1.976	2.041	2.096	2.142	2.181	2.217	2.248	2.276	2.302	2.326	2.348
34	1.307	1.615	1.779	1.890	1.972	2.038	2.092	2.137	2.177	2.212	2.243	2.272	2.297	2.321	2.343
36	1.306	1.613	1.777	1.887	1.969	2.034	2.088	2.134	2.173	2.208	2.239	2.267	2.293	2.316	2.338
38	1.304	1.611	1.774	1.884	1.966	2.031	2.085	2.130	2.170	2.205	2.236	2.264	2.289	2.313	2.334
40	1.303	1.609	1.772	1.882	1.964	2.028	2.082	2.127	2.167	2.201	2.232	2.260	2.286	2.309	2.331
42	1.302	1.608	1.770	1.880	1.961	2.026	2.079	2.125	2.164	2.199	2.229	2.257	2.283	2.306	2.327
44	1.301	1.606	1.769	1.878	1.959	2.024	2.077	2.122	2.161	2.196	2.227	2.255	2.280	2.303	2.324
46	1.300	1.605	1.767	1.876	1.957	2.022	2.075	2.120	2.159	2.194	2.224	2.252	2.277	2.300	2.322
48	1.299	1.604	1.766	1.874	1.956	2.020	2.073	2.118	2.157	2.191	2.222	2.250	2.275	2.298	2.319
50	1.299	1.603	1.764	1.873	1.954	2.018	2.071	2.116	2.155	2.189	2.220	2.248	2.273	2.296	2.317
55	1.297	1.600	1.762	1.870	1.951	2.014	2.067	2.112	2.151	2.185	2.216	2.243	2.268	2.291	2.312
60	1.296	1.598	1.759	1.867	1.948	2.011	2.064	2.109	2.147	2.181	2.212	2.239	2.264	2.287	2.308
65	1.295	1.597	1.757	1.865	1.945	2.009	2.061	2.106	2.144	2.178	2.209	2.236	2.261	2.284	2.305
70	1.294	1.595	1.756	1.863	1.943	2.007	2.059	2.103	2.142	2.176	2.206	2.233	2.258	2.281	2.302
75	1.293	1.594	1.754	1.861	1.941	2.005	2.057	2.101	2.140	2.173	2.204	2.231	2.256	2.278	2.299
80	1.292	1.593	1.753	1.860	1.940	2.003	2.055	2.099	2.138	2.171	2.202	2.229	2.254	2.276	2.297
85	1.292	1.592	1.752	1.859	1.938	2.001	2.054	2.098	2.136	2.170	2.200	2.227	2.252	2.274	2.295
90	1.291	1.591	1.751	1.857	1.937	2.000	2.052	2.096	2.135	2.168	2.198	2.225	2.250	2.273	2.293
95	1.291	1.590	1.750	1.856	1.936	1.999	2.051	2.095	2.133	2.167	2.197	2.224	2.248	2.271	2.292
100	1.290	1.590	1.749	1.856	1.935	1.998	2.050	2.094	2.132	2.166	2.196	2.223	2.247	2.270	2.290
125	1.288	1.587	1.746	1.852	1.931	1.994	2.046	2.089	2.127	2.161	2.191	2.218	2.242	2.265	2.285
150	1.287	1.586	1.744	1.850	1.929	1.991	2.043	2.087	2.124	2.158	2.187	2.214	2.239	2.261	2.282
175	1.286	1.584	1.742	1.848	1.927	1.989	2.041	2.084	2.122	2.155	2.185	2.212	2.236	2.259	2.279
200	1.286	1.583	1.741	1.847	1.926	1.988	2.039	2.083	2.121	2.154	2.183	2.210	2.235	2.257	2.277
250	1.285	1.582	1.740	1.845	1.924	1.986	2.037	2.081	2.118	2.151	2.181	2.208	2.232	2.254	2.275
∞	1.282	1.577	1.734	1.838	1.916	1.978	2.029	2.072	2.109	2.142	2.172	2.198	2.222	2.244	2.264

Percentile 95 of the one-tailed Dunnett's *t* distribution (table 1, cont.)

$\nu \backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2.920	3.805	4.336	4.712	5.002	5.237	5.434	5.603	5.751	5.882	6.000	6.106	6.204	6.293	6.379
3	2.353	2.938	3.279	3.518	3.702	3.849	3.973	4.079	4.172	4.254	4.328	4.394	4.456	4.512	4.564
4	2.132	2.611	2.885	3.076	3.221	3.339	3.437	3.521	3.594	3.659	3.718	3.770	3.819	3.863	3.904
5	2.015	2.440	2.682	2.848	2.976	3.078	3.163	3.236	3.300	3.356	3.407	3.453	3.495	3.533	3.569
6	1.943	2.336	2.558	2.711	2.827	2.920	2.998	3.064	3.122	3.174	3.220	3.261	3.299	3.334	3.367
7	1.895	2.267	2.476	2.619	2.728	2.815	2.888	2.950	3.004	3.052	3.095	3.134	3.169	3.202	3.232
8	1.860	2.217	2.416	2.553	2.657	2.740	2.809	2.868	2.919	2.965	3.006	3.043	3.076	3.107	3.136
9	1.833	2.180	2.372	2.504	2.604	2.684	2.750	2.807	2.856	2.900	2.939	2.974	3.007	3.037	3.064
10	1.812	2.151	2.338	2.466	2.562	2.640	2.704	2.759	2.807	2.849	2.887	2.921	2.953	2.982	3.008
11	1.796	2.127	2.310	2.435	2.529	2.605	2.668	2.721	2.768	2.809	2.846	2.879	2.910	2.938	2.964
12	1.782	2.108	2.287	2.410	2.502	2.576	2.638	2.690	2.736	2.776	2.812	2.845	2.874	2.902	2.927
13	1.771	2.092	2.269	2.389	2.480	2.552	2.613	2.664	2.709	2.748	2.784	2.816	2.845	2.872	2.897
14	1.761	2.079	2.253	2.371	2.461	2.532	2.592	2.642	2.686	2.725	2.760	2.792	2.820	2.847	2.871
15	1.753	2.067	2.239	2.356	2.445	2.515	2.574	2.624	2.667	2.705	2.740	2.771	2.799	2.825	2.849
16	1.746	2.057	2.227	2.343	2.430	2.500	2.558	2.607	2.650	2.688	2.722	2.753	2.781	2.806	2.830
17	1.740	2.048	2.217	2.332	2.418	2.487	2.544	2.593	2.636	2.673	2.707	2.737	2.765	2.790	2.813
18	1.734	2.041	2.208	2.322	2.407	2.476	2.532	2.581	2.623	2.660	2.693	2.723	2.750	2.776	2.799
19	1.729	2.034	2.200	2.313	2.398	2.465	2.522	2.569	2.611	2.648	2.681	2.711	2.738	2.763	2.786
20	1.725	2.027	2.192	2.305	2.389	2.456	2.512	2.560	2.601	2.637	2.670	2.700	2.727	2.751	2.774
22	1.717	2.017	2.180	2.291	2.374	2.440	2.496	2.542	2.583	2.619	2.652	2.681	2.707	2.732	2.754
24	1.711	2.008	2.170	2.279	2.362	2.428	2.482	2.528	2.569	2.604	2.636	2.665	2.691	2.715	2.738
26	1.706	2.001	2.161	2.270	2.352	2.417	2.471	2.517	2.557	2.592	2.623	2.652	2.678	2.702	2.724
28	1.701	1.995	2.154	2.262	2.343	2.407	2.461	2.507	2.546	2.581	2.612	2.641	2.667	2.690	2.712
30	1.697	1.989	2.147	2.255	2.335	2.399	2.453	2.498	2.537	2.572	2.603	2.631	2.657	2.680	2.702
32	1.694	1.984	2.142	2.249	2.329	2.393	2.445	2.490	2.529	2.564	2.595	2.623	2.648	2.671	2.693
34	1.691	1.980	2.137	2.243	2.323	2.386	2.439	2.484	2.523	2.557	2.588	2.615	2.641	2.664	2.685
36	1.688	1.977	2.133	2.238	2.318	2.381	2.433	2.478	2.517	2.551	2.581	2.609	2.634	2.657	2.678
38	1.686	1.973	2.129	2.234	2.313	2.376	2.428	2.473	2.511	2.545	2.576	2.603	2.628	2.651	2.672
40	1.684	1.971	2.126	2.230	2.309	2.372	2.424	2.468	2.506	2.540	2.571	2.598	2.623	2.646	2.667
42	1.682	1.968	2.122	2.227	2.306	2.368	2.420	2.464	2.502	2.536	2.566	2.593	2.618	2.641	2.662
44	1.680	1.965	2.120	2.224	2.302	2.364	2.416	2.460	2.498	2.532	2.562	2.589	2.614	2.636	2.657
46	1.679	1.963	2.117	2.221	2.299	2.361	2.413	2.456	2.494	2.528	2.558	2.585	2.610	2.632	2.653
48	1.677	1.961	2.115	2.219	2.296	2.358	2.410	2.453	2.491	2.525	2.554	2.581	2.606	2.629	2.649
50	1.676	1.959	2.113	2.216	2.294	2.356	2.407	2.450	2.488	2.521	2.551	2.578	2.603	2.625	2.646
55	1.673	1.955	2.108	2.211	2.288	2.350	2.401	2.444	2.481	2.515	2.544	2.571	2.596	2.618	2.639
60	1.671	1.952	2.104	2.207	2.284	2.345	2.396	2.439	2.476	2.509	2.539	2.565	2.589	2.612	2.632
65	1.669	1.949	2.101	2.203	2.280	2.341	2.391	2.434	2.471	2.504	2.534	2.560	2.584	2.607	2.627
70	1.667	1.947	2.098	2.200	2.276	2.337	2.388	2.430	2.467	2.500	2.530	2.556	2.580	2.602	2.623
75	1.665	1.945	2.096	2.197	2.274	2.334	2.384	2.427	2.464	2.497	2.526	2.552	2.576	2.598	2.619
80	1.664	1.943	2.093	2.195	2.271	2.332	2.382	2.424	2.461	2.494	2.523	2.549	2.573	2.595	2.615
85	1.663	1.942	2.092	2.193	2.269	2.329	2.379	2.422	2.458	2.491	2.520	2.546	2.570	2.592	2.612
90	1.662	1.940	2.090	2.191	2.267	2.327	2.377	2.419	2.456	2.489	2.518	2.544	2.568	2.590	2.610
95	1.661	1.939	2.088	2.189	2.265	2.325	2.375	2.417	2.454	2.486	2.515	2.542	2.565	2.587	2.607
100	1.660	1.938	2.087	2.188	2.264	2.324	2.373	2.416	2.452	2.485	2.513	2.540	2.563	2.585	2.605
125	1.657	1.933	2.082	2.182	2.258	2.317	2.367	2.409	2.445	2.477	2.506	2.532	2.556	2.577	2.597
150	1.655	1.931	2.079	2.179	2.254	2.313	2.362	2.404	2.440	2.472	2.501	2.527	2.550	2.572	2.592
175	1.654	1.929	2.076	2.176	2.251	2.310	2.359	2.401	2.437	2.469	2.497	2.523	2.547	2.568	2.588
200	1.653	1.927	2.075	2.174	2.249	2.308	2.357	2.398	2.435	2.466	2.495	2.521	2.544	2.565	2.585
250	1.651	1.925	2.072	2.171	2.246	2.305	2.354	2.395	2.431	2.463	2.491	2.517	2.540	2.561	2.581
∞	1.645	1.916	2.062	2.160	2.234	2.292	2.340	2.381	2.417	2.448	2.476	2.502	2.525	2.546	2.565

Percentile 99 of the one-tailed Dunnett's t distribution (table 1, cont.)

$\nu \backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	6.965	8.881	10.05	10.88	11.52	12.04	12.47	12.85	13.18	13.47	13.73	13.97	14.19	14.39	14.58
3	4.541	5.483	6.044	6.440	6.746	6.993	7.201	7.380	7.536	7.675	7.800	7.913	8.017	8.112	8.201
4	3.747	4.408	4.797	5.070	5.280	5.451	5.593	5.716	5.824	5.919	6.005	6.083	6.154	6.220	6.280
5	3.365	3.900	4.211	4.430	4.597	4.733	4.847	4.945	5.030	5.106	5.174	5.236	5.293	5.345	5.394
6	3.143	3.608	3.876	4.064	4.208	4.324	4.422	4.506	4.579	4.644	4.703	4.756	4.804	4.849	4.891
7	2.998	3.418	3.660	3.829	3.958	4.062	4.149	4.224	4.290	4.348	4.400	4.448	4.491	4.531	4.568
8	2.896	3.286	3.510	3.665	3.784	3.880	3.960	4.029	4.089	4.143	4.191	4.235	4.275	4.311	4.345
9	2.821	3.189	3.399	3.545	3.656	3.746	3.821	3.886	3.942	3.993	4.038	4.078	4.116	4.150	4.182
10	2.764	3.115	3.315	3.453	3.559	3.644	3.716	3.777	3.830	3.878	3.920	3.959	3.994	4.027	4.057
11	2.718	3.057	3.248	3.381	3.483	3.564	3.633	3.691	3.742	3.788	3.828	3.865	3.899	3.930	3.959
12	2.681	3.009	3.194	3.322	3.420	3.499	3.565	3.621	3.671	3.715	3.754	3.789	3.822	3.852	3.880
13	2.650	2.970	3.150	3.274	3.369	3.446	3.509	3.564	3.612	3.654	3.692	3.727	3.758	3.787	3.814
14	2.624	2.936	3.112	3.234	3.326	3.400	3.463	3.516	3.562	3.604	3.641	3.674	3.705	3.733	3.759
15	2.602	2.908	3.080	3.199	3.289	3.362	3.423	3.475	3.520	3.561	3.597	3.630	3.660	3.687	3.713
16	2.583	2.884	3.053	3.169	3.258	3.329	3.389	3.440	3.484	3.524	3.559	3.591	3.621	3.648	3.673
17	2.567	2.863	3.029	3.143	3.231	3.301	3.359	3.409	3.453	3.492	3.526	3.558	3.587	3.613	3.638
18	2.552	2.844	3.008	3.121	3.206	3.275	3.333	3.382	3.425	3.463	3.498	3.529	3.557	3.583	3.607
19	2.539	2.828	2.989	3.101	3.185	3.253	3.310	3.358	3.401	3.438	3.472	3.503	3.530	3.556	3.580
20	2.528	2.813	2.973	3.083	3.166	3.233	3.289	3.337	3.379	3.416	3.449	3.479	3.507	3.532	3.556
22	2.508	2.788	2.945	3.052	3.134	3.199	3.254	3.301	3.342	3.378	3.411	3.440	3.467	3.492	3.515
24	2.492	2.768	2.921	3.027	3.107	3.172	3.225	3.271	3.311	3.347	3.379	3.408	3.434	3.458	3.481
26	2.479	2.751	2.902	3.006	3.085	3.148	3.201	3.247	3.286	3.321	3.352	3.381	3.407	3.430	3.453
28	2.467	2.736	2.885	2.988	3.066	3.129	3.181	3.225	3.264	3.299	3.330	3.358	3.383	3.407	3.429
30	2.457	2.723	2.871	2.973	3.050	3.112	3.163	3.207	3.246	3.280	3.310	3.338	3.363	3.387	3.408
32	2.449	2.713	2.859	2.960	3.036	3.097	3.148	3.192	3.230	3.263	3.294	3.321	3.346	3.369	3.390
34	2.441	2.703	2.848	2.948	3.024	3.084	3.135	3.178	3.216	3.249	3.279	3.306	3.331	3.354	3.375
36	2.434	2.695	2.839	2.938	3.013	3.073	3.123	3.166	3.203	3.236	3.266	3.293	3.317	3.340	3.361
38	2.429	2.687	2.830	2.929	3.003	3.063	3.113	3.155	3.192	3.225	3.254	3.281	3.305	3.328	3.348
40	2.423	2.680	2.823	2.921	2.995	3.054	3.103	3.145	3.182	3.215	3.244	3.271	3.295	3.317	3.337
42	2.418	2.674	2.816	2.913	2.987	3.046	3.095	3.137	3.173	3.206	3.235	3.261	3.285	3.307	3.328
44	2.414	2.669	2.810	2.907	2.980	3.038	3.087	3.129	3.165	3.197	3.226	3.252	3.276	3.298	3.319
46	2.410	2.664	2.804	2.900	2.973	3.032	3.080	3.122	3.158	3.190	3.219	3.245	3.268	3.290	3.310
48	2.407	2.659	2.799	2.895	2.967	3.025	3.074	3.115	3.151	3.183	3.212	3.238	3.261	3.283	3.303
50	2.403	2.655	2.794	2.890	2.962	3.020	3.068	3.109	3.145	3.177	3.205	3.231	3.255	3.276	3.296
55	2.396	2.646	2.784	2.879	2.950	3.008	3.055	3.096	3.132	3.163	3.191	3.217	3.240	3.261	3.281
60	2.390	2.639	2.776	2.870	2.941	2.998	3.045	3.085	3.121	3.152	3.180	3.205	3.228	3.249	3.269
65	2.385	2.632	2.769	2.862	2.932	2.989	3.036	3.076	3.111	3.142	3.170	3.195	3.218	3.239	3.259
70	2.381	2.627	2.762	2.855	2.926	2.982	3.028	3.068	3.103	3.134	3.162	3.187	3.209	3.230	3.250
75	2.377	2.622	2.757	2.850	2.919	2.975	3.022	3.062	3.096	3.127	3.154	3.179	3.202	3.223	3.242
80	2.374	2.618	2.753	2.845	2.914	2.970	3.016	3.056	3.090	3.121	3.148	3.173	3.196	3.216	3.236
85	2.371	2.614	2.749	2.840	2.910	2.965	3.011	3.051	3.085	3.115	3.143	3.167	3.190	3.211	3.230
90	2.368	2.611	2.745	2.836	2.906	2.961	3.007	3.046	3.080	3.111	3.138	3.162	3.185	3.205	3.225
95	2.366	2.608	2.742	2.833	2.902	2.957	3.003	3.042	3.076	3.106	3.133	3.158	3.180	3.201	3.220
100	2.364	2.606	2.739	2.830	2.899	2.954	2.999	3.038	3.072	3.103	3.130	3.154	3.176	3.197	3.216
125	2.357	2.596	2.728	2.818	2.886	2.941	2.986	3.025	3.058	3.088	3.115	3.139	3.161	3.181	3.200
150	2.351	2.590	2.721	2.810	2.878	2.932	2.977	3.015	3.049	3.078	3.105	3.129	3.151	3.171	3.190
175	2.348	2.585	2.716	2.805	2.872	2.926	2.971	3.009	3.042	3.072	3.098	3.122	3.144	3.164	3.182
200	2.345	2.582	2.712	2.801	2.868	2.921	2.966	3.004	3.037	3.066	3.093	3.117	3.138	3.158	3.177
250	2.341	2.577	2.707	2.795	2.862	2.915	2.959	2.997	3.030	3.059	3.085	3.109	3.131	3.151	3.169
∞	2.326	2.558	2.685	2.772	2.837	2.890	2.933	2.970	3.003	3.031	3.057	3.080	3.101	3.120	3.138

Percentile 90 of the two-tailed Dunnett's *t* distribution (table 2)

ν \ p	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2.920	3.721	4.182	4.500	4.740	4.932	5.091	5.226	5.344	5.447	5.540	5.623	5.699	5.769	5.833
3	2.353	2.912	3.232	3.453	3.621	3.755	3.866	3.961	4.044	4.117	4.183	4.242	4.295	4.345	4.390
4	2.132	2.598	2.863	3.046	3.185	3.296	3.389	3.468	3.536	3.597	3.652	3.701	3.746	3.787	3.825
5	2.015	2.433	2.669	2.832	2.956	3.055	3.137	3.207	3.268	3.323	3.371	3.415	3.455	3.491	3.525
6	1.943	2.332	2.551	2.701	2.815	2.906	2.982	3.047	3.103	3.153	3.198	3.238	3.275	3.309	3.340
7	1.895	2.264	2.470	2.612	2.720	2.806	2.877	2.938	2.991	3.038	3.080	3.119	3.153	3.185	3.215
8	1.860	2.215	2.413	2.548	2.651	2.733	2.802	2.860	2.911	2.956	2.996	3.032	3.065	3.096	3.124
9	1.833	2.178	2.369	2.500	2.599	2.679	2.745	2.801	2.850	2.893	2.932	2.967	2.999	3.028	3.055
10	1.812	2.149	2.335	2.463	2.559	2.636	2.700	2.755	2.802	2.844	2.882	2.916	2.947	2.975	3.002
11	1.796	2.126	2.308	2.433	2.527	2.602	2.664	2.718	2.764	2.805	2.842	2.875	2.905	2.933	2.959
12	1.782	2.107	2.286	2.408	2.500	2.574	2.635	2.687	2.733	2.773	2.809	2.841	2.871	2.898	2.923
13	1.771	2.091	2.267	2.387	2.478	2.550	2.611	2.662	2.706	2.746	2.781	2.813	2.842	2.869	2.894
14	1.761	2.078	2.252	2.370	2.459	2.531	2.590	2.640	2.684	2.723	2.758	2.789	2.818	2.844	2.868
15	1.753	2.066	2.238	2.355	2.443	2.514	2.572	2.622	2.665	2.703	2.738	2.769	2.797	2.823	2.847
16	1.746	2.056	2.226	2.342	2.429	2.499	2.557	2.606	2.649	2.687	2.720	2.751	2.779	2.805	2.828
17	1.740	2.048	2.216	2.331	2.417	2.486	2.543	2.592	2.634	2.672	2.705	2.735	2.763	2.788	2.812
18	1.734	2.040	2.207	2.321	2.406	2.475	2.531	2.579	2.621	2.659	2.692	2.722	2.749	2.774	2.797
19	1.729	2.033	2.199	2.312	2.397	2.464	2.521	2.568	2.610	2.647	2.680	2.710	2.737	2.762	2.785
20	1.725	2.027	2.192	2.304	2.388	2.455	2.511	2.559	2.600	2.636	2.669	2.699	2.726	2.750	2.773
22	1.717	2.016	2.179	2.290	2.373	2.440	2.495	2.542	2.583	2.619	2.651	2.680	2.706	2.731	2.753
24	1.711	2.008	2.169	2.279	2.361	2.427	2.481	2.528	2.568	2.604	2.636	2.664	2.691	2.715	2.737
26	1.706	2.000	2.161	2.269	2.351	2.416	2.470	2.516	2.556	2.591	2.623	2.651	2.677	2.701	2.723
28	1.701	1.994	2.153	2.261	2.342	2.407	2.461	2.506	2.546	2.581	2.612	2.640	2.666	2.690	2.712
30	1.697	1.989	2.147	2.254	2.335	2.399	2.452	2.498	2.537	2.572	2.603	2.631	2.656	2.680	2.701
32	1.694	1.984	2.142	2.248	2.328	2.392	2.445	2.490	2.529	2.564	2.594	2.622	2.648	2.671	2.693
34	1.691	1.980	2.137	2.243	2.323	2.386	2.439	2.483	2.522	2.557	2.587	2.615	2.640	2.664	2.685
36	1.688	1.976	2.132	2.238	2.318	2.381	2.433	2.478	2.516	2.550	2.581	2.609	2.634	2.657	2.678
38	1.686	1.973	2.129	2.234	2.313	2.376	2.428	2.472	2.511	2.545	2.575	2.603	2.628	2.651	2.672
40	1.684	1.970	2.125	2.230	2.309	2.372	2.424	2.468	2.506	2.540	2.570	2.598	2.623	2.645	2.667
42	1.682	1.968	2.122	2.227	2.305	2.368	2.419	2.464	2.502	2.535	2.566	2.593	2.618	2.641	2.662
44	1.680	1.965	2.119	2.224	2.302	2.364	2.416	2.460	2.498	2.531	2.562	2.589	2.613	2.636	2.657
46	1.679	1.963	2.117	2.221	2.299	2.361	2.412	2.456	2.494	2.528	2.558	2.585	2.609	2.632	2.653
48	1.677	1.961	2.114	2.218	2.296	2.358	2.409	2.453	2.491	2.524	2.554	2.581	2.606	2.628	2.649
50	1.676	1.959	2.112	2.216	2.294	2.355	2.407	2.450	2.488	2.521	2.551	2.578	2.603	2.625	2.646
55	1.673	1.955	2.108	2.211	2.288	2.350	2.400	2.444	2.481	2.515	2.544	2.571	2.595	2.618	2.638
60	1.671	1.952	2.104	2.206	2.283	2.345	2.395	2.438	2.476	2.509	2.538	2.565	2.589	2.612	2.632
65	1.669	1.949	2.101	2.203	2.280	2.341	2.391	2.434	2.471	2.504	2.534	2.560	2.584	2.606	2.627
70	1.667	1.947	2.098	2.200	2.276	2.337	2.387	2.430	2.467	2.500	2.529	2.556	2.580	2.602	2.623
75	1.665	1.945	2.095	2.197	2.273	2.334	2.384	2.427	2.464	2.497	2.526	2.552	2.576	2.598	2.619
80	1.664	1.943	2.093	2.195	2.271	2.331	2.381	2.424	2.461	2.494	2.523	2.549	2.573	2.595	2.615
85	1.663	1.941	2.091	2.193	2.269	2.329	2.379	2.421	2.458	2.491	2.520	2.546	2.570	2.592	2.612
90	1.662	1.940	2.090	2.191	2.267	2.327	2.377	2.419	2.456	2.488	2.517	2.544	2.568	2.589	2.610
95	1.661	1.939	2.088	2.189	2.265	2.325	2.375	2.417	2.454	2.486	2.515	2.541	2.565	2.587	2.607
100	1.660	1.938	2.087	2.188	2.263	2.323	2.373	2.415	2.452	2.484	2.513	2.539	2.563	2.585	2.605
125	1.657	1.933	2.082	2.182	2.257	2.317	2.367	2.409	2.445	2.477	2.506	2.532	2.555	2.577	2.597
150	1.655	1.930	2.079	2.179	2.253	2.313	2.362	2.404	2.440	2.472	2.501	2.527	2.550	2.572	2.592
175	1.654	1.928	2.076	2.176	2.251	2.310	2.359	2.401	2.437	2.469	2.497	2.523	2.547	2.568	2.588
200	1.653	1.927	2.074	2.174	2.249	2.308	2.357	2.398	2.434	2.466	2.495	2.520	2.544	2.565	2.585
250	1.651	1.925	2.072	2.171	2.246	2.305	2.353	2.395	2.431	2.463	2.491	2.517	2.540	2.561	2.581
∞	1.645	1.916	2.062	2.160	2.234	2.292	2.340	2.381	2.417	2.448	2.476	2.502	2.525	2.546	2.565

Percentile 95 of the two-tailed Dunnett's t distribution (table 2, cont.)

$\nu \backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	4.303	5.418	6.065	6.513	6.852	7.124	7.349	7.540	7.707	7.854	7.985	8.103	8.211	8.310	8.401
3	3.182	3.867	4.263	4.538	4.748	4.916	5.056	5.176	5.280	5.372	5.455	5.529	5.597	5.660	5.717
4	2.776	3.3101	3.618	3.832	3.994	4.125	4.235	4.328	4.410	4.482	4.546	4.605	4.658	4.707	4.752
5	2.571	3.030	3.293	3.476	3.615	3.727	3.821	3.900	3.970	4.032	4.087	4.137	4.183	4.225	4.264
6	2.447	2.863	3.099	3.264	3.388	3.489	3.573	3.645	3.707	3.763	3.812	3.857	3.899	3.936	3.971
7	2.365	2.752	2.971	3.123	3.238	3.331	3.409	3.475	3.533	3.584	3.630	3.671	3.709	3.744	3.777
8	2.306	2.673	2.880	3.023	3.132	3.219	3.292	3.354	3.409	3.457	3.500	3.539	3.575	3.608	3.638
9	2.262	2.614	2.812	2.948	3.052	3.135	3.205	3.264	3.316	3.362	3.403	3.440	3.474	3.506	3.535
10	2.228	2.568	2.759	2.891	2.990	3.070	3.137	3.194	3.244	3.288	3.328	3.364	3.396	3.427	3.454
11	2.201	2.532	2.717	2.845	2.941	3.019	3.084	3.139	3.187	3.230	3.268	3.303	3.334	3.364	3.390
12	2.179	2.502	2.683	2.807	2.901	2.977	3.040	3.094	3.140	3.182	3.219	3.253	3.284	3.312	3.338
13	2.160	2.478	2.655	2.776	2.868	2.942	3.003	3.056	3.102	3.142	3.179	3.212	3.242	3.269	3.295
14	2.145	2.457	2.631	2.750	2.840	2.912	2.973	3.024	3.069	3.109	3.145	3.177	3.206	3.233	3.258
15	2.131	2.439	2.610	2.727	2.816	2.887	2.947	2.997	3.041	3.080	3.115	3.147	3.176	3.202	3.227
16	2.120	2.424	2.592	2.708	2.795	2.866	2.924	2.974	3.017	3.056	3.090	3.121	3.150	3.176	3.200
17	2.110	2.410	2.577	2.691	2.777	2.847	2.904	2.953	2.996	3.034	3.068	3.099	3.127	3.153	3.176
18	2.101	2.399	2.563	2.676	2.761	2.830	2.887	2.935	2.977	3.015	3.048	3.079	3.107	3.132	3.156
19	2.093	2.388	2.551	2.663	2.747	2.815	2.871	2.919	2.961	2.998	3.031	3.061	3.089	3.114	3.137
20	2.086	2.379	2.540	2.651	2.735	2.802	2.857	2.905	2.946	2.983	3.016	3.046	3.073	3.098	3.121
22	2.074	2.363	2.522	2.631	2.713	2.779	2.834	2.880	2.921	2.957	2.989	3.019	3.045	3.070	3.092
24	2.064	2.349	2.507	2.614	2.695	2.760	2.814	2.860	2.900	2.936	2.968	2.996	3.023	3.047	3.069
26	2.056	2.338	2.494	2.600	2.680	2.745	2.798	2.843	2.883	2.918	2.950	2.978	3.004	3.028	3.050
28	2.048	2.329	2.483	2.588	2.668	2.731	2.784	2.829	2.868	2.903	2.934	2.962	2.988	3.011	3.033
30	2.042	2.321	2.474	2.578	2.657	2.720	2.772	2.817	2.856	2.890	2.921	2.949	2.974	2.997	3.019
32	2.037	2.314	2.466	2.569	2.647	2.710	2.762	2.806	2.845	2.879	2.909	2.937	2.962	2.985	3.006
34	2.032	2.308	2.458	2.561	2.639	2.701	2.753	2.797	2.835	2.869	2.899	2.926	2.951	2.974	2.996
36	2.028	2.302	2.452	2.555	2.632	2.693	2.745	2.788	2.826	2.860	2.890	2.917	2.942	2.965	2.986
38	2.024	2.297	2.447	2.548	2.625	2.686	2.737	2.781	2.819	2.852	2.882	2.909	2.934	2.956	2.977
40	2.021	2.293	2.442	2.543	2.619	2.680	2.731	2.774	2.812	2.845	2.875	2.902	2.926	2.949	2.970
42	2.018	2.289	2.437	2.538	2.614	2.675	2.725	2.768	2.806	2.839	2.868	2.895	2.920	2.942	2.963
44	2.015	2.285	2.433	2.533	2.609	2.670	2.720	2.763	2.800	2.833	2.862	2.889	2.913	2.936	2.956
46	2.013	2.282	2.429	2.529	2.605	2.665	2.715	2.758	2.795	2.828	2.857	2.884	2.908	2.930	2.951
48	2.011	2.279	2.426	2.526	2.601	2.661	2.711	2.753	2.790	2.823	2.852	2.879	2.903	2.925	2.945
50	2.009	2.276	2.423	2.522	2.597	2.657	2.707	2.749	2.786	2.819	2.848	2.874	2.898	2.920	2.941
55	2.004	2.270	2.416	2.515	2.589	2.649	2.698	2.740	2.777	2.809	2.838	2.864	2.888	2.910	2.930
60	2.000	2.265	2.410	2.508	2.583	2.642	2.691	2.733	2.769	2.801	2.830	2.856	2.880	2.901	2.922
65	1.997	2.261	2.405	2.503	2.577	2.636	2.685	2.726	2.762	2.794	2.823	2.849	2.873	2.894	2.914
70	1.994	2.258	2.401	2.499	2.572	2.631	2.679	2.721	2.757	2.789	2.817	2.843	2.867	2.888	2.908
75	1.992	2.254	2.398	2.495	2.568	2.626	2.675	2.716	2.752	2.784	2.812	2.838	2.861	2.883	2.903
80	1.990	2.252	2.394	2.491	2.564	2.623	2.671	2.712	2.748	2.780	2.808	2.833	2.857	2.878	2.898
85	1.988	2.249	2.392	2.489	2.561	2.619	2.668	2.709	2.744	2.776	2.804	2.829	2.853	2.874	2.894
90	1.987	2.247	2.389	2.486	2.558	2.616	2.664	2.705	2.741	2.772	2.801	2.826	2.849	2.870	2.890
95	1.985	2.245	2.387	2.484	2.556	2.614	2.662	2.703	2.738	2.769	2.797	2.823	2.846	2.867	2.887
100	1.984	2.244	2.385	2.481	2.554	2.611	2.659	2.700	2.735	2.767	2.795	2.820	2.843	2.864	2.884
125	1.979	2.237	2.378	2.473	2.545	2.602	2.650	2.690	2.725	2.757	2.784	2.809	2.832	2.853	2.873
150	1.976	2.233	2.373	2.468	2.540	2.596	2.644	2.684	2.719	2.750	2.777	2.802	2.825	2.846	2.865
175	1.974	2.230	2.370	2.464	2.535	2.592	2.639	2.679	2.714	2.745	2.772	2.797	2.820	2.841	2.860
200	1.972	2.228	2.367	2.461	2.532	2.589	2.636	2.676	2.711	2.741	2.769	2.794	2.816	2.837	2.856
250	1.969	2.225	2.363	2.458	2.528	2.585	2.631	2.671	2.706	2.736	2.764	2.788	2.811	2.832	2.851
∞	1.960	2.212	2.349	2.442	2.511	2.567	2.613	2.652	2.686	2.716	2.743	2.767	2.790	2.810	2.829

Percentile 99 of the two-tailed Dunnett's *t* distribution (table 2, cont.)

$\nu \backslash p$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	9.925	12.39	13.83	14.83	15.59	16.19	16.70	17.13	17.50	17.83	18.11	18.39	18.63	18.85	19.06
3	5.841	6.975	7.640	8.106	8.461	8.748	8.986	9.191	9.369	9.526	9.668	9.796	9.912	10.02	10.12
4	4.604	5.365	5.810	6.122	6.362	6.555	6.717	6.855	6.976	7.084	7.180	7.267	7.347	7.420	7.488
5	4.032	4.628	4.975	5.219	5.406	5.557	5.684	5.793	5.888	5.972	6.047	6.116	6.179	6.237	6.290
6	3.707	4.213	4.506	4.712	4.870	4.998	5.105	5.196	5.277	5.348	5.412	5.470	5.523	5.572	5.618
7	3.499	3.949	4.208	4.390	4.529	4.642	4.737	4.818	4.889	4.952	5.008	5.060	5.107	5.150	5.190
8	3.355	3.766	4.003	4.168	4.295	4.398	4.483	4.557	4.622	4.679	4.731	4.777	4.820	4.860	4.896
9	3.250	3.633	3.853	4.007	4.124	4.219	4.299	4.367	4.427	4.481	4.528	4.572	4.611	4.648	4.682
10	3.169	3.532	3.739	3.884	3.995	4.084	4.159	4.223	4.280	4.330	4.375	4.415	4.453	4.487	4.519
11	3.106	3.452	3.650	3.787	3.893	3.978	4.049	4.110	4.164	4.211	4.254	4.293	4.328	4.361	4.391
12	3.055	3.388	3.578	3.710	3.811	3.893	3.961	4.019	4.071	4.116	4.157	4.194	4.228	4.260	4.289
13	3.012	3.335	3.518	3.646	3.744	3.822	3.888	3.945	3.994	4.038	4.077	4.113	4.146	4.176	4.204
14	2.977	3.291	3.469	3.593	3.687	3.764	3.827	3.882	3.930	3.973	4.011	4.045	4.077	4.106	4.133
15	2.947	3.253	3.427	3.547	3.639	3.714	3.776	3.829	3.876	3.917	3.954	3.988	4.019	4.047	4.074
16	2.921	3.221	3.391	3.508	3.598	3.671	3.732	3.784	3.829	3.869	3.906	3.939	3.969	3.996	4.022
17	2.898	3.193	3.359	3.475	3.563	3.634	3.693	3.744	3.789	3.828	3.864	3.896	3.925	3.952	3.978
18	2.878	3.168	3.332	3.445	3.532	3.601	3.660	3.709	3.753	3.792	3.827	3.858	3.887	3.914	3.938
19	2.861	3.146	3.307	3.419	3.504	3.573	3.630	3.679	3.722	3.760	3.794	3.825	3.854	3.880	3.904
20	2.845	3.127	3.286	3.396	3.479	3.547	3.603	3.652	3.694	3.731	3.765	3.796	3.824	3.849	3.873
22	2.819	3.094	3.249	3.356	3.438	3.503	3.558	3.605	3.647	3.683	3.716	3.746	3.773	3.798	3.821
24	2.797	3.067	3.219	3.324	3.403	3.468	3.521	3.567	3.608	3.643	3.675	3.704	3.731	3.755	3.778
26	2.779	3.044	3.194	3.297	3.375	3.438	3.491	3.536	3.575	3.610	3.642	3.670	3.696	3.720	3.742
28	2.763	3.025	3.172	3.274	3.351	3.413	3.465	3.509	3.548	3.582	3.613	3.641	3.667	3.690	3.712
30	2.750	3.009	3.154	3.254	3.330	3.391	3.442	3.486	3.524	3.558	3.589	3.616	3.642	3.665	3.686
32	2.738	2.995	3.138	3.237	3.312	3.373	3.423	3.466	3.504	3.538	3.568	3.595	3.620	3.643	3.664
34	2.728	2.982	3.124	3.222	3.296	3.356	3.406	3.449	3.486	3.519	3.549	3.576	3.601	3.623	3.644
36	2.719	2.971	3.112	3.209	3.283	3.342	3.391	3.434	3.471	3.503	3.533	3.559	3.584	3.606	3.627
38	2.712	2.962	3.101	3.197	3.270	3.329	3.378	3.420	3.457	3.489	3.518	3.545	3.569	3.591	3.612
40	2.704	2.953	3.091	3.187	3.259	3.317	3.366	3.408	3.444	3.476	3.505	3.531	3.555	3.577	3.598
42	2.698	2.945	3.082	3.177	3.249	3.307	3.355	3.397	3.433	3.465	3.493	3.520	3.543	3.565	3.585
44	2.692	2.938	3.074	3.169	3.240	3.298	3.346	3.387	3.423	3.454	3.483	3.509	3.532	3.554	3.574
46	2.687	2.931	3.067	3.161	3.232	3.289	3.337	3.378	3.413	3.445	3.473	3.499	3.522	3.544	3.564
48	2.682	2.925	3.061	3.154	3.225	3.282	3.329	3.370	3.405	3.436	3.465	3.490	3.513	3.535	3.555
50	2.678	2.920	3.055	3.147	3.218	3.274	3.322	3.362	3.397	3.428	3.456	3.482	3.505	3.526	3.546
55	2.668	2.908	3.041	3.133	3.203	3.259	3.306	3.346	3.380	3.411	3.439	3.464	3.487	3.508	3.528
60	2.660	2.898	3.031	3.122	3.191	3.246	3.292	3.332	3.366	3.397	3.424	3.449	3.472	3.493	3.512
65	2.654	2.890	3.021	3.112	3.180	3.235	3.281	3.321	3.355	3.385	3.412	3.437	3.460	3.480	3.499
70	2.648	2.883	3.014	3.103	3.172	3.226	3.272	3.311	3.345	3.375	3.402	3.426	3.449	3.469	3.488
75	2.643	2.877	3.007	3.096	3.164	3.218	3.264	3.302	3.336	3.366	3.393	3.417	3.440	3.460	3.479
80	2.639	2.872	3.001	3.090	3.157	3.211	3.256	3.295	3.329	3.358	3.385	3.409	3.432	3.452	3.471
85	2.635	2.867	2.996	3.084	3.151	3.205	3.250	3.289	3.322	3.352	3.378	3.402	3.424	3.445	3.463
90	2.632	2.863	2.991	3.080	3.146	3.200	3.245	3.283	3.316	3.346	3.372	3.396	3.418	3.438	3.457
95	2.629	2.859	2.987	3.075	3.142	3.195	3.240	3.278	3.311	3.340	3.367	3.391	3.413	3.433	3.451
100	2.626	2.856	2.984	3.071	3.138	3.191	3.235	3.273	3.306	3.336	3.362	3.386	3.408	3.428	3.446
125	2.616	2.844	2.970	3.056	3.122	3.175	3.218	3.256	3.289	3.318	3.343	3.367	3.388	3.408	3.427
150	2.609	2.835	2.961	3.046	3.112	3.164	3.207	3.244	3.277	3.306	3.331	3.355	3.376	3.395	3.414
175	2.604	2.829	2.954	3.039	3.104	3.156	3.199	3.236	3.268	3.297	3.323	3.346	3.367	3.386	3.404
200	2.601	2.825	2.949	3.034	3.099	3.150	3.193	3.230	3.262	3.291	3.316	3.339	3.360	3.380	3.398
250	2.596	2.819	2.942	3.027	3.091	3.142	3.185	3.222	3.254	3.282	3.307	3.330	3.351	3.370	3.388
∞	2.576	2.795	2.915	2.998	3.061	3.111	3.152	3.188	3.219	3.247	3.271	3.294	3.314	3.333	3.350

Reading off the tables

Percentiles 90, 95 and 99 of Dunnett's t statistic are presented in two tables: table 1, on three pages, concerns one-tailed (*i.e.* signed) tests of differences between means, with parameters ν = 2(1)20(2)50(5)100(25)250,∞ and number of means p = 2(1)16. The three pages for table 2 concern two-tailed (unsigned) tests, with the same parameter combinations.

Illustration 1. What is the appropriate critical value of Dunnett's t for comparing 8 group means, using a variance estimate (or mean square) with 32 *df*, according to a bidirectional hypothesis at a significance level of 5 %? In table 2, for bidirectional or unsigned tests, we take percentile 95, line $\nu = 32$ and column $p = 8$: the critical value is $t_D(8,32[.95]) = 2.762$. For a one-tailed test (relevant to a directional hypothesis) with the same parameter values, we would have $t_D = 2.445$.

Illustration 2. A research student wishes to compare the means of four treatments to that of a control treatment, and verify whether any one of the particular treatments brings about an increase in results; he chooses a 0.05 significance level. Each treatment group, including the control, is composed of $n_j = 12$ participants, the reference Mean square (here, $MS_{\text{within-groups}}$) having $\nu = 5\times(12-1) = 55$ *df.* A directional hypothesis is called for and, in table 1, we find $t_D(5,55[.95]) = 2.211$.

Full example

A popular experiment in American high schools is to study the influence of music on plants' growth. At one campus, the biology teacher selected 48 young cactus sprouts, all similar in shape and body, and he distributes them equally and randomly in six cubicles. The cubicles were identical, except for the musical contexts within (Silence, Classical music, Jazz, Country, Rock'n-roll, Heavy-metal). The cactus plants were left alone day and night in their respective sustained musical surroundings, for eight consecutive weeks. At that time, the volume increase in each plant was measured (in cm^3) : the results are summarized in a table on the next page. Considering these data, can we confirm that music of a sort or another had some influence on plants' growth, using a global significance level of 0.05?

Type of music	n	$\bar{x}$	s_x
(Silence)	8	16.03	3.22
Classical	8	17.11	2.90
Jazz	8	16.41	3.08
Country	8	15.60	3.00
Rock'n-roll	8	15.94	3.17
Heavy-metal	8	16.09	2.98

Solution: The question asked suggests that we compare the mean of every musical environment to that of the control condition, *i.e.* "silence". As the sought-for influence could either be positive or negative, the appropriate procedure for this situation is Dunnett's bilateral t test. Group sizes (n_j) being equal, the reference $MS_{w\text{-}g}$ is computed as a simple average of each group's variance, $(3.22^2+2.90^2+\cdots+2.98^2)/6 \approx 9.366$, with $\nu=6\times(8-1) = 42$ *df*. For $p = 6$ means in the comparison set and $\alpha = 0.05$, the two-tailed Dunnett's table (table 2) furnishes $t_D(6,42[.95]) = \pm 2.614$. The appropriate test statistic is:

$$t_D = \frac{\bar{x}_j - \bar{x}_0}{\sqrt{MS_{error}\left[\frac{1}{n_j}+\frac{1}{n_0}\right]}} \; ;$$

it simplifies to $t_D = (\bar{x}_j - \bar{x}_0)/\sqrt{[2MS_{error}/n]}$ when $n_j = n_0 = n$. Using $\bar{x}_0 = 16.03$, $MS_{w\text{-}g} = 9.366$ and $n = 8$, the five successive tests equal 0.706, 0.248, −0.281, −0.059, and 0.039, for the respective musical environments, from Classical to Heavy-metal. Not one test even approaches the critical value, so that we can not confirm any music's influence on growth..., at least plants' growth!

Mathematical presentation

Dunnett's t distribution, as its name implies, is akin to Student's t: it is a multivariate extension of it, elaborated for sets of p variables, and it allows to compare each of $p-1$ means with the p^{th} mean, which represents a control group, a neutral treatment or some analogous reference point. Let p independent r.v.'s $\{x_0, x_1, x_2, \cdots, x_{p-1}\}$, each distributed as $N(\mu,\sigma^2)$, and s^2, an independent estimate of σ^2 with ν *df*. Then, the set:

$$t_1 = (x_1 - x_0)/s \;;\; t_2 = (x_2 - x_0)/s \;;\; \cdots \;;\; t_{p-1} = (x_{p-1} - x_0)/s$$

obeys to Dunnett's multivariate t distribution with parameters p and ν[1]. In the context of ANOVA, each x_i is a mean, averaged over n individual observations, and the variance estimate is a mean square (MS), for example the $MS_{\text{within-groups}}$ in a one-way design with p independent samples. For this typical case, the formula[2] is:

$$t_j = \frac{\bar{x}_j - \bar{x}_0}{\sqrt{\dfrac{2\,MS_{\text{error}}}{n}}} \;,$$

the resulting value being compared to the critical value, *i.e.* percentile $100(1-\alpha)$ of Dunnett's t with parameters p and ν, $t_{p,\nu[1-\alpha]}$.

Dunnett's procedure, which consists in comparing $p-1$ variables with the same p^{th} variable, aims at controlling the global error rate (α) by taking into account the number ($p-1$) of comparisons and the actual correlation between the $p-1$ test statistics. This correlation is of the form $\rho[(x_i - x_0), (x_j - x_0)]$ and is equal to $+½$ for all i,j's (under the assumption of independent sampling for each group). Thanks to this correlation value and the constrained number of tests, Dunnett's procedure is more powerful than, for instance, Dunn-Šidák's (*see* section on Student's t distribution). Moreover, in accordance with the hypotheses to be tested, Dunnett's procedure allows for unilateral (one-tailed) as well as bilateral (two-tailed) rejection.

Calculation and moments

The joint probability $P(t) = \Pr\{\, t_1 \le t,\ t_2 \le t,\ \cdots,\ t_k \le t \,\}$ or, equivalently, $\Pr\{\max t_i \le t\}$, where $k = p-1$ and $\rho(t_i, t_j) = ½$, is given by:

$$P(t) = \int_0^{\infty} f(u) \left[\int_{-\infty}^{\infty} g_{t,u}(w)\, dw \right] du \;.$$

The external density $f(u)$ depicts the variation of the s.d. estimate, with parameter ν:

[1] A third parameter, the correlation coefficient among the $p-1$ test statistics t, $\rho(t_i,t_j)$, has (constant) value $\rho = ½$ in the Dunnett setting.

[2] For cases where sample sizes are unequal, the quantity under the square root is computed rather with $\sqrt{[MS_{\text{error}}(n_j^{-1} + n_0^{-1})]}$. The available critical values are then approximately correct if the ratio of sample sizes does not exceed 3/2.

$$f(u) = \frac{2(\nu/2)^{\frac{\nu}{2}}}{\Gamma(\nu/2)} u^{\nu-1} e^{-\nu \cdot u^2/2} ;$$

$\Gamma(x)$ denotes the *Gamma* function. As for $g(w)$, the internal p.d.f. conditioned on t and u, it shows the variation of the maximum of k normal variables with mutual correlation $\rho_{i,j} = ½$. For the one-tailed (or signed difference) version of the test, this p.d.f. is:

$$g_{t,u}(w) = [\Phi(w+\sqrt{2}tu)]^k \varphi(w) .$$

In that expression, Φ is the standard normal integral or d.f., and φ the associated p.d.f. For the two-tailed (or unsigned difference) version, the sought-for probability is $\Pr\{ |t_1|\le t, |t_2|\le t, \cdots, |t_k|\le t \}$, or $\Pr\{\max |t_i| \le t\}$. In this case, the p.d.f. is:

$$g_{t,u}(w) = [\Phi(w+\sqrt{2}tu) - \Phi(w-\sqrt{2}tu)]^k \varphi(w) .$$

Considering the normal integral Φ as a simple function, the computation of $P(t)$ for determining critical values confronts us with a double integration, the shape of the integrands changing substantially for various combinations of k and ν. Recall, for instance, that $f(s)$ above is a-modal for $\nu = 1$, spread out and unimodal for intermediate ν, with the mode at $s = \sqrt{(1-1/\nu)}$, and then pointed for higher values of ν. The percentiles in our tables were fixed up through the evaluation of the double integral with a repeated application of Simpson's rule.

Moments. Dunnett's t really is identical to Student's t variable, albeit it enters a particular $p-1$ multivariate structure with common correlation $\rho = ½$. The *individual* moments of each t-component are thus the same as Student's (*see* section on Student's t distribution).

On the other hand, for establishing the probability criterion in Dunnett's procedure, the key variable is given by $t_{(k)} \equiv \max t_i$, $1 \le i \le k$, for the signed, one-tailed version (or by $\max|t_i|$, for the unsigned version), with the set $\{ t_1, t_2, \cdots, t_k \}$ obeying the multivariate t distribution with parameters k, ν and $\rho_{i,j} = ½$. As it was the case for the Studentized range q variable, $t_{(k)}$ is equivalent to the product $M_k \times 1/s_\nu$, where M_k is the maximum of k correlated normal r.v.'s with correlation $\rho_{i,j} = ½$ and $1/s$ is the reciprocal of a standard deviation estimate having ν *df.* The p.d.f. of $1/s$ can be obtained (*see* section on the Studentized range), but the moments of M_k are generally not known[3] and, consequently, the moments of Dunnett's t statistic, either in its signed or unsigned version, can not be determined.

[3] The basic bibliographical reference for these questions is: S. Kotz, N. Balakrishnan & N. L. Johnson: *Continuous multivariate distributions. Volume 1: Models and applications* (2^{nd} edition). New York, Wiley, 2000.

$\overline{E}^2$ (monotonic variation) distribution

- ✓ Percentiles 95, 97.5, 99 and 99.5 of $\overline{E}^2$ (or $\overline{\chi}^2$) for Simple order (table 1)
- ✓ Percentiles 95, 97.5, 99 and 99.5 of $\overline{E}^2$ (or $\overline{\chi}^2$) for Dominance (table 2)
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation

  Calculation and moments

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Percentile 95 of $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Simple order (table 1)

$\nu_e \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.8100	.8111	.7730	.7278	.6834	.6424	.6053	.5719	.5419	.5149	.4905	.4684	.4482	.4299
3	.6486	.6866	.6713	.6430	.6120	.5814	.5527	.5261	.5017	.4793	.4588	.4400	.4226	.4066
4	.5319	.5895	.5900	.5742	.5530	.5304	.5082	.4869	.4669	.4483	.4308	.4147	.3997	.3857
5	.4481	.5144	.5249	.5178	.5038	.4873	.4700	.4529	.4365	.4208	.4061	.3921	.3791	.3668
6	.3862	.4554	.4721	.4710	.4623	.4504	.4370	.4233	.4097	.3966	.3839	.3719	.3605	.3496
7	.3390	.4080	.4287	.4317	.4270	.4185	.4082	.3972	.3860	.3749	.3640	.3535	.3436	.3340
8	.3018	.3694	.3923	.3984	.3966	.3908	.3830	.3741	.3648	.3554	.3461	.3369	.3281	.3196
9	.2719	.3373	.3615	.3697	.3701	.3665	.3606	.3535	.3458	.3378	.3298	.3218	.3140	.3065
10	.2473	.3103	.3351	.3448	.3469	.3449	.3407	.3350	.3286	.3219	.3149	.3080	.3011	.2943
11	.2267	.2872	.3123	.3230	.3264	.3258	.3228	.3184	.3131	.3073	.3014	.2952	.2892	.2832
12	.2093	.2673	.2923	.3038	.3082	.3086	.3067	.3032	.2989	.2941	.2889	.2835	.2781	.2728
13	.1944	.2500	.2748	.2867	.2918	.2931	.2921	.2895	.2860	.2819	.2774	.2727	.2679	.2631
14	.1814	.2347	.2591	.2714	.2771	.2791	.2788	.2769	.2741	.2707	.2668	.2627	.2584	.2541
15	.1700	.2212	.2452	.2576	.2638	.2664	.2667	.2654	.2632	.2603	.2570	.2534	.2496	.2457
16	.1600	.2091	.2327	.2452	.2518	.2548	.2555	.2548	.2531	.2507	.2478	.2447	.2413	.2378
17	.1511	.1983	.2214	.2339	.2407	.2441	.2453	.2450	.2437	.2418	.2393	.2365	.2336	.2304
18	.1431	.1886	.2111	.2236	.2306	.2343	.2358	.2359	.2350	.2334	.2314	.2290	.2263	.2235
19	.1360	.1797	.2017	.2141	.2213	.2252	.2271	.2275	.2269	.2257	.2239	.2218	.2195	.2170
20	.1295	.1717	.1932	.2055	.2127	.2168	.2189	.2196	.2194	.2184	.2169	.2151	.2130	.2108
22	.1182	.1576	.1780	.1901	.1974	.2018	.2043	.2055	.2057	.2052	.2042	.2029	.2013	.1994
24	.1087	.1456	.1651	.1768	.1841	.1887	.1915	.1930	.1936	.1935	.1929	.1919	.1907	.1893
26	.1006	.1353	.1539	.1652	.1725	.1772	.1802	.1819	.1828	.1830	.1828	.1821	.1812	.1800
28	.0937	.1264	.1441	.1551	.1623	.1670	.1701	.1721	.1732	.1736	.1736	.1733	.1726	.1717
30	.0876	.1185	.1355	.1461	.1532	.1579	.1611	.1632	.1645	.1652	.1654	.1652	.1648	.1641
32	.0823	.1116	.1279	.1382	.1451	.1498	.1531	.1553	.1567	.1575	.1579	.1579	.1576	.1571
34	.0776	.1055	.1210	.1310	.1378	.1425	.1457	.1480	.1495	.1505	.1510	.1512	.1511	.1507
36	.0734	.0999	.1149	.1245	.1312	.1358	.1391	.1414	.1430	.1441	.1447	.1450	.1450	.1448
38	.0696	.0950	.1094	.1187	.1251	.1297	.1330	.1354	.1371	.1382	.1389	.1393	.1395	.1394
40	.0662	.0905	.1043	.1134	.1197	.1242	.1275	.1299	.1316	.1328	.1336	.1341	.1343	.1343
42	.0631	.0864	.0997	.1085	.1147	.1191	.1223	.1247	.1265	.1278	.1286	.1292	.1295	.1296
44	.0603	.0827	.0955	.1040	.1100	.1144	.1176	.1200	.1218	.1231	.1240	.1247	.1250	.1252
46	.0577	.0792	.0917	.0999	.1058	.1101	.1132	.1157	.1174	.1188	.1198	.1204	.1209	.1211
48	.0554	.0761	.0881	.0961	.1018	.1060	.1092	.1116	.1134	.1148	.1158	.1165	.1170	.1173
50	.0532	.0732	.0848	.0926	.0982	.1023	.1054	.1078	.1096	.1110	.1120	.1128	.1133	.1137
55	.0484	.0668	.0775	.0848	.0901	.0940	.0970	.0994	.1012	.1026	.1037	.1045	.1051	.1055
60	.0445	.0614	.0714	.0783	.0832	.0870	.0899	.0921	.0939	.0953	.0965	.0973	.0980	.0985
65	.0411	.0568	.0662	.0726	.0773	.0809	.0837	.0859	.0877	.0891	.0902	.0911	.0918	.0924
70	.0382	.0529	.0617	.0678	.0722	.0756	.0783	.0805	.0822	.0836	.0847	.0856	.0863	.0869
75	.0357	.0495	.0578	.0635	.0677	.0710	.0736	.0757	.0773	.0787	.0798	.0807	.0815	.0821
80	.0335	.0465	.0543	.0597	.0638	.0669	.0694	.0714	.0730	.0744	.0755	.0764	.0772	.0778
85	.0315	.0438	.0512	.0564	.0603	.0633	.0657	.0676	.0692	.0705	.0716	.0725	.0733	.0739
90	.0298	.0414	.0485	.0534	.0571	.0600	.0623	.0642	.0657	.0670	.0681	.0690	.0697	.0704
95	.0282	.0393	.0460	.0507	.0543	.0570	.0593	.0611	.0626	.0638	.0649	.0658	.0665	.0672
100	.0268	.0374	.0438	.0483	.0517	.0544	.0565	.0583	.0597	.0610	.0620	.0629	.0636	.0643
125	.0215	.0300	.0353	.0390	.0418	.0440	.0459	.0474	.0487	.0497	.0507	.0515	.0522	.0528
150	.0179	.0251	.0295	.0327	.0351	.0370	.0386	.0399	.0410	.0420	.0428	.0436	.0442	.0448
175	.0154	.0216	.0254	.0281	.0302	.0319	.0333	.0345	.0355	.0364	.0371	.0378	.0384	.0389
200	.0135	.0189	.0223	.0247	.0266	.0281	.0293	.0304	.0313	.0321	.0327	.0333	.0339	.0344
250	.0108	.0151	.0179	.0198	.0214	.0226	.0236	.0245	.0252	.0259	.0265	.0270	.0275	.0279
∞	2.706	3.820	4.528	5.049	5.460	5.800	6.088	6.339	6.560	6.758	6.937	7.100	7.250	7.389

Percentile 97.5 of $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Simple order (table 1, cont.)

$\nu_e \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.9025	.8949	.8603	.8177	.7740	.7322	.6934	.6578	.6253	.5958	.5688	.5442	.5216	.5009
3	.7715	.7898	.7690	.7378	.7041	.6709	.6394	.6101	.5829	.5578	.5346	.5133	.4935	.4753
4	.6584	.6971	.6897	.6690	.6439	.6180	.5925	.5683	.5454	.5241	.5041	.4855	.4682	.4521
5	.5693	.6201	.6228	.6104	.5922	.5721	.5516	.5315	.5123	.4941	.4768	.4606	.4453	.4310
6	.4995	.5567	.5665	.5603	.5476	.5321	.5156	.4991	.4828	.4672	.4522	.4380	.4245	.4117
7	.4441	.5042	.5189	.5174	.5089	.4972	.4839	.4702	.4564	.4430	.4300	.4175	.4055	.3941
8	.3993	.4603	.4783	.4803	.4751	.4663	.4557	.4444	.4327	.4211	.4097	.3987	.3881	.3779
9	.3625	.4232	.4433	.4480	.4454	.4390	.4306	.4211	.4112	.4012	.3913	.3816	.3721	.3629
10	.3318	.3915	.4130	.4196	.4190	.4146	.4080	.4002	.3918	.3831	.3744	.3658	.3573	.3491
11	.3057	.3640	.3864	.3945	.3956	.3927	.3876	.3812	.3741	.3666	.3589	.3513	.3437	.3363
12	.2835	.3401	.3630	.3722	.3745	.3729	.3691	.3639	.3578	.3514	.3447	.3378	.3311	.3244
13	.2642	.3191	.3422	.3523	.3556	.3551	.3523	.3480	.3429	.3373	.3314	.3254	.3193	.3132
14	.2473	.3005	.3237	.3343	.3384	.3388	.3369	.3335	.3292	.3244	.3192	.3138	.3083	.3029
15	.2325	.2840	.3070	.3181	.3228	.3239	.3228	.3201	.3166	.3124	.3078	.3030	.2981	.2931
16	.2193	.2691	.2919	.3033	.3086	.3103	.3098	.3078	.3048	.3012	.2972	.2929	.2885	.2840
17	.2075	.2557	.2783	.2898	.2956	.2978	.2978	.2963	.2939	.2908	.2873	.2835	.2795	.2754
18	.1969	.2436	.2658	.2775	.2836	.2862	.2867	.2857	.2838	.2811	.2781	.2747	.2711	.2673
19	.1874	.2326	.2544	.2662	.2725	.2755	.2764	.2758	.2743	.2720	.2694	.2663	.2631	.2597
20	.1787	.2225	.2439	.2557	.2622	.2655	.2668	.2666	.2654	.2635	.2612	.2585	.2556	.2525
22	.1635	.2047	.2254	.2371	.2439	.2476	.2494	.2498	.2492	.2480	.2462	.2441	.2418	.2392
24	.1507	.1895	.2094	.2210	.2279	.2320	.2342	.2350	.2349	.2342	.2329	.2313	.2294	.2273
26	.1398	.1765	.1956	.2069	.2139	.2182	.2207	.2219	.2222	.2218	.2209	.2197	.2182	.2165
28	.1303	.1651	.1834	.1945	.2015	.2059	.2086	.2101	.2107	.2107	.2101	.2092	.2080	.2066
30	.1221	.1551	.1727	.1835	.1905	.1950	.1978	.1995	.2004	.2006	.2003	.1997	.1988	.1976
32	.1148	.1462	.1632	.1737	.1806	.1851	.1881	.1900	.1910	.1914	.1914	.1910	.1903	.1894
34	.1083	.1383	.1546	.1649	.1716	.1762	.1793	.1813	.1825	.1831	.1832	.1830	.1825	.1818
36	.1025	.1312	.1469	.1569	.1636	.1681	.1713	.1733	.1747	.1754	.1757	.1757	.1753	.1748
38	.0973	.1248	.1400	.1497	.1562	.1607	.1639	.1661	.1675	.1684	.1688	.1689	.1687	.1683
40	.0927	.1189	.1336	.1431	.1495	.1540	.1572	.1594	.1609	.1619	.1624	.1626	.1626	.1623
42	.0884	.1136	.1278	.1370	.1433	.1478	.1510	.1532	.1548	.1558	.1565	.1568	.1569	.1567
44	.0845	.1088	.1225	.1314	.1376	.1420	.1452	.1475	.1491	.1503	.1510	.1514	.1515	.1515
46	.0810	.1044	.1176	.1263	.1324	.1367	.1399	.1422	.1439	.1450	.1458	.1463	.1466	.1466
48	.0777	.1003	.1131	.1216	.1275	.1318	.1349	.1373	.1390	.1402	.1410	.1416	.1419	.1420
50	.0747	.0965	.1089	.1172	.1230	.1272	.1303	.1327	.1344	.1356	.1365	.1371	.1375	.1377
55	.0681	.0881	.0997	.1075	.1130	.1170	.1201	.1224	.1242	.1255	.1265	.1272	.1277	.1280
60	.0625	.0811	.0920	.0992	.1045	.1084	.1113	.1136	.1154	.1167	.1178	.1186	.1191	.1195
65	.0578	.0751	.0853	.0922	.0971	.1009	.1038	.1060	.1078	.1091	.1102	.1110	.1117	.1121
70	.0538	.0700	.0795	.0860	.0908	.0944	.0971	.0993	.1011	.1024	.1035	.1044	.1051	.1056
75	.0503	.0655	.0745	.0807	.0852	.0886	.0913	.0935	.0952	.0965	.0976	.0985	.0992	.0998
80	.0472	.0615	.0701	.0759	.0803	.0836	.0862	.0882	.0899	.0913	.0924	.0933	.0940	.0946
85	.0444	.0580	.0662	.0717	.0759	.0791	.0816	.0836	.0852	.0866	.0876	.0886	.0893	.0899
90	.0420	.0549	.0626	.0680	.0719	.0750	.0774	.0794	.0810	.0823	.0834	.0843	.0850	.0856
95	.0398	.0521	.0595	.0646	.0684	.0713	.0737	.0756	.0771	.0784	.0795	.0804	.0812	.0818
100	.0379	.0496	.0566	.0615	.0652	.0680	.0703	.0721	.0737	.0749	.0760	.0769	.0776	.0783
125	.0304	.0399	.0456	.0497	.0528	.0552	.0571	.0587	.0601	.0612	.0622	.0630	.0637	.0644
150	.0254	.0334	.0382	.0417	.0443	.0464	.0481	.0495	.0507	.0518	.0526	.0534	.0541	.0546
175	.0218	.0287	.0329	.0359	.0382	.0401	.0416	.0428	.0439	.0448	.0456	.0463	.0469	.0475
200	.0191	.0251	.0289	.0315	.0336	.0352	.0366	.0377	.0387	.0395	.0403	.0409	.0415	.0420
250	.0153	.0202	.0232	.0254	.0270	.0284	.0295	.0304	.0313	.0320	.0326	.0331	.0336	.0341
∞	3.841	5.098	5.891	6.471	6.928	7.304	7.624	7.901	8.145	8.363	8.561	8.740	8.905	9.058

Percentile 99 of $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Simple order (table 1, cont.)

$\nu_e \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.9604	.9531	.9289	.8954	.8578	.8195	.7822	.7469	.7138	.6831	.6546	.6283	.6039	.5813
3	.8730	.8781	.8573	.8278	.7951	.7621	.7301	.6996	.6709	.6440	.6190	.5957	.5740	.5538
4	.7783	.7995	.7872	.7645	.7379	.7103	.6832	.6570	.6322	.6087	.5867	.5661	.5467	.5286
5	.6937	.7276	.7236	.7075	.6866	.6640	.6411	.6188	.5973	.5768	.5575	.5391	.5219	.5056
6	.6221	.6646	.6674	.6569	.6409	.6225	.6034	.5844	.5658	.5479	.5308	.5145	.4991	.4844
7	.5622	.6100	.6181	.6122	.6003	.5855	.5696	.5534	.5373	.5216	.5065	.4919	.4781	.4648
8	.5119	.5628	.5748	.5727	.5641	.5523	.5391	.5253	.5113	.4976	.4842	.4712	.4587	.4467
9	.4694	.5219	.5368	.5376	.5317	.5225	.5116	.4998	.4877	.4756	.4637	.4521	.4408	.4300
10	.4331	.4862	.5032	.5063	.5027	.4956	.4866	.4765	.4660	.4554	.4448	.4344	.4243	.4144
11	.4018	.4549	.4734	.4783	.4765	.4712	.4638	.4553	.4462	.4368	.4274	.4180	.4089	.3999
12	.3746	.4272	.4467	.4531	.4529	.4490	.4430	.4358	.4279	.4196	.4112	.4028	.3945	.3863
13	.3508	.4026	.4229	.4304	.4314	.4287	.4239	.4179	.4110	.4037	.3962	.3886	.3811	.3737
14	.3298	.3806	.4013	.4097	.4118	.4102	.4064	.4013	.3954	.3890	.3822	.3754	.3686	.3618
15	.3111	.3608	.3818	.3909	.3938	.3931	.3902	.3860	.3809	.3752	.3692	.3631	.3569	.3506
16	.2944	.3430	.3641	.3737	.3773	.3774	.3753	.3718	.3674	.3624	.3570	.3515	.3458	.3401
17	.2793	.3268	.3479	.3580	.3622	.3629	.3614	.3586	.3548	.3504	.3456	.3406	.3354	.3302
18	.2657	.3120	.3331	.3435	.3481	.3494	.3485	.3462	.3430	.3392	.3349	.3304	.3257	.3209
19	.2534	.2985	.3195	.3301	.3351	.3369	.3365	.3347	.3320	.3286	.3248	.3207	.3164	.3120
20	.2422	.2861	.3069	.3177	.3231	.3252	.3253	.3239	.3216	.3187	.3153	.3116	.3077	.3037
22	.2224	.2642	.2844	.2954	.3013	.3041	.3049	.3043	.3028	.3006	.2979	.2948	.2916	.2882
24	.2056	.2453	.2650	.2761	.2823	.2856	.2869	.2869	.2860	.2843	.2822	.2798	.2771	.2742
26	.1911	.2289	.2481	.2591	.2655	.2691	.2709	.2714	.2709	.2698	.2682	.2662	.2639	.2615
28	.1786	.2146	.2332	.2440	.2506	.2545	.2566	.2574	.2573	.2566	.2554	.2538	.2520	.2499
30	.1675	.2020	.2199	.2306	.2373	.2413	.2437	.2448	.2451	.2447	.2438	.2426	.2411	.2393
32	.1578	.1907	.2081	.2186	.2253	.2295	.2320	.2334	.2339	.2338	.2332	.2323	.2310	.2296
34	.1491	.1807	.1975	.2078	.2144	.2187	.2214	.2230	.2237	.2238	.2235	.2228	.2218	.2206
36	.1414	.1716	.1879	.1980	.2046	.2089	.2117	.2134	.2144	.2147	.2145	.2140	.2133	.2123
38	.1344	.1634	.1792	.1891	.1956	.1999	.2028	.2047	.2057	.2062	.2063	.2060	.2054	.2046
40	.1280	.1560	.1713	.1809	.1873	.1917	.1947	.1966	.1978	.1984	.1986	.1985	.1980	.1974
42	.1222	.1491	.1640	.1734	.1798	.1841	.1871	.1892	.1905	.1912	.1915	.1915	.1912	.1907
44	.1170	.1429	.1573	.1665	.1728	.1771	.1801	.1822	.1836	.1845	.1849	.1850	.1848	.1845
46	.1121	.1372	.1512	.1602	.1663	.1706	.1737	.1758	.1773	.1782	.1787	.1789	.1789	.1786
48	.1077	.1319	.1455	.1543	.1603	.1646	.1676	.1698	.1713	.1723	.1730	.1733	.1733	.1731
50	.1036	.1270	.1402	.1488	.1547	.1590	.1620	.1642	.1658	.1669	.1675	.1679	.1680	.1679
55	.0945	.1162	.1285	.1366	.1423	.1464	.1495	.1517	.1534	.1546	.1554	.1559	.1562	.1563
60	.0869	.1071	.1186	.1263	.1318	.1358	.1387	.1410	.1427	.1439	.1449	.1455	.1459	.1461
65	.0805	.0993	.1102	.1175	.1227	.1265	.1294	.1317	.1334	.1347	.1357	.1364	.1369	.1372
70	.0749	.0926	.1028	.1098	.1147	.1185	.1213	.1235	.1252	.1265	.1276	.1283	.1289	.1293
75	.0701	.0867	.0964	.1030	.1078	.1114	.1141	.1163	.1180	.1193	.1204	.1212	.1218	.1223
80	.0658	.0815	.0907	.0970	.1016	.1051	.1078	.1099	.1116	.1129	.1140	.1148	.1155	.1160
85	.0620	.0769	.0857	.0917	.0961	.0994	.1021	.1041	.1058	.1071	.1082	.1091	.1097	.1103
90	.0587	.0728	.0812	.0869	.0912	.0944	.0969	.0990	.1006	.1019	.1030	.1039	.1046	.1051
95	.0557	.0691	.0771	.0826	.0867	.0898	.0923	.0943	.0959	.0972	.0982	.0991	.0998	.1004
100	.0529	.0658	.0735	.0787	.0827	.0857	.0881	.0900	.0916	.0929	.0939	.0948	.0955	.0961
125	.0425	.0530	.0593	.0637	.0670	.0696	.0717	.0734	.0748	.0760	.0770	.0778	.0786	.0792
150	.0356	.0444	.0498	.0535	.0564	.0586	.0605	.0620	.0632	.0643	.0652	.0660	.0667	.0673
175	.0305	.0382	.0428	.0461	.0486	.0506	.0523	.0536	.0548	.0558	.0566	.0573	.0580	.0585
200	.0268	.0335	.0376	.0405	.0428	.0446	.0460	.0473	.0483	.0492	.0500	.0507	.0513	.0518
250	.0215	.0269	.0302	.0326	.0345	.0359	.0372	.0382	.0391	.0398	.0405	.0411	.0416	.0421
∞	5.412	6.823	7.709	8.356	8.865	9.284	9.638	9.946	10.22	10.46	10.68	10.88	11.06	11.23

Percentile 99.5 of $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Simple order (table 1, cont.)

$\nu_e \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.9801	.9751	.9584	.9327	.9013	.8673	.8330	.7994	.7673	.7369	.7084	.6817	.6567	.6334
3	.9192	.9200	.9019	.8758	.8460	.8149	.7840	.7540	.7254	.6983	.6728	.6488	.6263	.6052
4	.8413	.8542	.8410	.8188	.7929	.7658	.7387	.7123	.6870	.6629	.6401	.6186	.5983	.5793
5	.7648	.7892	.7824	.7653	.7438	.7206	.6972	.6741	.6518	.6305	.6101	.5909	.5726	.5553
6	.6961	.7293	.7286	.7162	.6989	.6795	.6594	.6394	.6197	.6008	.5826	.5653	.5488	.5331
7	.6363	.6757	.6800	.6719	.6583	.6422	.6250	.6076	.5904	.5736	.5573	.5418	.5268	.5126
8	.5846	.6281	.6364	.6319	.6215	.6083	.5937	.5786	.5635	.5486	.5340	.5200	.5065	.4935
9	.5399	.5861	.5975	.5960	.5883	.5775	.5652	.5521	.5388	.5255	.5125	.4998	.4875	.4757
10	.5011	.5488	.5627	.5635	.5581	.5494	.5390	.5277	.5160	.5042	.4925	.4811	.4699	.4591
11	.4672	.5157	.5313	.5342	.5307	.5238	.5151	.5053	.4950	.4845	.4740	.4637	.4535	.4436
12	.4374	.4861	.5031	.5076	.5057	.5004	.4931	.4846	.4756	.4663	.4568	.4474	.4382	.4291
13	.4111	.4596	.4776	.4834	.4828	.4788	.4728	.4655	.4576	.4493	.4408	.4323	.4238	.4155
14	.3876	.4357	.4545	.4613	.4619	.4590	.4541	.4478	.4409	.4334	.4258	.4180	.4103	.4027
15	.3666	.4141	.4334	.4410	.4426	.4407	.4367	.4314	.4253	.4187	.4118	.4047	.3977	.3906
16	.3478	.3945	.4141	.4224	.4248	.4238	.4206	.4161	.4107	.4048	.3986	.3922	.3857	.3793
17	.3307	.3766	.3965	.4053	.4084	.4080	.4056	.4018	.3971	.3919	.3862	.3804	.3745	.3686
18	.3152	.3603	.3802	.3895	.3931	.3934	.3916	.3884	.3844	.3797	.3746	.3693	.3639	.3584
19	.3011	.3452	.3652	.3748	.3790	.3798	.3785	.3759	.3724	.3682	.3637	.3588	.3539	.3488
20	.2882	.3314	.3513	.3612	.3658	.3671	.3663	.3642	.3611	.3574	.3533	.3489	.3443	.3397
22	.2653	.3067	.3264	.3367	.3419	.3439	.3440	.3427	.3405	.3376	.3343	.3306	.3268	.3228
24	.2458	.2855	.3048	.3153	.3209	.3235	.3242	.3236	.3221	.3198	.3172	.3141	.3109	.3075
26	.2290	.2669	.2858	.2963	.3023	.3054	.3066	.3065	.3055	.3038	.3017	.2992	.2964	.2935
28	.2143	.2506	.2690	.2796	.2857	.2891	.2907	.2911	.2905	.2893	.2876	.2856	.2833	.2808
30	.2013	.2362	.2541	.2646	.2708	.2745	.2764	.2771	.2770	.2761	.2748	.2732	.2712	.2691
32	.1899	.2233	.2407	.2511	.2574	.2613	.2635	.2645	.2646	.2641	.2631	.2618	.2602	.2583
34	.1796	.2117	.2287	.2389	.2453	.2493	.2517	.2529	.2533	.2530	.2524	.2513	.2500	.2484
36	.1704	.2013	.2178	.2278	.2342	.2383	.2409	.2423	.2429	.2429	.2424	.2416	.2405	.2392
38	.1621	.1919	.2079	.2177	.2241	.2283	.2309	.2325	.2333	.2335	.2332	.2326	.2317	.2306
40	.1546	.1833	.1988	.2085	.2148	.2190	.2218	.2235	.2244	.2248	.2247	.2243	.2236	.2227
42	.1477	.1754	.1905	.2000	.2063	.2105	.2133	.2152	.2162	.2167	.2168	.2165	.2160	.2152
44	.1414	.1682	.1829	.1922	.1984	.2026	.2055	.2074	.2086	.2092	.2094	.2093	.2089	.2083
46	.1357	.1615	.1758	.1849	.1911	.1953	.1982	.2002	.2015	.2022	.2025	.2025	.2023	.2018
48	.1303	.1554	.1693	.1782	.1843	.1885	.1914	.1935	.1948	.1957	.1961	.1962	.1960	.1956
50	.1254	.1497	.1632	.1720	.1779	.1821	.1851	.1872	.1886	.1895	.1900	.1902	.1901	.1899
55	.1146	.1371	.1498	.1581	.1639	.1680	.1710	.1731	.1746	.1757	.1764	.1768	.1769	.1768
60	.1055	.1265	.1384	.1463	.1518	.1559	.1588	.1610	.1626	.1638	.1646	.1651	.1654	.1654
65	.0977	.1174	.1287	.1362	.1415	.1454	.1483	.1505	.1521	.1533	.1542	.1549	.1553	.1555
70	.0910	.1095	.1202	.1273	.1324	.1362	.1391	.1412	.1429	.1442	.1451	.1458	.1463	.1466
75	.0852	.1026	.1127	.1195	.1244	.1281	.1309	.1331	.1347	.1360	.1370	.1378	.1383	.1387
80	.0801	.0965	.1062	.1127	.1174	.1209	.1236	.1258	.1275	.1288	.1298	.1306	.1312	.1316
85	.0755	.0911	.1003	.1065	.1111	.1145	.1172	.1193	.1209	.1222	.1233	.1241	.1247	.1252
90	.0714	.0863	.0951	.1010	.1054	.1087	.1113	.1134	.1150	.1163	.1174	.1182	.1189	.1194
95	.0678	.0820	.0903	.0961	.1003	.1035	.1060	.1080	.1097	.1109	.1120	.1129	.1135	.1141
100	.0645	.0780	.0861	.0916	.0957	.0988	.1012	.1032	.1048	.1061	.1071	.1080	.1087	.1092
125	.0519	.0630	.0696	.0742	.0777	.0804	.0825	.0843	.0857	.0869	.0879	.0888	.0895	.0901
150	.0434	.0528	.0584	.0624	.0654	.0677	.0696	.0712	.0725	.0736	.0745	.0754	.0761	.0767
175	.0373	.0454	.0503	.0538	.0565	.0585	.0602	.0617	.0628	.0639	.0647	.0655	.0661	.0667
200	.0327	.0399	.0442	.0473	.0497	.0515	.0531	.0543	.0554	.0564	.0572	.0579	.0585	.0590
250	.0262	.0320	.0356	.0381	.0400	.0416	.0429	.0439	.0449	.0457	.0464	.0470	.0475	.0480
∞	6.635	8.144	9.092	9.784	10.33	10.77	11.15	11.48	11.77	12.03	12.26	12.47	12.66	12.84

Percentile 95 of $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Dominance (table 2)

$\nu_e \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.8100	.8781	.9033	.9173	.9265	.9332	.9384	.9426	.9461	.9490	.9515	.9537	.9556	.9574
3	.6486	.7629	.8122	.8414	.8611	.8756	.8868	.8958	.9032	.9094	.9147	.9193	.9233	.9269
4	.5319	.6653	.7298	.7701	.7983	.8193	.8358	.8491	.8601	.8694	.8774	.8843	.8903	.8957
5	.4481	.5865	.6593	.7068	.7410	.7671	.7878	.8047	.8188	.8308	.8411	.8500	.8579	.8650
6	.3862	.5231	.5996	.6515	.6898	.7196	.7435	.7633	.7799	.7941	.8065	.8172	.8267	.8352
7	.3390	.4713	.5490	.6033	.6443	.6767	.7031	.7251	.7437	.7598	.7738	.7861	.7970	.8067
8	.3018	.4285	.5057	.5612	.6039	.6380	.6662	.6899	.7102	.7277	.7431	.7567	.7688	.7796
9	.2719	.3926	.4686	.5243	.5678	.6032	.6326	.6576	.6791	.6978	.7143	.7290	.7421	.7539
10	.2473	.3621	.4363	.4917	.5356	.5717	.6020	.6279	.6503	.6700	.6874	.7030	.7169	.7295
11	.2267	.3359	.4081	.4628	.5067	.5431	.5740	.6005	.6237	.6441	.6622	.6785	.6932	.7065
12	.2093	.3132	.3832	.4370	.4806	.5171	.5483	.5753	.5990	.6200	.6387	.6555	.6708	.6846
13	.1944	.2934	.3611	.4138	.4570	.4934	.5247	.5520	.5760	.5974	.6166	.6339	.6496	.6640
14	.1814	.2759	.3414	.3930	.4356	.4717	.5030	.5304	.5547	.5764	.5959	.6136	.6297	.6444
15	.1700	.2603	.3237	.3741	.4160	.4518	.4830	.5104	.5348	.5567	.5765	.5945	.6108	.6259
16	.1600	.2464	.3077	.3569	.3980	.4334	.4644	.4918	.5162	.5383	.5582	.5764	.5930	.6083
17	.1511	.2339	.2933	.3411	.3816	.4165	.4472	.4744	.4989	.5210	.5410	.5593	.5762	.5917
18	.1431	.2226	.2801	.3267	.3664	.4008	.4311	.4582	.4826	.5047	.5248	.5432	.5602	.5759
19	.1360	.2123	.2680	.3135	.3523	.3862	.4162	.4431	.4673	.4894	.5095	.5280	.5450	.5608
20	.1295	.2029	.2569	.3013	.3393	.3726	.4022	.4289	.4529	.4749	.4950	.5135	.5307	.5465
22	.1182	.1865	.2373	.2794	.3159	.3481	.3769	.4030	.4267	.4484	.4684	.4868	.5040	.5199
24	.1087	.1725	.2204	.2605	.2955	.3266	.3545	.3799	.4032	.4246	.4443	.4627	.4798	.4957
26	.1006	.1604	.2058	.2440	.2775	.3075	.3346	.3594	.3821	.4031	.4226	.4408	.4577	.4736
28	.0937	.1499	.1929	.2295	.2616	.2906	.3168	.3409	.3631	.3837	.4029	.4208	.4376	.4533
30	.0876	.1407	.1816	.2165	.2474	.2753	.3008	.3242	.3459	.3661	.3849	.4025	.4191	.4347
32	.0823	.1326	.1715	.2050	.2347	.2617	.2863	.3091	.3303	.3500	.3684	.3858	.4021	.4175
34	.0776	.1253	.1625	.1946	.2232	.2492	.2732	.2953	.3159	.3352	.3533	.3703	.3864	.4016
36	.0734	.1188	.1544	.1852	.2128	.2379	.2611	.2827	.3028	.3216	.3393	.3560	.3718	.3868
38	.0696	.1130	.1471	.1767	.2033	.2276	.2501	.2711	.2907	.3091	.3264	.3428	.3583	.3731
40	.0662	.1077	.1404	.1689	.1946	.2182	.2400	.2604	.2795	.2975	.3144	.3305	.3458	.3603
42	.0631	.1028	.1343	.1618	.1866	.2095	.2307	.2505	.2691	.2867	.3033	.3191	.3341	.3483
44	.0603	.0984	.1287	.1552	.1793	.2014	.2220	.2413	.2595	.2767	.2929	.3084	.3231	.3371
46	.0577	.0944	.1235	.1492	.1725	.1940	.2140	.2328	.2506	.2673	.2833	.2984	.3128	.3266
48	.0554	.0906	.1188	.1436	.1662	.1870	.2065	.2249	.2422	.2586	.2742	.2890	.3032	.3167
50	.0532	.0872	.1144	.1384	.1603	.1806	.1996	.2175	.2344	.2504	.2657	.2802	.2941	.3074
55	.0484	.0796	.1047	.1269	.1473	.1663	.1841	.2009	.2168	.2320	.2465	.2604	.2737	.2864
60	.0445	.0732	.0965	.1172	.1363	.1540	.1708	.1866	.2017	.2161	.2299	.2431	.2558	.2681
65	.0411	.0678	.0895	.1089	.1268	.1435	.1593	.1743	.1886	.2023	.2154	.2280	.2402	.2519
70	.0382	.0631	.0835	.1017	.1185	.1343	.1492	.1635	.1771	.1901	.2026	.2147	.2263	.2376
75	.0357	.0591	.0782	.0953	.1112	.1262	.1404	.1539	.1669	.1793	.1913	.2028	.2140	.2248
80	.0335	.0555	.0735	.0898	.1048	.1190	.1325	.1454	.1578	.1697	.1811	.1922	.2029	.2133
85	.0315	.0523	.0694	.0848	.0991	.1126	.1255	.1378	.1496	.1610	.1720	.1826	.1929	.2029
90	.0298	.0495	.0657	.0803	.0940	.1069	.1192	.1309	.1422	.1532	.1637	.1740	.1839	.1935
95	.0282	.0470	.0624	.0763	.0894	.1017	.1134	.1247	.1356	.1461	.1562	.1661	.1756	.1849
100	.0268	.0447	.0594	.0727	.0852	.0970	.1082	.1191	.1295	.1396	.1494	.1589	.1681	.1771
125	.0215	.0359	.0479	.0588	.0690	.0787	.0881	.0971	.1058	.1143	.1225	.1306	.1384	.1460
150	.0179	.0300	.0401	.0493	.0580	.0663	.0742	.0819	.0894	.0967	.1039	.1108	.1176	.1243
175	.0154	.0258	.0345	.0425	.0500	.0572	.0641	.0709	.0775	.0839	.0901	.0962	.1022	.1081
200	.0135	.0226	.0303	.0373	.0439	.0503	.0565	.0625	.0683	.0740	.0796	.0851	.0904	.0957
250	.0108	.0181	.0243	.0300	.0354	.0406	.0456	.0505	.0552	.0599	.0645	.0690	.0735	.0778
∞	2.706	4.577	6.171	7.653	9.075	10.46	11.81	13.14	14.45	15.74	17.02	18.29	19.56	20.81

Percentile 97.5 $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Dominance (table 2, cont.)

$\nu_e \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.9025	.9356	.9478	.9547	.9592	.9625	.9651	.9672	.9690	.9704	.9717	.9729	.9739	.9748
3	.7715	.8454	.8770	.8956	.9082	.9175	.9246	.9304	.9351	.9391	.9425	.9455	.9481	.9504
4	.6584	.7576	.8047	.8338	.8541	.8692	.8810	.8906	.8984	.9051	.9108	.9157	.9201	.9239
5	.5693	.6809	.7383	.7754	.8019	.8220	.8379	.8508	.8616	.8707	.8786	.8854	.8914	.8967
6	.4995	.6160	.6796	.7221	.7533	.7774	.7967	.8125	.8258	.8371	.8469	.8555	.8630	.8697
7	.4441	.5611	.6281	.6744	.7089	.7360	.7580	.7762	.7916	.8048	.8163	.8264	.8353	.8433
8	.3993	.5145	.5832	.6317	.6686	.6980	.7221	.7422	.7594	.7742	.7871	.7985	.8087	.8178
9	.3625	.4747	.5437	.5935	.6321	.6631	.6888	.7105	.7290	.7452	.7593	.7719	.7831	.7932
10	.3318	.4404	.5089	.5594	.5989	.6311	.6580	.6809	.7006	.7179	.7331	.7466	.7587	.7696
11	.3057	.4105	.4782	.5287	.5688	.6018	.6296	.6534	.6740	.6922	.7082	.7226	.7355	.7472
12	.2835	.3843	.4507	.5011	.5414	.5749	.6033	.6278	.6492	.6680	.6848	.6998	.7134	.7257
13	.2642	.3612	.4262	.4760	.5164	.5502	.5790	.6040	.6259	.6453	.6627	.6783	.6924	.7053
14	.2473	.3407	.4041	.4533	.4935	.5273	.5564	.5818	.6041	.6240	.6418	.6579	.6725	.6858
15	.2325	.3223	.3842	.4326	.4724	.5062	.5354	.5610	.5837	.6039	.6221	.6385	.6535	.6673
16	.2193	.3058	.3661	.4136	.4531	.4867	.5159	.5416	.5645	.5849	.6034	.6202	.6355	.6496
17	.2075	.2909	.3495	.3962	.4352	.4686	.4977	.5234	.5464	.5671	.5858	.6028	.6184	.6328
18	.1969	.2774	.3344	.3802	.4186	.4517	.4807	.5064	.5294	.5502	.5691	.5863	.6021	.6167
19	.1874	.2650	.3206	.3654	.4032	.4359	.4648	.4904	.5134	.5343	.5533	.5707	.5866	.6014
20	.1787	.2537	.3078	.3516	.3889	.4212	.4498	.4753	.4983	.5192	.5383	.5558	.5719	.5868
22	.1635	.2337	.2850	.3270	.3630	.3945	.4226	.4478	.4706	.4914	.5105	.5281	.5443	.5594
24	.1507	.2167	.2654	.3056	.3403	.3710	.3984	.4231	.4457	.4663	.4853	.5029	.5192	.5344
26	.1398	.2019	.2482	.2868	.3203	.3500	.3767	.4010	.4232	.4436	.4624	.4799	.4962	.5115
28	.1303	.1890	.2331	.2702	.3025	.3313	.3573	.3810	.4028	.4229	.4416	.4589	.4751	.4903
30	.1221	.1777	.2198	.2553	.2865	.3144	.3398	.3629	.3843	.4041	.4225	.4396	.4557	.4708
32	.1148	.1676	.2079	.2420	.2721	.2992	.3238	.3465	.3674	.3868	.4049	.4219	.4378	.4528
34	.1083	.1586	.1972	.2300	.2591	.2853	.3093	.3314	.3519	.3709	.3887	.4055	.4212	.4360
36	.1025	.1506	.1875	.2192	.2473	.2727	.2960	.3176	.3376	.3563	.3738	.3903	.4058	.4205
38	.0973	.1433	.1788	.2093	.2365	.2611	.2838	.3048	.3244	.3427	.3599	.3761	.3914	.4059
40	.0927	.1366	.1708	.2002	.2265	.2505	.2726	.2931	.3122	.3302	.3470	.3630	.3781	.3924
42	.0884	.1306	.1635	.1919	.2174	.2407	.2622	.2822	.3009	.3185	.3350	.3507	.3656	.3797
44	.0845	.1251	.1568	.1843	.2090	.2316	.2526	.2721	.2904	.3076	.3238	.3392	.3539	.3678
46	.0810	.1200	.1506	.1772	.2012	.2232	.2436	.2627	.2805	.2974	.3133	.3285	.3429	.3565
48	.0777	.1153	.1449	.1707	.1940	.2154	.2353	.2539	.2714	.2879	.3035	.3184	.3325	.3460
50	.0747	.1110	.1396	.1646	.1873	.2081	.2275	.2456	.2628	.2789	.2943	.3089	.3228	.3361
55	.0681	.1015	.1279	.1512	.1723	.1918	.2101	.2272	.2434	.2588	.2734	.2874	.3007	.3135
60	.0625	.0934	.1181	.1398	.1596	.1779	.1951	.2114	.2268	.2414	.2553	.2687	.2815	.2938
65	.0578	.0866	.1096	.1300	.1486	.1659	.1822	.1976	.2122	.2262	.2395	.2523	.2645	.2763
70	.0538	.0807	.1023	.1214	.1390	.1554	.1708	.1855	.1994	.2127	.2255	.2377	.2495	.2609
75	.0503	.0755	.0959	.1139	.1306	.1461	.1608	.1748	.1881	.2008	.2130	.2248	.2361	.2470
80	.0472	.0710	.0902	.1073	.1231	.1379	.1519	.1652	.1779	.1901	.2018	.2131	.2240	.2346
85	.0444	.0669	.0852	.1014	.1165	.1306	.1439	.1566	.1688	.1805	.1918	.2026	.2131	.2233
90	.0420	.0634	.0807	.0962	.1105	.1240	.1367	.1489	.1606	.1718	.1827	.1931	.2033	.2131
95	.0398	.0601	.0766	.0914	.1051	.1180	.1302	.1419	.1532	.1640	.1744	.1845	.1942	.2037
100	.0379	.0572	.0730	.0871	.1002	.1126	.1243	.1356	.1464	.1568	.1668	.1766	.1860	.1952
125	.0304	.0461	.0589	.0705	.0813	.0915	.1013	.1107	.1198	.1286	.1371	.1454	.1534	.1613
150	.0254	.0385	.0494	.0592	.0684	.0771	.0855	.0936	.1014	.1090	.1163	.1235	.1305	.1374
175	.0218	.0331	.0425	.0510	.0590	.0666	.0739	.0810	.0879	.0945	.1010	.1074	.1136	.1197
200	.0191	.0291	.0373	.0448	.0519	.0587	.0651	.0714	.0775	.0835	.0893	.0950	.1005	.1060
250	.0153	.0233	.0300	.0361	.0418	.0473	.0526	.0578	.0628	.0677	.0725	.0771	.0817	.0863
∞	3.841	5.901	7.635	9.237	10.76	12.24	13.68	15.09	16.48	17.84	19.20	20.53	21.86	23.17

Percentile 99 of $\overline{E}^2$ (or $\overline{\chi}^2$ when $\nu_e \to \infty$) for Dominance (table 2, cont.)

$\nu_e \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.9604	.9729	.9775	.9800	.9817	.9830	.9840	.9848	.9854	.9860	.9865	.9870	.9874	.9877
3	.8730	.9131	.9302	.9403	.9471	.9522	.9561	.9593	.9619	.9641	.9660	.9677	.9691	.9704
4	.7783	.8428	.8731	.8918	.9048	.9145	.9220	.9282	.9332	.9375	.9411	.9443	.9471	.9496
5	.6937	.7746	.8156	.8418	.8605	.8746	.8858	.8948	.9024	.9088	.9143	.9190	.9232	.9270
6	.6221	.7128	.7614	.7936	.8171	.8351	.8494	.8612	.8711	.8795	.8867	.8930	.8986	.9036
7	.5622	.6580	.7118	.7486	.7758	.7970	.8141	.8283	.8402	.8504	.8593	.8671	.8739	.8801
8	.5119	.6098	.6671	.7070	.7372	.7610	.7804	.7966	.8104	.8222	.8325	.8416	.8496	.8568
9	.4694	.5676	.6268	.6690	.7014	.7273	.7486	.7664	.7817	.7949	.8065	.8168	.8259	.8341
10	.4331	.5303	.5906	.6343	.6683	.6958	.7186	.7379	.7544	.7689	.7816	.7928	.8029	.8120
11	.4018	.4974	.5580	.6027	.6378	.6665	.6905	.7109	.7286	.7440	.7577	.7698	.7807	.7906
12	.3746	.4681	.5286	.5738	.6097	.6392	.6642	.6855	.7041	.7204	.7348	.7478	.7594	.7700
13	.3508	.4420	.5019	.5473	.5837	.6139	.6395	.6616	.6809	.6979	.7131	.7267	.7390	.7501
14	.3298	.4185	.4777	.5230	.5597	.5903	.6165	.6391	.6590	.6766	.6924	.7066	.7194	.7311
15	.3111	.3973	.4556	.5007	.5374	.5683	.5949	.6180	.6383	.6565	.6727	.6874	.7007	.7128
16	.2944	.3781	.4354	.4801	.5168	.5478	.5746	.5980	.6188	.6373	.6539	.6690	.6827	.6953
17	.2793	.3606	.4169	.4610	.4976	.5287	.5556	.5793	.6003	.6191	.6361	.6515	.6656	.6785
18	.2657	.3447	.3998	.4434	.4797	.5107	.5377	.5616	.5828	.6019	.6191	.6348	.6492	.6624
19	.2534	.3301	.3840	.4270	.4630	.4939	.5209	.5448	.5662	.5855	.6029	.6189	.6335	.6470
20	.2422	.3166	.3694	.4118	.4474	.4781	.5051	.5290	.5505	.5699	.5875	.6037	.6185	.6322
22	.2224	.2927	.3433	.3843	.4191	.4493	.4760	.4999	.5214	.5409	.5588	.5752	.5903	.6043
24	.2056	.2721	.3206	.3602	.3940	.4237	.4500	.4737	.4951	.5147	.5326	.5492	.5645	.5787
26	.1911	.2542	.3006	.3389	.3717	.4007	.4266	.4500	.4712	.4907	.5087	.5253	.5407	.5550
28	.1786	.2385	.2830	.3199	.3518	.3801	.4055	.4285	.4495	.4688	.4867	.5033	.5187	.5331
30	.1675	.2246	.2673	.3029	.3339	.3614	.3863	.4089	.4296	.4487	.4665	.4830	.4984	.5128
32	.1578	.2123	.2532	.2876	.3177	.3445	.3688	.3910	.4114	.4303	.4478	.4642	.4795	.4939
34	.1491	.2012	.2406	.2738	.3029	.3291	.3528	.3745	.3946	.4132	.4306	.4468	.4620	.4763
36	.1414	.1912	.2291	.2612	.2895	.3149	.3381	.3594	.3791	.3975	.4146	.4306	.4457	.4599
38	.1344	.1822	.2187	.2497	.2772	.3020	.3246	.3454	.3648	.3828	.3997	.4155	.4305	.4446
40	.1280	.1739	.2091	.2392	.2659	.2900	.3121	.3325	.3515	.3692	.3858	.4015	.4162	.4302
42	.1222	.1664	.2004	.2295	.2554	.2789	.3005	.3205	.3391	.3565	.3729	.3883	.4029	.4167
44	.1170	.1595	.1924	.2206	.2458	.2687	.2898	.3093	.3275	.3446	.3607	.3759	.3903	.4040
46	.1121	.1531	.1850	.2124	.2368	.2592	.2797	.2989	.3167	.3335	.3494	.3643	.3785	.3920
48	.1077	.1473	.1781	.2047	.2285	.2503	.2704	.2891	.3066	.3231	.3387	.3534	.3674	.3807
50	.1036	.1418	.1717	.1976	.2208	.2420	.2616	.2800	.2971	.3133	.3286	.3431	.3569	.3701
55	.0945	.1299	.1576	.1817	.2035	.2235	.2420	.2594	.2758	.2912	.3059	.3198	.3331	.3458
60	.0869	.1197	.1456	.1682	.1887	.2075	.2251	.2417	.2572	.2720	.2861	.2995	.3123	.3245
65	.0805	.1111	.1353	.1566	.1759	.1938	.2104	.2262	.2410	.2552	.2687	.2815	.2939	.3057
70	.0749	.1036	.1264	.1465	.1647	.1817	.1975	.2125	.2268	.2403	.2532	.2656	.2775	.2889
75	.0701	.0970	.1186	.1376	.1549	.1710	.1861	.2004	.2141	.2270	.2395	.2514	.2628	.2739
80	.0658	.0913	.1117	.1297	.1462	.1615	.1760	.1897	.2027	.2152	.2271	.2386	.2496	.2603
85	.0620	.0862	.1055	.1226	.1384	.1530	.1668	.1800	.1925	.2045	.2160	.2270	.2377	.2480
90	.0587	.0816	.1000	.1163	.1313	.1454	.1586	.1712	.1832	.1948	.2059	.2165	.2268	.2368
95	.0557	.0775	.0950	.1106	.1250	.1385	.1512	.1633	.1748	.1860	.1967	.2070	.2169	.2266
100	.0529	.0737	.0905	.1055	.1192	.1322	.1444	.1560	.1672	.1779	.1882	.1982	.2078	.2172
125	.0425	.0595	.0732	.0855	.0969	.1077	.1179	.1277	.1371	.1462	.1550	.1635	.1718	.1799
150	.0356	.0498	.0614	.0719	.0817	.0909	.0996	.1081	.1162	.1241	.1318	.1392	.1464	.1535
175	.0305	.0429	.0529	.0620	.0705	.0786	.0863	.0937	.1008	.1078	.1146	.1211	.1276	.1338
200	.0268	.0376	.0465	.0545	.0621	.0692	.0761	.0827	.0891	.0953	.1013	.1072	.1130	.1187
250	.0215	.0302	.0374	.0439	.0501	.0559	.0615	.0669	.0722	.0773	.0823	.0872	.0920	.0967
∞	5.412	7.672	9.561	11.30	12.94	14.52	16.06	17.56	19.04	20.49	21.92	23.33	24.73	26.12

Percentile 99.5 of $\bar{E}^2$ (or $\bar{\chi}^2$ when $\nu_e \to \infty$) for Dominance (table 2, cont.)

$\nu_e \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	.9801	.9861	.9882	.9894	.9902	.9908	.9913	.9916	.9920	.9922	.9925	.9927	.9929	.9931
3	.9192	.9441	.9548	.9610	.9653	.9685	.9710	.9730	.9746	.9760	.9773	.9783	.9792	.9801
4	.8413	.8872	.9087	.9219	.9311	.9380	.9434	.9477	.9513	.9544	.9570	.9593	.9613	.9630
5	.7648	.8272	.8586	.8787	.8929	.9037	.9122	.9191	.9248	.9297	.9339	.9375	.9408	.9436
6	.6961	.7699	.8092	.8351	.8538	.8682	.8797	.8891	.8970	.9037	.9094	.9145	.9189	.9229
7	.6363	.7173	.7624	.7930	.8156	.8332	.8473	.8590	.8688	.8772	.8845	.8909	.8965	.9016
8	.5846	.6698	.7191	.7533	.7790	.7993	.8157	.8294	.8410	.8510	.8597	.8673	.8741	.8801
9	.5399	.6272	.6794	.7163	.7445	.7669	.7853	.8008	.8139	.8253	.8352	.8440	.8518	.8588
10	.5011	.5891	.6431	.6821	.7122	.7364	.7564	.7733	.7878	.8004	.8115	.8213	.8301	.8379
11	.4672	.5550	.6100	.6504	.6820	.7076	.7290	.7471	.7628	.7765	.7885	.7992	.8088	.8175
12	.4374	.5243	.5799	.6212	.6538	.6806	.7030	.7222	.7389	.7535	.7664	.7779	.7883	.7977
13	.4111	.4966	.5523	.5942	.6276	.6553	.6786	.6986	.7161	.7315	.7452	.7574	.7685	.7785
14	.3876	.4716	.5271	.5693	.6033	.6315	.6555	.6763	.6945	.7105	.7248	.7377	.7493	.7599
15	.3666	.4488	.5039	.5462	.5805	.6092	.6338	.6551	.6739	.6905	.7054	.7188	.7309	.7420
16	.3478	.4281	.4826	.5248	.5593	.5884	.6133	.6351	.6543	.6714	.6867	.7006	.7132	.7247
17	.3307	.4092	.4630	.5049	.5395	.5687	.5940	.6161	.6357	.6532	.6690	.6832	.6962	.7081
18	.3152	.3918	.4448	.4865	.5209	.5503	.5757	.5981	.6180	.6358	.6519	.6666	.6799	.6922
19	.3011	.3758	.4279	.4692	.5036	.5329	.5585	.5811	.6012	.6193	.6357	.6506	.6642	.6768
20	.2882	.3610	.4123	.4531	.4873	.5166	.5422	.5649	.5852	.6035	.6201	.6353	.6492	.6620
22	.2653	.3346	.3841	.4239	.4575	.4866	.5122	.5350	.5555	.5741	.5910	.6065	.6208	.6340
24	.2458	.3118	.3594	.3981	.4311	.4597	.4852	.5079	.5285	.5472	.5643	.5801	.5946	.6082
26	.2290	.2918	.3377	.3753	.4074	.4356	.4607	.4833	.5038	.5226	.5398	.5557	.5704	.5842
28	.2143	.2742	.3184	.3548	.3862	.4138	.4386	.4609	.4813	.5000	.5172	.5332	.5480	.5618
30	.2013	.2586	.3012	.3365	.3670	.3941	.4184	.4405	.4607	.4792	.4964	.5123	.5272	.5411
32	.1899	.2447	.2857	.3199	.3496	.3761	.4000	.4217	.4416	.4600	.4771	.4930	.5078	.5218
34	.1796	.2322	.2717	.3048	.3338	.3596	.3830	.4044	.4241	.4423	.4592	.4750	.4898	.5037
36	.1704	.2209	.2590	.2911	.3193	.3445	.3675	.3885	.4078	.4258	.4426	.4582	.4729	.4868
38	.1621	.2106	.2474	.2786	.3060	.3307	.3531	.3737	.3928	.4105	.4271	.4426	.4572	.4709
40	.1546	.2013	.2369	.2671	.2938	.3178	.3398	.3600	.3788	.3962	.4126	.4279	.4424	.4561
42	.1477	.1927	.2271	.2565	.2825	.3059	.3274	.3473	.3657	.3829	.3990	.4142	.4285	.4421
44	.1414	.1848	.2182	.2467	.2720	.2949	.3159	.3354	.3535	.3704	.3863	.4013	.4155	.4289
46	.1357	.1776	.2099	.2376	.2622	.2846	.3052	.3243	.3421	.3587	.3744	.3892	.4032	.4165
48	.1303	.1709	.2022	.2291	.2532	.2750	.2952	.3139	.3313	.3477	.3632	.3777	.3916	.4047
50	.1254	.1646	.1951	.2213	.2447	.2661	.2858	.3041	.3212	.3374	.3526	.3670	.3806	.3936
55	.1146	.1509	.1793	.2038	.2258	.2460	.2647	.2821	.2985	.3140	.3286	.3425	.3557	.3683
60	.1055	.1393	.1658	.1888	.2096	.2287	.2465	.2631	.2788	.2936	.3076	.3210	.3338	.3460
65	.0977	.1293	.1542	.1759	.1956	.2137	.2306	.2465	.2614	.2757	.2892	.3021	.3144	.3262
70	.0910	.1207	.1441	.1647	.1833	.2005	.2166	.2318	.2461	.2598	.2728	.2852	.2971	.3085
75	.0852	.1131	.1353	.1548	.1725	.1889	.2043	.2188	.2325	.2456	.2581	.2701	.2816	.2927
80	.0801	.1065	.1275	.1460	.1628	.1785	.1932	.2071	.2203	.2329	.2450	.2565	.2676	.2783
85	.0755	.1005	.1205	.1381	.1542	.1692	.1833	.1966	.2093	.2215	.2331	.2442	.2550	.2653
90	.0714	.0952	.1143	.1311	.1465	.1609	.1744	.1872	.1994	.2111	.2223	.2331	.2435	.2535
95	.0678	.0905	.1086	.1247	.1395	.1532	.1662	.1786	.1903	.2016	.2124	.2229	.2329	.2426
100	.0645	.0861	.1035	.1190	.1331	.1464	.1588	.1707	.1821	.1930	.2034	.2135	.2233	.2327
125	.0519	.0695	.0838	.0966	.1084	.1194	.1299	.1400	.1496	.1589	.1678	.1765	.1849	.1930
150	.0434	.0583	.0704	.0813	.0914	.1009	.1099	.1186	.1269	.1350	.1428	.1504	.1578	.1649
175	.0373	.0502	.0607	.0702	.0790	.0873	.0952	.1029	.1102	.1173	.1243	.1310	.1375	.1439
200	.0327	.0441	.0534	.0617	.0695	.0769	.0840	.0908	.0974	.1038	.1100	.1160	.1219	.1277
250	.0262	.0354	.0430	.0498	.0561	.0622	.0680	.0736	.0790	.0843	.0894	.0944	.0994	.1042
∞	6.635	9.021	11.01	12.83	14.55	16.21	17.81	19.38	20.91	22.42	23.90	25.36	26.81	28.24

Reading off the tables

The distribution of statistic $\overline{E}^2$ for monotonic variation depends upon three parameters: the number (k) of levels (groups, conditions) to be compared, the *df* (ν_e) of the error or reference mean square (MS_{error}), and the tested *model of variation* (indicated by H_1). Tables 1 and 2 furnish critical values for significance thresholds $\alpha = 0.005$, 0.01, 0.025 and 0.05, $k = 2$ to 15, and $\nu_e = 2$ to ∞, respectively for the *Simple order* model (table 1) and for the *Dominance* model (table 2). For $\nu_e \to \infty$ or if the true error variance is postulated or known (*see* illustration 3), the appropriate statistic is $\overline{\chi}^2$, instead of $\overline{E}^2$.

Let us suppose that we obtain the averages $\{ Y_1, Y_2, \cdots, Y_k \}$, in correspondance with the increasing (or decreasing) values of a regressor variable $\{ X_1, X_2, \cdots, X_k \}$. The test of the "Simple order model" consists in verifying whether the data statistically conform or not with the following multiple inequality relations (or their converse):

$$H_1 \{ \text{Simple order} \} : \qquad Y_1 \leq Y_2 \leq \cdots \leq Y_k \tag{1}$$

Note that two or more sample estimates of the Y_j's can be in apparent violation of the model (e.g. $\hat{Y}_1 > \hat{Y}_2$); this event, which has a computable probability, does not hamper the $\overline{E}^2$ test's validity. Another model, the "Dominance" (or "Simple tree") model, refers to the following global statistical inequality:

$$H_1 \{ \text{Dominance} \} : \qquad (Y_1, Y_2, \cdots, Y_{k-1}) < Y_k \tag{2}$$

or its converse (*i.e.* "Subjection").

The section on Mathematical presentation, in later pages, goes over the theory and explains the calculations for the $\overline{E}^2$ and $\overline{\chi}^2$ tests; these are also run through in the examples.

Illustration 1. Find the critical value of $\overline{E}^2$, at the 1 % level, for the Simple order model, with $k = 10$ levels (or groups) and $\nu_e = 40$. In table 1, for percentile 99, at column $k = 10$, line $\nu_e = 40$, we read 0.1978. Any obtained $\overline{E}^2$ equal to or higher than 0.1978 would confirm the Simple order model.

Illustration 2. A researcher investigates six different conditions, with 5 subjects per condition, and he hypothesizes that condition 1 dominates the other five. What critical value of $\overline{E}^2$ must be used, at the $\alpha = 0.05$ level? In the present case, the MS_{error} is in fact $MS_{within\text{-}groups}$ based on $\nu_e = k(n-1)$ *df.* Looking up table 2 for the Dominance variation model, at percentile

95, $k = 6$ and $\nu_e = 6\times(5-1) = 24$, he finds 0.2955, a value to be exceeded in order to ascertain statistically the proposed model.

Illustration 3. Taking up the case given in illustration 2, the researcher now possesses the value of σ_e^2, the true error variance (which is generally to be preferred to a sample estimate such as MS_{error}). What critical value must he use? For the present case, which is the same as if $\nu_e = \infty$, the second version of the test applies, namely $\bar{\chi}^2$. In table 2, for percentile 95, column $k = 8$, we find at the foot of the column (for $\nu_e = \infty$) the value $\bar{\chi}^2_{[.95]} = 9.075$.

Full examples

Example 1 [from Laurencelle 1993]. Fifty participants are randomly assigned to k = 5 groups, 10 per group. Following a controlled fasting diet, participants must drink 200 ml of a mixture composed of water and coffee concentrate; the mixture contains 0, 50, 100, 150 and 200 ml of concentrate for groups 1 to 5 respectively. Then, all participants are asked to carry out a standardized visuomotor task. The groups' averages are:

Group No. (ml of coffee)	1 (0)	2 (50)	3 (100)	4 (150)	5 (200)
Mean (Y_i)	24.2	26.1	25.9	27.0	27.1

The reference mean square (MS_{error}), here $MS_{w\text{-}g}$ with $\nu_e = k(n-1) = 5\times(10-1) = 45$ *df*, has previously been obtained and is equal to 5.50. With a 1 % significance level, can we assert that the quantity of ingested coffee concentrate stimulates or improves performance? *Solution*: The question asked refers to the so-called Simple order model, according to which the measure of performance (Y_i) should increase or at least not decrease for increasing quantities of concentrate (or increasing group number). The steps for performing the test, using statistic $\bar{E}^2$, are the following:

Step 1 [Enforcing the model by amalgamation of means] The first step in performing the test consists in looking up the observed Y_i series. If that series immediately conforms to the model, here the Simple order model, we put $r = k$ and go to step 2. If the model is not fully exemplified in the series, we must enforce it by applying two rules: 1) replace the values of two or more adjacent Y_i's with their (weighted) average, and 2), subject to rule 1, keep up the highest variability amongst Y_i's. Here, the regular increase is contradicted by the sub-series { 24.2 26.1 25.9 }. The best solution seems to pool Y_2 and Y_3 and substitute their average, (26.1+25.9)/2 = 26.0. The revised series (after "amalgamation") is:

$$Y_i^* = \{\ 24.2,\ 26.0,\ 26.0,\ 27.0,\ 27.1\ \}\ ;$$

it now contains only $r = 4$ distinct values and it fully conforms to the Simple order model.

Step 2 [Computing the mean squares] There are three mean squares required to establish the $\bar{E}^2$ statistic: $MS_{Conditions}$, MS_{Model}, and MS_{error}. Let the reference or error mean square (MS_{error}) be known (= 5.50), with its *df* ($\nu_e = 45$). Quantity MS_{Cond}, standing for Conditions or Levels or Groups, refers to the observed variance among the k means Y_i's, while MS_{Model} refers to the variance of the revised series of (partially amalgamated) k means Y_i^*, having r distinct values. The calculation proceeds as follows (remember that each original mean Y_i is based on n raw observations, here $n = 10$):

$$MS_{Cond} = \frac{n \sum_{i=1}^{k} (Y_i - \bar{Y})^2}{k-1}\ ;\quad MS_{Model} = \frac{n \sum_{i=1}^{k} (Y_i^* - \bar{Y})^2}{r-1}\ ; \tag{3}$$

in these formulae, $\bar{Y}$ designates the average of all k values Y_i or Y_i^*. For our data, $MS_{Cond} = 10 \times 5.452/(5-1) = 13.630$, and $MS_{Model} = 10 \times 5.432/(4-1) = 18.107$. We may recast these computations in the following, handier formulae: $MS_{Cond} = n \cdot s^2(Y_i)$ and $MS_{Model} = n \cdot s^2(Y_i^*) \times [(k-1)/(r-1)]$.

Note that, at this step, if the original Y_i series conforms to the chosen model, then $r = k$ and, consequently, quantities MS_{Cond} and MS_{Model} are equal.

Step 3 [Concluding the test] The formula for establishing the $\bar{E}^2$ statistic is:

$$\bar{E}^2 = \frac{(r-1)MS_{Model}}{(k-1)MS_{Cond} + \nu_e MS_{error}}\ . \tag{4}$$

For our example, we calculate $\bar{E}^2 = [(4-1) \times 18.107]/[\ (5-1) \times 13.630 + 45 \times 5.50] \approx 0.1799$. In table 1, on the page for percentile 99 (or $\alpha = 0.01$), column $k = 5$ shows 0.1665 for $\nu_e = 44$ and 0.1602 for $\nu_e = 46$, giving 0.1634 after linear interpolation for $\nu_e = 45$. The obtained $\bar{E}^2$ (= 0.1799) being higher than the critical value, we may reject the null hypothesis (of no variation) at the 1 % significance threshold and assert that our obtained data agree with the Simple order model[1].

[1] The reader may note, with this example, that the overall ANOVA F quotient, *i.e.* $F = MS_{Cond}/MS_{error}$, is equal to 2.478 and that, with its 4 and 45 *df*, it is not significant at the 1 %, indeed at the 5 % level.

Example 2 [based on Laurencelle 1993]. A microbiologist inoculates 48 laboratory rats with a microbial suspension, then allocates them randomly in four groups of 12. Animals in Group 1 receive an antibiotics treatment, while the three other groups are given diverse medicinal and vitamin-rich supplements, without antibiotics. A "vitality index" (based on body weight and overall muscular exertion) is calculated for each animal following seven days of treatment. The measured averages are Y_i = { 26.4, 24.2, 27.1, 22.7 }. The $MS_{w\text{-}g}$ (= MS_{error}) equals 8.72, with $\nu_e = k(n-1) = 44$. At the 5 % significance level, can we confirm that treatment with antibiotics is superior to the other compared treatments? *Solution*: The present situation calls for the Dominance (ou Simple tree) model, *i.e.* the dominance of Group 1 over the three others. The steps for the test, globally similar to those for the Simple order model, are concisely given hereafter.

> Step 1 [Enforcing the model by amalgamation of means] The Dominance model is infringed in our data by Group 3's average, Y_3 = 27.1, which should be *dominated* by Y_1. It is advantageous to combine Y_3 with Y_2, their values being closer, in order to preserve high variance. Consequently, the amalgamated series becomes Y_i^* = { 26.4, 25.65, 25.65, 22.7 } and now contains r = 3 distinct components.
>
> Step 2 [Computing the mean squares] Using the simplified formulae, we compute $MS_{Cond} = n \cdot s^2(Y_i) = 12 \times 4.0867 = 49.040$ and $MS_{Model} = n \cdot s^2(Y_i^*) \times [(k-1)/(r-1)] = 12 \times 2.685 \times 3/2 = 48.330$. We already have MS_{error} = 8.72 and ν_e = 44.
>
> Step 3 [Concluding the test] Formula (4) given above can be carried out. We thus calculate $[(3-1) \times 48.330]/[(4-1) \times 49.040 + 44 \times 8.72] \approx 0.1821$. In table 2, for percentile 95, column k = 4 and line ν_e = 44, we read 0.1287. The calculated $\bar{E}^2$ being higher, we reject the null hypothesis and conclude that our data globally support the superior efficacy of the antibiotics treament[2].

Mathematical presentation

The most widespread model of relation between a dependent (Y) variable and an independent (X) variable is unquestionably polynomial linear regression (*see* section on Orthogonal polynomials), including the foremost simple linear regression. Other functional models – Fourier series model, mixed auto-regressive models, etc. – may appear in appropriate contexts.

[2] In this example of the Dominance model, a simple t test for an appropriate *contrast* would also do the job. The statistic, $t = \sqrt{n}\,[(k-1)Y_1 - Y_2 - Y_3 - \cdots - Y_k]/\sqrt{[k(k-1)MS_{error}]} = \sqrt{12}\,[3 \times 26.4 - 24.2 - 27.1 - 22.7]/\sqrt{[4 \times 3 \times 8.72]} \approx 1.769$, is t-distributed with 44 *df.* The critical value (one-tailed) at 5 % (*see* section on Student's t distribution) is $t_{44[.95]}$ = 1.680, our result obtaining significance again. The relative powers of these two approaches have not been established, to our knowledge.

However, it may happen that the researcher be unable or unprepared for setting up precisely the equivalence function f in a model: $\hat{Y} = f(X)$ but, instead, that he postulates only an order relation tying up the dependent variable Y_i with an ordered set of fixed values (X_i) of the independent variable. Consider the bivariate series (X,Y) on k points:

$$\begin{Bmatrix} X_1 & X_2 & \cdots & X_k \\ Y_1 & Y_2 & \cdots & Y_k \end{Bmatrix},$$

where the X_i values are ordered, e.g. $X_1 < X_2 < \cdots < X_k$, and each Y_i denotes the average of n observations made under treatment or condition level X_i. In this context, the researcher could postulate the *Simple order* model:

(Simple order) $\{ \mu_{Y_1} \leq \mu_{Y_2} \leq \cdots \leq \mu_{Y_k} \}$,

in which at least one inequality is strict, or the *Dominance* model:

(Dominance) $\{ (\mu_{Y_1}, \mu_{Y_2}, \cdots, \mu_{Y_{k-1}}) < \mu_{Y_k} \}$,

or any other model characterized by ordinal relations, without any specified metric. The ability to test a hypothesis of this kind dispenses the researcher from forcibly fitting his data to a model with a given metric, linear or non-linear. The general null hypothesis, as opposed to any one of these ordinal or hierarchical models, may be written: H_0: $\mu_{Y_1} = \mu_{Y_2} = \cdots = \mu_{Y_k} = \mu_Y$. Thus, the researcher may judge the pertinency of the Simple order model without resorting to the subterfuge of simple or first-degree linear regression, or without making do with the global F test of ANOVA, an ambiguous and less powerful statistic in this context.

The tests of monotonic variation[3] permit us to decide whether the observed variation in the vector $\{ Y_1, Y_2, \cdots, Y_k \}$ matches a given variation model or not. The required parameters are the number (k) of levels (or groups, or means), the degrees of freedom (ν_e) associated with the estimate of error variance, and the chosen model of monotonic variation. Here, we examine only the Simple order and Dominance models; likewise, the proposed developments all require that the k means (Y_i) be based on an equal number (n) of observations.

Let us have an experimental design comprising k groups or samples of n observations each, a "Between-groups" or "Conditions" mean square (MS_{Cond}), measuring the observed variation among means and having $k-1$ df, finally an "Error" mean square (MS_{error}), represented here by the "Within-groups" mean square ($MS_{w\text{-}g}$) with $\nu_e = k(n-1)$ df. Now, we can calculate

[3] Authors also use the expressions: "isotonic variation" and "isotonic regression".

one more mean square (MS_{Model}) pertaining to the chosen monotonic model and based on $r-1$ *df* ($r-1 \le k-1$), r being defined lower. The test statistic is then:

$$\bar{E}^2 = \frac{(r-1)MS_{Model}}{(k-1)MS_{Cond} + \nu_e \times MS_{error}} \ .$$

If the true error variance (σ_e^2) is known (or $\nu_e \to \infty$), the test changes to:

$$\bar{\chi}^2 = \frac{(r-1)MS_{Model}}{\sigma_e^2} \ .$$

Values of these test statistics that exceed the appropriate critical value, with parameters k and ν_e and significance level α, confirm statistically the variation model.

The computations necessary for obtaining MS_{Model} and determining r ($\le k$) are laid out below.

Computation of MS_{Model}. The observed vector of means (Y_i) does or does not conform to the chosen model (here, Simple order or Dominance). If it does, then $MS_{Model} = MS_{Cond}$ and $r = k$. If it does not, we must enforce the model on our data by reducing the "errors", *i.e.* by pooling adjacent levels of the independant variable (X_i) and amalgamating (averaging) their corresponding means (Y_i); each amalgamated mean (Y_i^*) is the *weighted average* of the component means. The amalgamation process is continued until the revised data array, { Y_1^*, Y_2^*, ..., Y_k^* }, respects every ordinal relations in the model. When the choice exists, the selection of levels to be pooled should be done with the aim of keeping the variance among means as high as possible. Each pooling of two levels entails the loss of 1 *df*, and parameter r denotes the number of distinct means after amalgamation. Noting by $\bar{Y}$ the average of all Y_i (and all Y_i^*), we now obtain:

$$MS_{Model} = n \sum_{j=1}^{k} (Y_i^* - \bar{Y})^2 \,/\, (r-1)$$

$$= n \operatorname{var}(\ Y_i^*\ ) \times [(k-1)/(r-1)] \ .$$

To illustrate the amalgamation process for the Simple order model, let us take the vector { 11, 9, 14, 19, 22, 21 }, with $k = 6$ components (or means, say with n = 1): the MS_{Cond} (= s^2) is 29.6. We wish to obtain increasing or non-decreasing values Y_i^*. By pooling up neighbors that are closest, *i.e.* (11, 9) and (22, 21), we obtain the revised vector { 10, 10, 14, 19, 21.5, 21.5 }, which now conforms to Simple order with r = 4 distinct values, MS_{Model} = 48.5 and $(r-1)MS_{Model}$ = 145.5

For the Dominance model, the same vector, { 11 | 9, 14, 19, 22, 21 }, may be utilized, again with $k = 6$ and $MS_{Cond} = 29.6$: the last five levels should dominate level 1. Let us amalgamate transgressor value "9" with a neighboring good value, "14", to obtain { 11 | 11.5, 11.5, 19, 22, 21 }, $r = 5$, $MS_{Model} = 33.875$ and $(r-1)MS_{Model} = 135.5$. Had we chosen to pool "9" with "22" instead of "14", the loss of variance ($MS_{Model} = 15.875$), and of power, would have been dreadful.

Calculation and moments

Extensive calculations are required to determine the critical values (C) of the $\overline{E}^2$ test statistic. The master formulae are:

$$\begin{aligned} \Pr\{\overline{E}^2 \geq C\} &= \sum_{r=2}^{k} P_{Model}(k,r) \Pr\{\beta_{(r-1)/2,(\nu_e+k-r)/2} \geq C\} , && \text{if } C > 0 \\ \Pr\{\overline{E}^2 \geq 0\} &= P_{Model}(k,1) , && \text{if } C = 0 . \end{aligned} \quad (5)$$

Probability "$\Pr\{\overline{E}^2 \geq C\}$" is thus a weighted sum of component probabilities. Firstly, the weight function, "$P_{Model}(k,r)$", depends on the chosen variation model (*see* below). Secondly, the component probabilities are integrals from the *Beta* (β) distribution, with parameters a and b: the β r.v. is akin to F (*see* section on F distribution) through the transformation $\beta_{a,b} \rightarrow F_{\nu_1,\nu_2}$, where $F = a\beta/[b(1-\beta)]$, $\nu_1 = 2a$ and $\nu_2 = 2b$. For the $\overline{\chi}^2$ test, the β r.v. in the right-hand term of (5) is simply replaced with a Chi-square r.v. having $r-1$ *df*, *i.e.* χ^2_{r-1}.

To complete the probability calculations and determine critical values, we must specify the appropriate weight function. For the Simple order model, the elements of the weight function are established recursively, with:

$$P_{s.o.}(k,1) = 1/k! \ ; \ P_{s.o.}(k,k) = 1/k \qquad \text{(Simple order)}$$

and:

$$P_{s.o.}(k,r) = \frac{1}{k} P_{s.o.}(k-1,r-1) + \frac{k-1}{k} P_{s.o.}(k-1,r) \ , \quad 1 < r < k .$$

For the Dominance model, we rather have:

$$P_d(k,k) = 1/k \ ; \ P_d(k,1) = 1 - \sum_{r=2}^{k} P_d(k,r) \qquad \text{(Dominance)}$$

and:

$$P_d(k,r) = \binom{k-1}{k-r} P_d(k-r+1,1) \, N^{+}_{r-1}(1/\sqrt{k-r+1}) \ , \quad 1 < r < k .$$

Quantity $N_r^+(\rho)$, in this last expression, denotes the area of the hypervolume confined by $x_1>0$, $x_2>0$, $\cdots$, $x_r>0$, *i.e.* the integral of the positive hyperquadrant of a multivariate normal distribution in r dimensions for which the correlations of the x_i across dimensions are all equal to ρ. The figures given below illustrate this function.

Supplementary formulae and data, notably on the amalgamation and computation algorithms, may be found in the references.

Moments. Barlow, Bartholomew, Bremner and Brunk (1972) give some developments on the moments of statistic $\bar{\chi}^2$ with respect to the Simple order model along with indications for approximating its distribution with a standard χ^2 distribution.

Values of the positive multinormal integral $N_k^+(\rho)$, according to the number (k) of dimensions and their common inter-correlation (ρ) [decimal point omitted]

k / ρ	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
2	2500	2659	2820	2985	3155	3333	3524	3734	3976	4282	5000
3	1250	1489	1731	1977	2232	2500	2786	3101	3464	3923	5000
4	0625	0871	1130	1403	1692	2000	2335	2707	3140	3693	5000
5	0313	0529	0774	1045	1342	1667	2026	2432	2910	3527	5000
6	0156	0331	0551	0808	1100	1429	1800	2226	2735	3400	5000
7	0^2781	0214	0404	0642	0924	1250	1626	2066	2596	3297	5000
8	0^2391	0141	0304	0522	0791	1111	1488	1935	2482	3211	5000
9	0^2192	0^2955	0234	0432	0688	1000	1375	1827	2386	3138	5000
10	0^3977	0^2659	0184	0364	0605	0909	1281	1735	2303	3075	5000
11	0^3488	0^2462	0146	0310	0538	0833	1201	1656	2231	3019	5000
12	0^3244	0^2330	0118	0267	0483	0769	1132	1587	2168	2969	5000
13	0^3122	0^2239	0^2967	0233	0437	0714	1072	1526	2111	2925	5000
14	0^4610	0^2175	0^2799	0204	0398	0667	1019	1471	2060	2884	5000
15	0^4305	0^2130	0^2668	0181	0364	0625	0972	1422	2014	2847	5000

F_{max} distribution

✓ Percentiles 75, 90, 95 and 99 of the F_{max} distribution
✓ Reading off the table
✓ Full examples
✓ Mathematical presentation
Calculation and moments

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Percentile 75 of the F_{max} distribution

$\nu \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	7.000	15.44	25.05	35.55	46.75	58.54	70.83	83.56	96.68	110.2	123.9	138.0	152.4	167.0
3	4.476	8.128	11.68	15.14	18.54	21.88	25.16	28.40	31.59	34.73	37.85	40.92	43.96	46.97
4	3.524	5.789	7.809	9.674	11.43	13.09	14.67	16.19	17.65	19.07	20.45	21.78	23.08	24.36
5	3.024	4.663	6.050	7.284	8.411	9.456	10.44	11.36	12.24	13.08	13.89	14.67	15.41	16.14
6	2.713	4.003	5.054	5.967	6.785	7.532	8.222	8.868	9.476	10.05	10.60	11.13	11.63	12.11
7	2.499	3.568	4.415	5.137	5.775	6.351	6.878	7.368	7.825	8.255	8.662	9.050	9.419	9.773
8	2.343	3.259	3.970	4.567	5.088	5.555	5.980	6.371	6.735	7.076	7.397	7.700	7.989	8.265
9	2.222	3.027	3.641	4.151	4.592	4.984	5.338	5.663	5.964	6.245	6.508	6.757	6.993	7.217
10	2.127	2.847	3.388	3.833	4.215	4.553	4.857	5.135	5.391	5.629	5.852	6.062	6.260	6.449
11	2.049	2.702	3.187	3.582	3.920	4.217	4.483	4.726	4.948	5.155	5.348	5.529	5.700	5.862
12	1.983	2.583	3.023	3.379	3.682	3.947	4.184	4.399	4.596	4.778	4.948	5.108	5.258	5.400
13	1.928	2.483	2.886	3.211	3.486	3.726	3.939	4.132	4.309	4.472	4.624	4.766	4.900	5.026
14	1.880	2.397	2.771	3.069	3.321	3.540	3.735	3.910	4.070	4.218	4.355	4.484	4.604	4.718
15	1.838	2.324	2.671	2.948	3.181	3.382	3.561	3.722	3.869	4.004	4.129	4.246	4.356	4.459
16	1.801	2.259	2.585	2.843	3.059	3.246	3.412	3.561	3.696	3.820	3.936	4.043	4.144	4.239
17	1.769	2.202	2.509	2.751	2.954	3.128	3.282	3.421	3.546	3.662	3.768	3.868	3.961	4.049
18	1.739	2.152	2.442	2.670	2.860	3.024	3.169	3.298	3.415	3.523	3.622	3.715	3.802	3.883
19	1.713	2.106	2.382	2.598	2.778	2.932	3.068	3.189	3.299	3.400	3.493	3.580	3.661	3.737
20	1.689	2.066	2.328	2.533	2.704	2.850	2.978	3.092	3.196	3.291	3.379	3.460	3.537	3.608
21	1.667	2.028	2.279	2.475	2.637	2.775	2.897	3.006	3.104	3.194	3.277	3.353	3.425	3.493
22	1.647	1.995	2.235	2.422	2.576	2.708	2.824	2.927	3.021	3.106	3.184	3.257	3.325	3.389
23	1.628	1.963	2.194	2.373	2.521	2.647	2.758	2.856	2.945	3.026	3.101	3.170	3.235	3.295
24	1.611	1.935	2.157	2.329	2.471	2.592	2.697	2.791	2.876	2.953	3.025	3.091	3.152	3.210
25	1.595	1.908	2.123	2.288	2.424	2.540	2.642	2.732	2.813	2.887	2.955	3.018	3.077	3.132
26	1.581	1.884	2.091	2.251	2.382	2.493	2.590	2.677	2.755	2.826	2.891	2.951	3.008	3.060
27	1.567	1.861	2.061	2.216	2.342	2.449	2.543	2.626	2.701	2.769	2.832	2.890	2.944	2.995
28	1.554	1.840	2.034	2.183	2.305	2.409	2.499	2.579	2.652	2.717	2.778	2.833	2.885	2.934
29	1.542	1.820	2.008	2.153	2.271	2.371	2.459	2.536	2.606	2.669	2.727	2.781	2.831	2.877
30	1.530	1.801	1.984	2.124	2.239	2.336	2.420	2.495	2.563	2.624	2.680	2.732	2.780	2.825
35	1.482	1.723	1.884	2.006	2.106	2.190	2.263	2.327	2.385	2.437	2.485	2.530	2.571	2.609
40	1.444	1.662	1.807	1.916	2.005	2.079	2.144	2.201	2.252	2.298	2.340	2.379	2.415	2.448
45	1.414	1.614	1.745	1.845	1.925	1.992	2.050	2.101	2.147	2.189	2.226	2.261	2.293	2.323
50	1.388	1.574	1.695	1.787	1.860	1.922	1.975	2.021	2.063	2.101	2.135	2.166	2.196	2.223
55	1.367	1.541	1.654	1.738	1.806	1.863	1.912	1.955	1.993	2.028	2.060	2.089	2.115	2.140
60	1.349	1.512	1.618	1.697	1.761	1.814	1.859	1.899	1.935	1.967	1.996	2.023	2.048	2.071
120	1.235	1.339	1.404	1.452	1.490	1.521	1.548	1.571	1.592	1.610	1.627	1.642	1.657	1.670
240	1.161	1.228	1.271	1.301	1.325	1.345	1.361	1.376	1.388	1.400	1.410	1.419	1.428	1.436
480	1.111	1.156	1.184	1.205	1.220	1.233	1.244	1.253	1.261	1.268	1.275	1.281	1.286	1.291
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Percentile 90 of the F_{max} distribution

$\nu \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	19.00	42.48	69.14	98.18	129.1	161.7	195.6	230.7	266.8	303.9	341.9	380.6	420.1	460.3
3	9.277	16.78	23.96	30.93	37.74	44.40	50.95	57.38	63.72	69.97	76.14	82.24	88.26	94.22
4	6.389	10.38	13.88	17.08	20.07	22.89	25.57	28.15	30.62	33.01	35.33	37.58	39.77	41.91
5	5.051	7.682	9.865	11.80	13.54	15.16	16.66	18.09	19.43	20.72	21.95	23.14	24.28	25.38
6	4.284	6.227	7.780	9.115	10.31	11.39	12.39	13.31	14.19	15.01	15.80	16.55	17.26	17.95
7	3.788	5.324	6.518	7.525	8.409	9.204	9.931	10.60	11.23	11.82	12.38	12.91	13.41	13.89
8	3.439	4.710	5.676	6.480	7.177	7.799	8.363	8.881	9.362	9.812	10.24	10.64	11.02	11.38
9	3.179	4.265	5.075	5.742	6.315	6.823	7.281	7.699	8.086	8.446	8.784	9.103	9.404	9.691
10	2.979	3.927	4.626	5.194	5.680	6.108	6.491	6.841	7.162	7.461	7.740	8.003	8.251	8.487
11	2.818	3.662	4.276	4.772	5.193	5.561	5.891	6.190	6.465	6.719	6.956	7.179	7.388	7.587
12	2.687	3.448	3.996	4.436	4.807	5.131	5.419	5.680	5.919	6.140	6.346	6.538	6.720	6.891
13	2.577	3.272	3.767	4.162	4.494	4.783	5.039	5.271	5.482	5.677	5.858	6.028	6.187	6.337
14	2.484	3.123	3.576	3.934	4.235	4.495	4.726	4.934	5.123	5.298	5.460	5.611	5.753	5.887
15	2.404	2.997	3.413	3.742	4.017	4.254	4.464	4.653	4.824	4.982	5.128	5.264	5.392	5.513
16	2.334	2.888	3.274	3.578	3.831	4.049	4.241	4.414	4.570	4.714	4.848	4.972	5.088	5.197
17	2.272	2.792	3.153	3.435	3.670	3.872	4.049	4.208	4.353	4.485	4.607	4.721	4.828	4.928
18	2.218	2.708	3.046	3.310	3.529	3.717	3.882	4.030	4.163	4.286	4.399	4.505	4.603	4.696
19	2.169	2.633	2.952	3.200	3.405	3.581	3.735	3.873	3.998	4.112	4.217	4.315	4.406	4.492
20	2.125	2.566	2.868	3.102	3.295	3.460	3.605	3.734	3.851	3.958	4.056	4.148	4.233	4.313
21	2.085	2.505	2.792	3.014	3.197	3.352	3.489	3.611	3.721	3.821	3.913	3.999	4.079	4.155
22	2.048	2.451	2.724	2.935	3.108	3.255	3.385	3.500	3.603	3.698	3.785	3.866	3.942	4.012
23	2.015	2.401	2.662	2.863	3.027	3.168	3.290	3.399	3.498	3.588	3.670	3.747	3.818	3.885
24	1.984	2.355	2.605	2.797	2.954	3.088	3.205	3.308	3.402	3.487	3.566	3.638	3.706	3.769
25	1.956	2.313	2.553	2.737	2.887	3.015	3.127	3.225	3.315	3.396	3.470	3.539	3.604	3.664
26	1.930	2.274	2.505	2.681	2.826	2.948	3.055	3.149	3.235	3.312	3.383	3.449	3.511	3.568
27	1.905	2.238	2.460	2.630	2.769	2.887	2.989	3.080	3.161	3.235	3.304	3.367	3.425	3.480
28	1.883	2.205	2.419	2.583	2.717	2.830	2.928	3.015	3.093	3.165	3.230	3.290	3.346	3.399
29	1.861	2.173	2.381	2.539	2.668	2.777	2.871	2.955	3.031	3.099	3.162	3.220	3.274	3.324
30	1.841	2.144	2.345	2.498	2.623	2.728	2.819	2.900	2.972	3.038	3.099	3.154	3.206	3.255
35	1.758	2.023	2.197	2.329	2.435	2.525	2.603	2.672	2.733	2.789	2.840	2.887	2.931	2.971
40	1.693	1.930	2.085	2.202	2.295	2.374	2.442	2.502	2.556	2.604	2.649	2.690	2.728	2.763
45	1.642	1.858	1.997	2.102	2.186	2.257	2.317	2.371	2.419	2.462	2.501	2.537	2.571	2.602
50	1.600	1.798	1.926	2.022	2.098	2.162	2.217	2.266	2.309	2.348	2.383	2.416	2.446	2.474
55	1.565	1.749	1.867	1.955	2.026	2.084	2.135	2.179	2.219	2.254	2.287	2.316	2.344	2.370
60	1.535	1.707	1.817	1.899	1.964	2.019	2.066	2.107	2.143	2.176	2.206	2.233	2.259	2.282
120	1.352	1.457	1.523	1.571	1.608	1.640	1.666	1.689	1.710	1.728	1.744	1.760	1.774	1.787
240	1.238	1.304	1.346	1.375	1.398	1.417	1.434	1.448	1.460	1.471	1.481	1.490	1.498	1.506
480	1.163	1.207	1.233	1.253	1.267	1.280	1.290	1.299	1.306	1.313	1.320	1.325	1.331	1.335
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Percentile 95 of the F_{max} distribution

$\nu \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	39.00	87.49	142.5	202.4	266.2	333.2	403.1	475.4	549.8	626.2	704.4	784.2	865.5	948.3
3	15.44	27.76	39.51	50.89	61.98	72.84	83.48	93.95	104.3	114.5	124.5	134.4	144.1	153.8
4	9.605	15.46	20.56	25.22	29.55	33.63	37.52	41.24	44.82	48.27	51.62	54.87	58.03	61.11
5	7.147	10.76	13.73	16.34	18.70	20.88	22.92	24.83	26.65	28.38	30.04	31.63	33.17	34.65
6	5.820	8.363	10.38	12.11	13.65	15.04	16.32	17.52	18.64	19.70	20.71	21.67	22.59	23.47
7	4.995	6.940	8.440	9.701	10.81	11.80	12.71	13.54	14.32	15.05	15.74	16.40	17.02	17.62
8	4.434	6.002	7.186	8.166	9.016	9.771	10.46	11.09	11.67	12.22	12.73	13.21	13.67	14.11
9	4.026	5.339	6.312	7.109	7.794	8.398	8.943	9.440	9.900	10.33	10.73	11.11	11.47	11.81
10	3.717	4.846	5.670	6.339	6.910	7.411	7.860	8.269	8.645	8.994	9.320	9.626	9.916	10.19
11	3.474	4.464	5.179	5.754	6.242	6.669	7.049	7.394	7.711	8.004	8.277	8.534	8.775	9.004
12	3.278	4.160	4.791	5.296	5.721	6.091	6.421	6.719	6.991	7.243	7.477	7.696	7.903	8.098
13	3.116	3.913	4.477	4.926	5.303	5.630	5.920	6.181	6.420	6.640	6.845	7.036	7.215	7.385
14	2.979	3.706	4.218	4.622	4.960	5.252	5.511	5.744	5.956	6.151	6.332	6.502	6.660	6.810
15	2.863	3.532	3.999	4.367	4.674	4.938	5.172	5.381	5.572	5.747	5.910	6.061	6.203	6.337
16	2.762	3.383	3.813	4.151	4.431	4.672	4.885	5.076	5.249	5.408	5.555	5.692	5.820	5.941
17	2.674	3.253	3.652	3.964	4.223	4.445	4.640	4.815	4.973	5.119	5.253	5.378	5.495	5.605
18	2.596	3.139	3.512	3.802	4.042	4.247	4.428	4.589	4.736	4.869	4.993	5.108	5.215	5.316
19	2.527	3.039	3.388	3.660	3.883	4.075	4.243	4.393	4.528	4.652	4.767	4.873	4.972	5.065
20	2.465	2.949	3.279	3.533	3.743	3.922	4.079	4.219	4.346	4.461	4.568	4.667	4.759	4.846
21	2.409	2.869	3.180	3.421	3.618	3.787	3.934	4.065	4.184	4.292	4.392	4.484	4.570	4.651
22	2.358	2.796	3.092	3.319	3.506	3.665	3.804	3.928	4.039	4.141	4.235	4.321	4.402	4.478
23	2.312	2.730	3.012	3.228	3.405	3.556	3.687	3.804	3.909	4.005	4.094	4.175	4.252	4.323
24	2.270	2.670	2.939	3.145	3.313	3.456	3.581	3.692	3.792	3.883	3.966	4.044	4.116	4.183
25	2.231	2.615	2.872	3.069	3.230	3.366	3.485	3.590	3.685	3.771	3.851	3.924	3.993	4.057
26	2.195	2.565	2.811	2.999	3.153	3.283	3.397	3.497	3.587	3.670	3.745	3.815	3.880	3.941
27	2.161	2.518	2.755	2.935	3.083	3.207	3.316	3.412	3.498	3.577	3.649	3.716	3.778	3.836
28	2.130	2.474	2.703	2.876	3.018	3.137	3.241	3.333	3.416	3.491	3.560	3.624	3.683	3.739
29	2.101	2.434	2.654	2.822	2.958	3.072	3.172	3.261	3.340	3.412	3.478	3.539	3.596	3.649
30	2.074	2.396	2.609	2.771	2.902	3.012	3.108	3.193	3.270	3.339	3.402	3.461	3.516	3.567
35	1.962	2.241	2.423	2.561	2.672	2.766	2.847	2.919	2.983	3.041	3.094	3.143	3.189	3.231
40	1.876	2.123	2.284	2.405	2.502	2.584	2.655	2.717	2.772	2.822	2.868	2.911	2.950	2.986
45	1.808	2.031	2.176	2.284	2.371	2.443	2.506	2.561	2.610	2.654	2.695	2.732	2.767	2.799
50	1.752	1.957	2.089	2.187	2.265	2.331	2.387	2.437	2.481	2.521	2.557	2.591	2.622	2.650
55	1.706	1.896	2.017	2.106	2.178	2.238	2.290	2.335	2.375	2.412	2.445	2.475	2.503	2.530
60	1.667	1.844	1.956	2.039	2.106	2.161	2.209	2.250	2.287	2.321	2.351	2.379	2.405	2.429
120	1.433	1.538	1.603	1.651	1.688	1.719	1.745	1.768	1.789	1.807	1.824	1.839	1.853	1.865
240	1.289	1.355	1.395	1.424	1.447	1.465	1.481	1.495	1.507	1.518	1.527	1.536	1.544	1.552
480	1.197	1.239	1.265	1.284	1.298	1.310	1.320	1.328	1.336	1.343	1.349	1.354	1.359	1.364
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Percentile 99 of the F_{max} distribution

$\nu \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
2	199.0	447.5	729.2	1036	1362	1705	2063	2432	2813	3204	3604	4012	4427	4850
3	47.47	84.57	119.8	153.9	187.0	219.4	251.2	282.3	313.1	343.3	373.1	402.6	431.7	460.5
4	23.16	36.7	48.43	59.09	69.00	78.33	87.2	95.68	103.9	111.7	119.4	126.7	133.9	140.9
5	14.94	22.07	27.9	33.01	37.62	41.86	45.82	49.54	53.06	56.42	59.63	62.72	65.70	68.57
6	11.07	15.60	19.16	22.20	24.89	27.33	29.57	31.66	33.62	35.47	37.22	38.90	40.50	42.04
7	8.886	12.10	14.55	16.61	18.40	20.01	21.47	22.82	24.08	25.26	26.38	27.44	28.45	29.41
8	7.496	9.938	11.77	13.28	14.58	15.74	16.79	17.75	18.64	19.47	20.25	20.99	21.69	22.35
9	6.542	8.494	9.931	11.11	12.11	13.00	13.79	14.52	15.19	15.82	16.40	16.95	17.47	17.97
10	5.847	7.465	8.639	9.588	10.40	11.11	11.74	12.32	12.85	13.34	13.80	14.23	14.63	15.02
11	5.320	6.697	7.685	8.477	9.146	9.731	10.26	10.73	11.16	11.56	11.93	12.28	12.61	12.93
12	4.907	6.103	6.953	7.629	8.199	8.694	9.134	9.531	9.894	10.23	10.55	10.84	11.11	11.37
13	4.574	5.631	6.375	6.964	7.457	7.885	8.264	8.606	8.917	9.204	9.471	9.720	9.954	10.18
14	4.300	5.246	5.907	6.428	6.862	7.238	7.570	7.868	8.140	8.390	8.622	8.838	9.041	9.232
15	4.070	4.927	5.521	5.987	6.374	6.708	7.003	7.268	7.508	7.729	7.933	8.124	8.303	8.471
16	3.875	4.657	5.197	5.618	5.968	6.268	6.533	6.770	6.985	7.182	7.365	7.535	7.694	7.844
17	3.707	4.427	4.921	5.305	5.624	5.896	6.136	6.351	6.545	6.723	6.888	7.041	7.184	7.319
18	3.561	4.228	4.683	5.037	5.328	5.578	5.797	5.993	6.170	6.333	6.482	6.622	6.752	6.874
19	3.432	4.054	4.477	4.804	5.073	5.303	5.505	5.685	5.847	5.996	6.133	6.261	6.380	6.492
20	3.318	3.901	4.295	4.599	4.849	5.062	5.249	5.416	5.566	5.703	5.830	5.948	6.057	6.160
21	3.217	3.765	4.134	4.418	4.652	4.851	5.024	5.179	5.319	5.446	5.564	5.673	5.774	5.870
22	3.125	3.643	3.990	4.257	4.476	4.662	4.825	4.970	5.100	5.219	5.329	5.430	5.525	5.613
23	3.042	3.533	3.861	4.113	4.319	4.494	4.647	4.783	4.905	5.016	5.119	5.214	5.302	5.385
24	2.967	3.434	3.745	3.983	4.178	4.343	4.487	4.615	4.730	4.835	4.931	5.020	5.104	5.181
25	2.899	3.343	3.639	3.865	4.050	4.206	4.342	4.463	4.572	4.671	4.762	4.846	4.924	4.998
26	2.836	3.261	3.543	3.758	3.933	4.082	4.211	4.325	4.428	4.522	4.608	4.688	4.762	4.831
27	2.778	3.185	3.454	3.660	3.827	3.968	4.091	4.200	4.298	4.387	4.469	4.544	4.615	4.680
28	2.724	3.115	3.373	3.570	3.729	3.864	3.981	4.085	4.178	4.263	4.341	4.413	4.480	4.542
29	2.674	3.050	3.298	3.486	3.639	3.768	3.880	3.979	4.069	4.149	4.224	4.292	4.356	4.416
30	2.628	2.990	3.229	3.409	3.556	3.680	3.787	3.882	3.967	4.045	4.116	4.181	4.242	4.299
35	2.438	2.746	2.946	3.097	3.219	3.322	3.411	3.489	3.560	3.623	3.682	3.735	3.785	3.832
40	2.296	2.565	2.739	2.869	2.974	3.062	3.138	3.205	3.265	3.320	3.369	3.415	3.457	3.497
45	2.186	2.425	2.579	2.694	2.787	2.864	2.931	2.990	3.042	3.089	3.133	3.172	3.209	3.244
50	2.097	2.314	2.452	2.556	2.638	2.708	2.767	2.820	2.866	2.908	2.947	2.982	3.015	3.046
55	2.024	2.222	2.349	2.443	2.518	2.581	2.635	2.682	2.724	2.762	2.797	2.829	2.858	2.886
60	1.963	2.146	2.262	2.349	2.418	2.475	2.525	2.568	2.606	2.641	2.673	2.702	2.729	2.754
120	1.606	1.709	1.774	1.821	1.858	1.888	1.915	1.937	1.958	1.976	1.992	2.007	2.021	2.034
240	1.397	1.459	1.497	1.525	1.547	1.565	1.580	1.593	1.605	1.615	1.625	1.633	1.641	1.649
480	1.266	1.306	1.330	1.347	1.361	1.372	1.381	1.389	1.396	1.403	1.409	1.414	1.419	1.423
∞	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

Reading off the table

The table of F_{max} comprises four pages, one for each of four upper percentiles ($P = 0.75, 0.90, 0.95, 0.99$) of the distribution. It is used mostly for testing the "homogeneity of variance" hypothesis required in ANOVA designs, where k variance estimates (s_j^2), one per group, are to be pooled. The test statistic is defined as the maximum quotient of variance estimates, *i.e.*:

$$F_{max} = \max s_j^2 \ / \ \min s_j^2 \ ,$$

with number (k) of variance estimates in the set and df (ν) in each estimate as parameters.

Illustration 1. Find $F_{max}(6,20[.95])$, *i.e.* the critical value of F_{max} for comparing $k = 6$ variances each with $df = 20$, at the 0.05 significance level. Looking up the page for percentile 95, at coordinates $\nu = 20$ and $k = 6$, we find $F_{max} = 3.743$. The observed F_{max} must exceed (or equal) this value so that we may reject the hypothesis of an equality of parametric variances (σ_j^2) on the basis of our data.

Illustration 2. What is the value of $F_{max}(10,200[.75])$? The value $\nu = 200$ lies in between 120 and 240, as can be seen on the page for percentile 75. We note $F_{max}(10,120[.75]) = 1.592$ and $F_{max}(10,2400[.75]) = 1.388$. As the gap between 120 and 240 is rather large, a form of harmonic interpolation is recommended, *i.e.* to interpolate $\ln F$ using the argument $1/\sqrt{\nu}$. For $\nu = 200$, we calculate $\ln F = \ln 1.592 + (\ln 1.388 - \ln 1.592) \times (200^{-½} - 120^{-½}) \ / \ (240^{-½} - 120^{-½}) \approx 0.35946$, then $F \approx \exp(0.35946) \approx 1.433$.

Full examples

Example 1. Some specialists in nutrition wanted to appraise the effects of three methods of freezing oranges on the preservation of vitamin C. Ten samples ($n = 10$) of 50 oranges were subjected to each freezing process. In each trial, a measure of vitamin C content was taken for the batch, giving finally the following summary table.

Mode of freezing	n	Mean	S.d.
A	10	5.74	1.32
B	10	8.01	2.09
C	10	7.15	1.77

Were the specialists in a position to perform ANOVA with these data in order to ascertain the best freezing mode? For this, they must have stipulated that the freezing modes have an influence, if any, only on the mean content of vitamin C in the oranges, not on its variability either among oranges or among batches of oranges. *Solution*: To carry out a test of homogeneity of variance is mandatory in this context. Here, there are $k = 3$ observed variances, $s_1^2 = 1.32^2 = 1.7424$, $s_2^2 = 4.3681$ and $s_3^2 = 3.1329$, each one based on $n = 10$ batches and having thus $n-1 = 9$ *df*. The maximum quotient is $F_{\max} = 4.3681/1.7424 \approx 2.507$. At the 0.10 significance threshold (some authors recommend a somewhat high threshold for this "validation" test), with $k = 3$ and $\nu = 9$, $F_{\max}(3,9[.90]) = 4.265$. The observed $F_{\max}$ does not exceed this critical value. Therefore, the variance estimates can be deemed homogeneous, and the specialists may have proceeded confidently with ANOVA.

Example 2. What critical value of $F_{\max}$ can we use in order to test variances from four samples of sizes $n_j = 13, 11, 15, 14$, at the 1 % significance level? *Solution*: Here, the respective *df*'s for the variance estimates are $\nu_j = n_j - 1 = 12$, 10, 14, and 13. With percentile 99 (corresponding to $\alpha = 0.01$) and $k = 4$, we may fix $\nu = \min(\nu_j) = 10$, or $\nu = \max(\nu_j) = 14$, or else $\text{ave}(\nu_j) = 12.25 \approx 13$ (after rounding up), bringing about the percentile values 8.639 $\{= F_{\max}(4,10[.99])\}$, 5.907 and 6.375. If the obtained $F_{\max}$ quotient is inferior to 5.907, then it is non-significant and the four samples can be thought to come from populations having the same variance. If superior to 8.639, it is significant and the equality hypothesis must be discarded. Current practice, in cases where ν_j's do not differ much from one another, is rather to average the ν_j's, with a prudent rounding up meant to insure the error level α. In this example, we would use $\bar{\nu} \approx 13$ and $F_{\max}(4,13[.99]) = 6.375$.

Mathematical presentation

In ANOVA, the valid interpretation of an obtained F quotient rests on a few parametric assumptions, one of them being that the variance estimates that are pooled to form the F denominator be mutually homogeneous, *i.e.* that they be independent estimates of a unique, parametric "error variance", say σ_e^2. To test for this condition and thus validate the F quotient of ANOVA, H. O. Hartley proposed the following statistic:

$$F_{\max} = \frac{\max s_j^2}{\min s_j^2} ,$$

namely, the maximum quotient of two variance estimates, among k available estimates. The test is set up for variances having equal *df*'s. The parameters of the F_{max} are the number (k) of variance estimates in the set, and the *df* (ν) of each estimate.

For cases where *df*'s vary from one estimate to another, – in ANOVA designs with unequal group sizes, for instance –, the way to proceed is uncertain, and we could not unearth a definitive study on the subject. Authors recommend either to use the average $\bar{\nu}$, or to go to the most unfavorable case and pick up $\min(\nu_j)$.

For testing the homogeneity of variance condition, the literature also presents Cochran's C test (*see* section on Cochran's C distribution), a χ^2 test due to Bartlett (*see* Supplementary examples, n° 10), and some other tests.

Calculation and moments

The F_{max} quotient represents as it were the range of compared variance estimates, its logarithm being the range of the k estimates on the logarithmic scale. The estimates themselves each have a Chi-square (χ^2) distribution with parameter ν, hence the d.f. of F_{max} is given by:

$$P_{F_{max}}(t) = k \int_0^\infty h(x)[H(tx) - H(x)]^{k-1}\, dx \, .$$

Functions $h(x)$ and $H(x)$ above refer to the p.d.f. and d.f. of χ^2 with parameter ν (*see* section on χ^2 distribution).

The integrand in the above expression has a changing shape and a domain that shrinks near zero or climbs up according to the values of ν and k: in one word, evaluation of it is difficult. To achieve it, we proceeded in two steps: first, locate the approximate value of the mode (of the integrand), and second, evaluate separately each part on either side of the mode, the unbounded upper part being summed until convergence. Area estimation was done using Simpson's rule. H. A. David (1952) proposed another method.

The integral $P_{F_{max}}(t)$ presents two special cases, one for which there are only two variance estimates ($k = 2$), the other where $\nu = 2$. When $k = 2$, the F_{max} statistic is in fact equilavent to $\max(F, 1/F)$, where $F = s_1^2/s_2^2$, the F r.v. having equal *df*'s (*see* section on F distribution). In that case:

$$P_{F_{max}}(t) = 2[P_F(t) - ½] \qquad \{ k = 2 \} \, ,$$

where $P_F(t)$ designates the d.f. of $F_{\nu,\nu}$. The inversion of this equality produces the percentile of F_{max} from that of F, for instance $F_{max}(2,\nu[P]) = F_{\nu,\nu[½(P+1)]}$. As for $\nu = 2$, the inside integral expressions $H(x)$ and $H(tx)$ become simpler, and we obtain:

$$P_{F_{\max}}(t) = \sum_{i=0}^{k-1} (-1)^i \binom{k-1}{i} \left(\frac{t-1}{k} i + 1 \right)^{-1} \quad \{ \nu = 2 \} .$$

Thus, $P_{F_{\max}}(t) = 1 - 2/(t+1)$ for $k = 2$, $P_{F_{\max}}(t) = 1 - 6/(t+2) + 3/(2t+1)$ for $k = 3$, and so on.

Moments. Moments and other distributional properties of $F_{\max}$ are not known. However, the d.f. of $F_{\max}$ given earlier allows the determination of the median, in the same way as for the percentiles offered as critical values. Moreover, even without utilizing any p.d.f., we may resort to the following theorem on mathematical expectations based on the d.f. $P(x)$ of r.v. x,

$$E(x^r) = \mu'_r = \int r x^{r-1} [1 - P(x)] \, dx \ , \ 0 < x < \infty \ ,$$

thanks to which we obtain the mean $\mu = E(x)$, and the variance $\sigma^2 = E(x^2) - \mu^2$ of x.

Note that, for $\nu = 2$, the moments do not exist since the integral diverges. Our computing algorithm does not reach a sure value for $\nu = 3$, and for $\nu = 4$ in the case of the second moment. The table below presents the mean (or expectation, μ), variance (σ^2) and median (Md) of $F_{\max}$ for some combinations of parameters ν and k, the precision of each approximation being stressed by the number of printed digits. The values given in italics for $k = 3$ were obtained by Monte Carlo estimation.

ν		$k = 2$	3	4	5	10
2	Md	3.000	6.342	10.20	14.43	39.24
3	μ	5.534	*9.7*	*14*	*18*	*37*
	σ^2	7089	–	–	–	–
	Md	2.356	4.200	6.042	7.865	16.65
4	μ	3.500	5.57	7.44	9.19	16.7
	σ^2	87.78	–	–	–	–
	Md	2.064	3.360	4.556	5.678	10.57
5	μ	2.798	4.19	5.39	6.47	10.84
	σ^2	14.36	28.4	45.3	61.8	163
	Md	1.895	2.908	3.799	4.605	7.910
10	μ	1.865	2.441	2.893	3.270	4.620
	σ^2	1.072	1.801	2.409	2.944	5.092
	Md	1.551	2.079	2.493	2.840	4.080
20	μ	1.503	1.808	2.033	2.214	2.812
	σ^2	0.249	0.362	0.438	0.496	0.680
	Md	1.358	1.665	1.888	2.066	2.651

Cochran's C distribution

- ✓ Percentiles 75, 90, 95 and 99 of Cochran's C distribution
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation

  Calculation and moments

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Percentile 75 of Cochran's C distribution

$\nu \setminus k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.9620	.8403	.7371	.6584	.5972	.5480	.5077	.4738	.4448	.4196	.3975	.3780	.3605	.3448
2	.8750	.7114	.6032	.5272	.4704	.4261	.3903	.3607	.3357	.3144	.2958	.2796	.2652	.2524
3	.8174	.6457	.5391	.4660	.4123	.3707	.3375	.3103	.2876	.2682	.2516	.2370	.2242	.2129
4	.7790	.6049	.5001	.4291	.3774	.3378	.3063	.2807	.2594	.2413	.2258	.2124	.2005	.1901
5	.7515	.5767	.4732	.4039	.3537	.3155	.2853	.2608	.2405	.2233	.2087	.1959	.1848	.1749
6	.7307	.5556	.4533	.3853	.3363	.2991	.2699	.2463	.2267	.2103	.1962	.1840	.1734	.1640
7	.7142	.5392	.4379	.3708	.3228	.2865	.2581	.2351	.2162	.2003	.1867	.1750	.1647	.1556
8	.7009	.5259	.4254	.3592	.3120	.2765	.2486	.2263	.2078	.1923	.1791	.1677	.1578	.1490
9	.6897	.5148	.4151	.3496	.3031	.2682	.2409	.2190	.2009	.1858	.1729	.1618	.1521	.1436
10	.6802	.5055	.4063	.3416	.2956	.2612	.2344	.2129	.1952	.1804	.1678	.1569	.1474	.1391
11	.6720	.4974	.3988	.3346	.2892	.2552	.2288	.2076	.1902	.1757	.1633	.1527	.1434	.1353
12	.6648	.4904	.3923	.3286	.2836	.2501	.2240	.2031	.1860	.1717	.1595	.1491	.1400	.1320
13	.6584	.4842	.3865	.3233	.2787	.2455	.2198	.1991	.1823	.1681	.1562	.1459	.1369	.1291
14	.6528	.4786	.3814	.3186	.2744	.2415	.2160	.1956	.1790	.1650	.1532	.1431	.1342	.1265
15	.6477	.4737	.3768	.3143	.2705	.2379	.2127	.1925	.1760	.1622	.1506	.1406	.1318	.1242
16	.6430	.4692	.3727	.3105	.2670	.2346	.2096	.1897	.1733	.1597	.1482	.1383	.1297	.1221
17	.6388	.4651	.3689	.3071	.2638	.2317	.2069	.1871	.1709	.1575	.1461	.1363	.1277	.1203
18	.6349	.4613	.3654	.3039	.2609	.2290	.2044	.1848	.1688	.1554	.1441	.1344	.1260	.1186
19	.6314	.4579	.3622	.3010	.2582	.2266	.2021	.1826	.1667	.1535	.1423	.1327	.1244	.1170
20	.6281	.4547	.3593	.2983	.2558	.2243	.2000	.1807	.1649	.1518	.1407	.1311	.1229	.1156
21	.6250	.4517	.3566	.2958	.2535	.2222	.1981	.1789	.1632	.1502	.1391	.1297	.1215	.1143
22	.6222	.4490	.3541	.2935	.2514	.2202	.1962	.1772	.1616	.1487	.1377	.1283	.1202	.1131
23	.6195	.4464	.3517	.2914	.2494	.2184	.1946	.1756	.1601	.1473	.1364	.1271	.1190	.1120
24	.6170	.4440	.3495	.2894	.2476	.2167	.1930	.1741	.1588	.1460	.1352	.1259	.1179	.1109
25	.6147	.4418	.3474	.2875	.2458	.2151	.1915	.1727	.1575	.1448	.1340	.1248	.1169	.1099
26	.6125	.4396	.3455	.2857	.2442	.2136	.1901	.1715	.1563	.1436	.1330	.1238	.1159	.1090
27	.6104	.4376	.3436	.2840	.2427	.2122	.1888	.1702	.1551	.1426	.1319	.1229	.1150	.1081
28	.6084	.4357	.3419	.2824	.2412	.2109	.1876	.1691	.1540	.1415	.1310	.1220	.1141	.1073
29	.6065	.4339	.3403	.2810	.2399	.2096	.1864	.1680	.1530	.1406	.1301	.1211	.1133	.1065
30	.6048	.4322	.3387	.2795	.2386	.2084	.1853	.1670	.1521	.1397	.1292	.1203	.1125	.1058
35	.5970	.4248	.3319	.2734	.2330	.2033	.1805	.1625	.1479	.1358	.1255	.1168	.1092	.1026
40	.5908	.4188	.3265	.2685	.2285	.1992	.1767	.1590	.1446	.1327	.1226	.1140	.1066	.1001
45	.5856	.4139	.3220	.2644	.2248	.1958	.1736	.1561	.1419	.1301	.1202	.1117	.1044	.0980
50	.5813	.4097	.3182	.2610	.2217	.1929	.1710	.1536	.1396	.1279	.1182	.1098	.1026	.0963
55	.5775	.4061	.3150	.2580	.2190	.1905	.1687	.1515	.1376	.1261	.1164	.1082	.1010	.0948
60	.5742	.4030	.3121	.2555	.2167	.1884	.1668	.1497	.1359	.1245	.1149	.1067	.0997	.0935
120	.5525	.3824	.2936	.2388	.2016	.1746	.1541	.1379	.1249	.1142	.1052	.0976	.0910	.0852
∞	.5000	.3333	.2500	.2000	.1667	.1429	.1250	.1111	.1000	.0909	.0833	.0769	.0714	.0667

Percentile 90 of Cochran's C distribution

$\nu \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.9939	.9345	.8533	.7783	.7142	.6599	.6138	.5743	.5400	.5099	.4834	.4598	.4387	.4196
2	.9500	.8175	.7076	.6240	.5591	.5075	.4653	.4302	.4006	.3750	.3529	.3334	.3162	.3008
3	.9027	.7431	.6287	.5463	.4843	.4359	.3971	.3651	.3383	.3154	.2957	.2785	.2633	.2498
4	.8647	.6935	.5788	.4984	.4390	.3931	.3565	.3267	.3017	.2806	.2625	.2467	.2328	.2205
5	.8348	.6578	.5439	.4654	.4080	.3640	.3292	.3008	.2773	.2574	.2403	.2255	.2126	.2011
6	.8108	.6307	.5178	.4410	.3852	.3427	.3092	.2820	.2595	.2406	.2243	.2103	.1980	.1872
7	.7912	.6093	.4974	.4220	.3676	.3263	.2939	.2676	.2460	.2277	.2121	.1987	.1869	.1766
8	.7747	.5918	.4809	.4068	.3535	.3132	.2816	.2562	.2352	.2175	.2025	.1895	.1781	.1682
9	.7608	.5772	.4673	.3942	.3418	.3024	.2716	.2468	.2263	.2092	.1946	.1820	.1710	.1613
10	.7487	.5648	.4557	.3835	.3321	.2934	.2632	.2389	.2189	.2022	.1880	.1757	.1650	.1556
11	.7381	.5540	.4458	.3744	.3237	.2856	.2560	.2322	.2126	.1963	.1824	.1704	.1599	.1508
12	.7288	.5446	.4371	.3665	.3164	.2789	.2498	.2264	.2072	.1911	.1775	.1658	.1556	.1466
13	.7205	.5363	.4294	.3595	.3100	.2731	.2443	.2213	.2024	.1867	.1733	.1618	.1517	.1430
14	.7130	.5289	.4226	.3533	.3044	.2678	.2395	.2168	.1982	.1827	.1695	.1582	.1484	.1397
15	.7062	.5222	.4165	.3478	.2993	.2632	.2352	.2128	.1945	.1792	.1662	.1551	.1454	.1369
16	.7001	.5162	.4110	.3428	.2947	.2590	.2313	.2092	.1911	.1760	.1632	.1522	.1427	.1343
17	.6944	.5107	.4060	.3382	.2906	.2552	.2278	.2059	.1880	.1731	.1605	.1496	.1402	.1320
18	.6892	.5056	.4014	.3341	.2868	.2517	.2246	.2029	.1852	.1705	.1580	.1473	.1380	.1299
19	.6844	.5010	.3972	.3303	.2833	.2485	.2216	.2002	.1827	.1681	.1558	.1452	.1360	.1279
20	.6800	.4967	.3933	.3268	.2802	.2456	.2189	.1977	.1804	.1659	.1537	.1432	.1341	.1262
21	.6758	.4927	.3897	.3235	.2772	.2429	.2165	.1954	.1782	.1639	.1518	.1414	.1324	.1245
22	.6719	.4890	.3864	.3205	.2745	.2404	.2141	.1932	.1762	.1620	.1500	.1397	.1308	.1230
23	.6683	.4855	.3832	.3177	.2719	.2381	.2120	.1912	.1743	.1603	.1484	.1382	.1293	.1216
24	.6649	.4822	.3803	.3150	.2695	.2359	.2100	.1894	.1726	.1586	.1468	.1367	.1280	.1203
25	.6617	.4792	.3776	.3126	.2673	.2338	.2081	.1876	.1710	.1571	.1454	.1354	.1267	.1191
26	.6587	.4763	.3750	.3103	.2652	.2319	.2063	.1860	.1694	.1557	.1440	.1341	.1255	.1179
27	.6558	.4736	.3725	.3081	.2632	.2301	.2047	.1845	.1680	.1543	.1428	.1329	.1243	.1169
28	.6531	.4710	.3702	.3060	.2613	.2284	.2031	.1830	.1666	.1531	.1416	.1318	.1233	.1158
29	.6505	.4686	.3681	.3040	.2596	.2268	.2016	.1816	.1654	.1519	.1405	.1307	.1223	.1149
30	.6480	.4663	.3660	.3022	.2579	.2253	.2002	.1803	.1641	.1507	.1394	.1297	.1213	.1140
35	.6374	.4563	.3570	.2942	.2507	.2187	.1942	.1747	.1589	.1458	.1348	.1254	.1172	.1101
40	.6287	.4482	.3499	.2878	.2449	.2134	.1893	.1703	.1548	.1419	.1311	.1219	.1139	.1069
45	.6215	.4415	.3439	.2825	.2401	.2091	.1854	.1666	.1514	.1388	.1281	.1191	.1112	.1044
50	.6154	.4359	.3389	.2781	.2361	.2055	.1820	.1635	.1485	.1361	.1256	.1167	.1090	.1023
55	.6101	.4310	.3347	.2742	.2327	.2024	.1792	.1609	.1461	.1338	.1235	.1147	.1071	.1005
60	.6055	.4268	.3309	.2709	.2297	.1997	.1767	.1586	.1440	.1318	.1216	.1129	.1054	.0989
120	.5749	.3991	.3066	.2494	.2105	.1823	.1608	.1440	.1304	.1191	.1097	.1017	.0949	.0889
∞	.5000	.3333	.2500	.2000	.1667	.1429	.1250	.1111	.1000	.0909	.0833	.0769	.0714	.0667

Percentile 95 of Cochran's C distribution

$\nu \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	.9985	.9670	.9065	.8413	.7808	.7270	.6799	.6385	.6021	.5698	.5410	.5152	.4920	.4709
2	.9750	.8710	.7680	.6838	.6162	.5612	.5157	.4775	.4450	.4169	.3924	.3709	.3518	.3347
3	.9392	.7978	.6839	.5981	.5322	.4800	.4378	.4028	.3734	.3482	.3265	.3075	.2907	.2758
4	.9058	.7457	.6288	.5441	.4804	.4308	.3910	.3584	.3311	.3080	.2880	.2707	.2554	.2419
5	.8773	.7070	.5895	.5064	.4448	.3972	.3594	.3285	.3028	.2811	.2624	.2462	.2320	.2195
6	.8534	.6771	.5599	.4783	.4185	.3726	.3363	.3068	.2823	.2616	.2440	.2286	.2152	.2034
7	.8332	.6531	.5365	.4564	.3981	.3536	.3185	.2901	.2666	.2468	.2299	.2152	.2024	.1912
8	.8160	.6334	.5176	.4388	.3817	.3384	.3043	.2768	.2541	.2350	.2187	.2046	.1923	.1815
9	.8011	.6168	.5018	.4242	.3682	.3259	.2927	.2660	.2439	.2254	.2096	.1960	.1841	.1737
10	.7880	.6026	.4884	.4119	.3569	.3154	.2830	.2569	.2354	.2173	.2020	.1888	.1772	.1671
11	.7765	.5903	.4769	.4013	.3472	.3065	.2747	.2491	.2281	.2105	.1955	.1826	.1714	.1616
12	.7663	.5795	.4669	.3921	.3387	.2987	.2675	.2424	.2218	.2046	.1900	.1774	.1664	.1568
13	.7570	.5699	.4580	.3840	.3313	.2919	.2612	.2365	.2163	.1994	.1851	.1728	.1620	.1526
14	.7487	.5613	.4501	.3768	.3248	.2858	.2556	.2313	.2115	.1949	.1808	.1687	.1582	.1489
15	.7411	.5536	.4430	.3704	.3189	.2805	.2506	.2267	.2072	.1908	.1770	.1651	.1547	.1457
16	.7342	.5466	.4366	.3645	.3136	.2756	.2461	.2226	.2033	.1872	.1735	.1618	.1516	.1427
17	.7278	.5402	.4308	.3593	.3088	.2712	.2421	.2188	.1998	.1839	.1704	.1589	.1489	.1401
18	.7219	.5344	.4254	.3544	.3044	.2672	.2384	.2154	.1966	.1809	.1676	.1562	.1463	.1377
19	.7165	.5290	.4205	.3500	.3004	.2635	.2350	.2122	.1936	.1781	.1650	.1538	.1440	.1355
20	.7114	.5240	.4160	.3459	.2967	.2602	.2319	.2094	.1910	.1756	.1627	.1515	.1419	.1334
21	.7067	.5193	.4118	.3422	.2933	.2570	.2290	.2067	.1885	.1733	.1605	.1495	.1399	.1316
22	.7022	.5150	.4079	.3387	.2901	.2541	.2264	.2042	.1862	.1712	.1585	.1476	.1381	.1299
23	.6981	.5110	.4043	.3354	.2872	.2514	.2239	.2019	.1840	.1692	.1566	.1458	.1364	.1283
24	.6942	.5072	.4009	.3323	.2844	.2489	.2216	.1998	.1821	.1673	.1548	.1441	.1349	.1268
25	.6905	.5036	.3977	.3295	.2818	.2466	.2194	.1978	.1802	.1656	.1532	.1426	.1334	.1254
26	.6870	.5003	.3947	.3268	.2794	.2444	.2174	.1959	.1784	.1639	.1516	.1411	.1320	.1241
27	.6837	.4971	.3918	.3243	.2771	.2423	.2155	.1942	.1768	.1624	.1502	.1398	.1307	.1228
28	.6806	.4941	.3891	.3219	.2750	.2403	.2137	.1925	.1752	.1609	.1488	.1385	.1295	.1217
29	.6776	.4913	.3866	.3196	.2729	.2385	.2120	.1909	.1738	.1596	.1476	.1373	.1284	.1206
30	.6747	.4886	.3842	.3175	.2710	.2367	.2103	.1894	.1724	.1583	.1463	.1361	.1273	.1196
35	.6623	.4769	.3738	.3082	.2626	.2291	.2034	.1830	.1664	.1527	.1411	.1312	.1226	.1151
40	.6522	.4674	.3654	.3008	.2560	.2231	.1979	.1779	.1617	.1483	.1370	.1273	.1189	.1116
45	.6438	.4596	.3585	.2946	.2505	.2181	.1933	.1737	.1578	.1447	.1336	.1241	.1159	.1088
50	.6367	.4530	.3527	.2895	.2459	.2140	.1895	.1702	.1546	.1416	.1307	.1214	.1134	.1064
55	.6305	.4473	.3478	.2851	.2420	.2104	.1863	.1672	.1518	.1390	.1283	.1191	.1112	.1043
60	.6251	.4424	.3434	.2813	.2385	.2073	.1835	.1646	.1494	.1368	.1262	.1172	.1094	.1025
120	.5890	.4100	.3152	.2565	.2165	.1874	.1653	.1480	.1340	.1225	.1128	.1046	.0975	.0913
∞	.5000	.3333	.2500	.2000	.1667	.1429	.1250	.1111	.1000	.0909	.0833	.0769	.0714	.0667

Percentile 99 of Cochran's C distribution

$\nu \backslash k$	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	$.9^{4}39$	.9934	.9676	.9279	.8829	.8376	.7945	.7544	.7175	.6837	.6528	.6245	.5986	.5747
2	.9950	.9423	.8643	.7886	.7218	.6645	.6152	.5728	.5359	.5036	.4752	.4499	.4273	.4069
3	.9794	.8832	.7815	.6958	.6259	.5685	.5210	.4810	.4469	.4176	.3920	.3695	.3496	.3318
4	.9586	.8335	.7213	.6329	.5635	.5080	.4627	.4251	.3934	.3663	.3429	.3224	.3043	.2882
5	.9373	.7934	.6762	.5876	.5196	.4660	.4227	.3871	.3572	.3318	.3100	.2909	.2742	.2594
6	.9172	.7607	.6411	.5531	.4866	.4348	.3932	.3592	.3309	.3068	.2862	.2682	.2525	.2386
7	.8989	.7336	.6129	.5259	.4609	.4106	.3705	.3378	.3106	.2877	.2680	.2510	.2360	.2229
8	.8823	.7108	.5898	.5038	.4401	.3912	.3523	.3208	.2946	.2725	.2536	.2373	.2230	.2104
9	.8674	.6912	.5703	.4854	.4230	.3752	.3374	.3068	.2814	.2601	.2419	.2262	.2125	.2004
10	.8540	.6743	.5537	.4698	.4085	.3617	.3249	.2951	.2705	.2498	.2321	.2169	.2036	.1920
11	.8418	.6595	.5392	.4563	.3960	.3502	.3142	.2851	.2611	.2410	.2238	.2091	.1962	.1848
12	.8307	.6464	.5266	.4446	.3852	.3402	.3049	.2765	.2530	.2334	.2167	.2023	.1898	.1787
13	.8206	.6347	.5154	.4343	.3757	.3315	.2968	.2689	.2460	.2268	.2105	.1964	.1842	.1734
14	.8113	.6241	.5054	.4251	.3673	.3237	.2896	.2623	.2398	.2209	.2050	.1912	.1792	.1687
15	.8028	.6146	.4964	.4169	.3598	.3168	.2832	.2563	.2342	.2157	.2001	.1866	.1748	.1645
16	.7949	.6059	.4883	.4094	.3530	.3105	.2775	.2510	.2293	.2111	.1957	.1824	.1709	.1608
17	.7876	.5979	.4809	.4026	.3468	.3049	.2723	.2462	.2248	.2069	.1917	.1787	.1674	.1574
18	.7808	.5906	.4741	.3965	.3412	.2998	.2676	.2418	.2207	.2030	.1881	.1753	.1642	.1544
19	.7744	.5839	.4678	.3908	.3360	.2950	.2632	.2378	.2169	.1996	.1848	.1722	.1612	.1516
20	.7684	.5776	.4620	.3856	.3313	.2907	.2592	.2341	.2135	.1963	.1818	.1693	.1585	.1490
21	.7629	.5717	.4567	.3807	.3269	.2867	.2556	.2307	.2103	.1934	.1790	.1667	.1560	.1467
22	.7576	.5663	.4517	.3762	.3228	.2830	.2521	.2275	.2074	.1906	.1765	.1643	.1538	.1445
23	.7526	.5612	.4470	.3720	.3190	.2795	.2490	.2246	.2047	.1881	.1741	.1620	.1516	.1425
24	.7480	.5564	.4427	.3681	.3155	.2763	.2460	.2219	.2021	.1857	.1719	.1600	.1496	.1406
25	.7435	.5519	.4386	.3644	.3121	.2733	.2432	.2193	.1998	.1835	.1698	.1580	.1478	.1389
26	.7393	.5476	.4347	.3610	.3090	.2705	.2407	.2169	.1975	.1814	.1678	.1562	.1461	.1372
27	.7353	.5436	.4311	.3577	.3061	.2678	.2382	.2147	.1955	.1795	.1660	.1544	.1444	.1357
28	.7315	.5398	.4277	.3546	.3033	.2653	.2359	.2125	.1935	.1777	.1643	.1528	.1429	.1342
29	.7279	.5362	.4244	.3517	.3007	.2629	.2337	.2105	.1916	.1759	.1626	.1513	.1414	.1328
30	.7244	.5328	.4213	.3490	.2982	.2607	.2317	.2086	.1899	.1743	.1611	.1498	.1401	.1315
35	.7092	.5178	.4079	.3371	.2875	.2510	.2228	.2005	.1823	.1672	.1545	.1436	.1342	.1260
40	.6966	.5058	.3972	.3275	.2790	.2433	.2158	.1940	.1763	.1617	.1493	.1387	.1296	.1216
45	.6861	.4957	.3883	.3197	.2720	.2369	.2100	.1887	.1714	.1571	.1450	.1347	.1258	.1180
50	.6771	.4873	.3809	.3131	.2661	.2316	.2052	.1843	.1673	.1533	.1415	.1314	.1226	.1150
55	.6693	.4800	.3745	.3075	.2611	.2271	.2011	.1805	.1638	.1500	.1384	.1285	.1199	.1125
60	.6625	.4736	.3689	.3026	.2568	.2232	.1975	.1773	.1608	.1472	.1358	.1260	.1176	.1103
120	.6162	.4318	.3327	.2710	.2288	.1981	.1748	.1564	.1416	.1294	.1191	.1104	.1029	.0964
∞	.5000	.3333	.2500	.2000	.1667	.1429	.1250	.1111	.1000	.0909	.0833	.0769	.0714	.0667

Reading off the table

The table comprises four pages devoted respectively to percentiles 75, 90, 95, and 99 of Cochran's C distribution. The C statistic, like Hartley's F_{max}, is principally used for verifying the hypothesis of equality of variances across k normal populations, on the basis of k variance estimates, each with degrees of freedom ν. It is the quotient of the maximum out of k estimates on their sum, *i.e.*:

$$C = \max s_j^2 \;/\; \sum s_j^2 \;.$$

Illustration 1. Find $C_{5,10[.95]}$, percentile 95 of C suitable for a set of $k = 5$ variance estimates, with ($\nu =$) 10 *df* each. On the page for percentile 95, line $\nu = 10$ and colum $k = 5$ point to C = 0.4119. Any quotient equal to or higher than 0.4119 would be declared significant at the $\alpha = 0.05$ significance level.

Illustration 2. What is the value of $C_{10,200[.75]}$? On the page for percentile 75, we observe that $df = 200$ would lie between $\nu = 120$ and infinity. We note that $C_{10,120[.75]} = 0.1249$ and $C_{10,\infty[.75]} = 0.1000$. The marker "$\infty$" compels us to use harmonic interpolation, and we calculate $0.1249 + (0.1000-0.1249) \times (200^{-1}-120^{-1})/(1/\infty-120^{-1}) \approx 0.1149$.

Full examples

Example 1. Taking up again the first example presented in the section on the F_{max} distribution, we have $k = 3$ sample variances, $s_1^2 = 1.7424$, $s_2^2 = 4.3681$ and $s_3^2 = 3.1329$, each one having 9 *df*. Are these three estimates homogeneous? *Solution*: The highest variance estimate is 4.3681, and the sum of the three equals 9.2434. Thus, $C = 4.3681/9.2434 \approx 0.4726$. Looking at percentile 90, which stands for a significance level (α) of 0.10, we read $C_{3,9[.90]} = 0.5772$. The obtained sample value does not exceed the critical value, so that the test is non-significant and we can tolerate the null hypothesis on the equality of variances in the respective three populations.

Example 2 [Revisiting the second example in the section on the F_{max} distribution]. What critical value of C may we use to assess the variances from $k = 4$ samples of sizes $n_j = 13$, 11, 15, 14, at the $\alpha = 0.01$ level? *Solution*: The *df*'s of the respective variance estimates are unequal, *i.e.* $\nu_j = n_j - 1 = 12, 10, 14, 13$. Using percentile 99 (corresponding to $\alpha = 0.01$) and k

= 4, we may decide to fix $\nu = \min(\nu_j)$, or $\nu = \max(\nu_j)$, or again $\nu = \text{ave}(\nu_j) = 12.25 \approx 13$ (after rounding up), and obtain respectively 0.5537 (= $C_{4,10[.99]}$), 0.5054 and 0.5154. If the empirical C quotient is inferior to 0.5054, then all four populations may be deemed to have equal true variances. If the obtained C is superior to 0.5537, it is significant and the equality hypothesis must be rejected. In current pratice, for cases where the ν_j's do not differ much one from the other, authors favor ave(ν_j), with a value rounded up to the next integer in order to comply with the predetermined α level. Here, we would rather use $\nu = 13$ ($\approx \bar{\nu}$) and $C_{4,13[.99]} = 0.5054$.

Mathematical presentation

The valid practice of ANOVA depends on some verifiable conditions, one of which is the so-called homogeneity of variance condition : the independent estimates of variance that are pooled to make up an error mean square (used as denominator in the *F* quotient) must be mutually compatible, *i.e.* originate from populations having the same parametric variance. To test for this condition, Cochran (1941) proposed the statistic :

$$C = \frac{\max s_j^2}{\sum_{i=1}^{k} s_j^2} ,$$

obtained by dividing the highest estimate of variance ($\max s_j^2$) by the sum of all estimates. The statistic was planned for comparing variance estimates with equal degrees of freedom (ν). Its parameters are the number (k) of estimates in the comparison set, and the df (ν) of each estimate.

For cases where *df*'s vary from one estimate to another, – in ANOVA designs with unequal group sizes, for instance –, the way to proceed is uncertain, and we could not find a conclusive study on that subject. Authors recommend either to use the average $\bar{\nu}$, or to go to the most unfavorable case and pick up min(ν_j).

For testing the homogeneity of variance condition, the literature also presents the $F_{\max}$ test by H. O. Hartley (*see* section on $F_{\max}$), a χ^2 test due to Bartlett (*see* Supplementary examples, n° 10), and other tests.

Calculation and moments

After R. A. Fisher found a solution for calculating the d.f. of C when $\nu = 2$, Cochran (1941) tought up a generalization for $\nu \geq 2$. The calculations required for evaluating $P(\mathrm{C})$ and for determining critical values are arduous. Let $P(\mathrm{C}) = \Pr\{\, y \leq \mathrm{C} \,|\, k,\nu)$, the integral at C for k variances estimates of σ^2 with ν *df* each. We must compute :

$$P(\mathrm{C}) = 1 - kP_1(\mathrm{C}) + \binom{k}{2}P_2(\mathrm{C}) - \binom{k}{3}P_3(\mathrm{C}) + \cdots + (-1)^{k-1}\binom{k}{k-1}P_{k-1}(\mathrm{C}) \ , \qquad (1)$$

where each component P_r designates a r-dimensional integral. The above expression is an illustration of the famous inclusion-and-exclusion method in the calculus of probability; its derivative would be, if obtained, the p.d.f. of C.

In order to evaluate $P(\mathrm{C})$, it is necessary, in principle, to compute the series of integrals P_r, which are:

$$P_1 = \int_{\mathrm{C}}^{1} f_1(x_1)\, dx_1 \ ,$$

$$P_2 = \int_{\mathrm{C}}^{1-\mathrm{C}} \int_{\mathrm{C}}^{1-x_1} f_2(x_1,x_2)\, dx_2\, dx_1 \quad (\text{if } \mathrm{C} \leq ½) \ ,$$

$$P_3 = \int_{\mathrm{C}}^{1-2\mathrm{C}} \int_{\mathrm{C}}^{1-\mathrm{C}-x_1} \int_{\mathrm{C}}^{1-x_1-x_2} f_3(x_1,x_2,x_3)\, dx_3\, dx_2\, dx_1 \quad (\text{if } \mathrm{C} \leq ⅓) \ ,$$

$$P_4 = \text{etc.}$$

Each integral $P_r(\mathrm{C})$ measures the probability that, among the k quotients $\mathrm{C}_i = s_i^2 / \sum s_i^2$, r of those are superior to C. In these integrals, the functions f_r are generalized *Beta* functions, that is:

$$f_1(x_1) = \mathrm{Q}_{\nu,k}\, x_1^{½\nu-1}(1-x_1)^{½\nu(k-1)-1} \ ,$$

$$f_2(x_1,x_2) = \mathrm{Q}_{\nu,k}\mathrm{Q}_{\nu,k-1}(x_1x_2)^{½\nu-1}(1-x_1-x_2)^{½\nu(k-2)-1} \ ,$$

$$f_3(x_1,x_2,x_3) = \mathrm{Q}_{\nu,k}\mathrm{Q}_{\nu,k-1}\mathrm{Q}_{\nu,k-2}(x_1x_2x_3)^{½\nu-1}(1-x_1-x_2-x_3)^{½\nu(k-3)-1} \ ,$$

and so on, where $\mathrm{Q}_{\nu,k} = \Gamma[\nu k/2]/\{\Gamma[\nu/2]\ \Gamma[\nu(k-1)/2]\}$.

Probabilities P_r, difficult to obtain when $r \geq 2$, have the property of being strongly decreasing, such that $P_1 > P_2 > \cdots > P_{k-1}$: it is why already published tables were based solely on P_1. With this makeshift, for given values of ν and k, C_α is found such that $P_1(\mathrm{C}) = (1-\alpha)/k$. However, the tables presented here are based on a more thorough evaluation of $P(\mathrm{C})$.

First, integral $P_1(\mathrm{C})$, in fact an integral of the *Beta* distribution, corresponds also to the d.f. of the F distribution, with parameters $\nu_1 = \nu$, $\nu_2 = \nu(k-1)$ and $F = \nu_2\mathrm{C}/[\nu_1(1-\mathrm{C})]$; in one form or another, its evaluation is reasonably simple. When $\mathrm{C} < ½$, calculation of $P_1(\mathrm{C})$ is but a first step for obtaining $P(\mathrm{C})$, since one must then calculate $P_2(\mathrm{C})$, $P_3(\mathrm{C})$,..., until $P_r(\mathrm{C})$, where $r = \lfloor 1/\mathrm{C} \rfloor$, or before, if the sum (1) stabilizes. In this context, we set up a recursive algorithm, programmed in Pascal, which performs the numerical integration of the multivariate function f_r within the appropriate bounds. Practically, however, this computation is quite laborious, even for the high-speed computing facilities now available; therefore we limited the numerical evaluation of the P_r functions at $r = 3$; when required, values of P_r beyond $r = 3$ were extrapolated. For the reasons given, critical values of $\mathrm{C} < ¼$, in our tables, may be in error in the last digits.

Two special cases of the distribution of C are noteworthy. For $k = 2$ estimates (or samples), the C statistic is equivalent to a simple F quotient, so that:

$$\mathrm{C}_{2,\nu[P]} = F_{\nu,\nu[½(1+P)]} \,/\, [1 + F_{\nu,\nu[½(1+P)]}]$$
$$= 1 \,/\, [1 + F_{\nu,\nu[½(1-P)]}] \ .$$

The other special case is Fisher's solution when $\nu = 2$. Analytical reduction of the f_r functions leads to the much simpler evaluation scheme :

$$P(\mathrm{C})_{\nu=2} = 1 - \binom{k}{1}(1-\mathrm{C})^{k-1} + \binom{k}{2}(1-2\mathrm{C})^{k-1} - \cdots +(-1)^t\binom{k}{t}(1-t\mathrm{C})^{k-1} \ ,$$

where $t = \lfloor 1/\mathrm{C} \rfloor$, *i.e.* t is the highest integer such that $t \le 1/\mathrm{C}$.

Moments. Moments and other properties of the C distribution are not known. It is nonetheless feasable to obtain the median by the same process used for the upper percentiles. Moreover, as $P(\mathrm{C})$ is the d.f. of r.v. C, we can resort to the following theorem on mathematical expectations:

$$\mathrm{E}\{x^r\} = \mu'_r = \int r \cdot x^{r-1}[1 - P(x)]\, dx \ , \quad 0 < x < \infty \ ,$$

to obtain the mean $\mu = \mathrm{E}\{x\}$ and the variance $\sigma^2 = \mathrm{E}(x^2) - \mu^2$ of C. The table on next page presents the mean, variance and median for some combinations of parameters k et ν; the number of digits displayed indicates the approximative accuracy of each value.

Expectations (μ), variances (σ^2) and medians (Md) of Cochran's C for some combinations of parameters k (number of groups) and ν (degrees of freedom)

ν		$k = 2$	3	4	5	10
2	μ	.7500	.6111	.5208	.4567	.2929
	σ^2	.0208	.0201	.0169	.0140	.0063
	Md	.7500	.5918	.5000	.4369	.2785
3	μ	.7122	.565	.473	.409	.255
	σ^2	.0175	.0153	.0122	.0097	.0036
	Md	.7020	.5452	.4548	.3928	.2429
4	μ	.6875	.535	.443	.381	.232
	σ^2	.0148	.0123	.0094	.0073	.0025
	Md	.6736	.5173	.4272	.3663	.2219
5	μ	.6698	.515	.423	.361	.217
	σ^2	.0128	.0102	.0076	.0058	.0019
	Md	.6545	.4981	.4082	.3481	.2078
10	μ	.6230	.462	.371	.312	.180
	σ^2	.0076	.0054	.0038	.0028	$.0^{3}78$
	Md	.6080	.4501	.3611	.3034	.1738
20	μ	.5881	.424	.335	.278	.155
	σ^2	.0041	.0027	.0018	.0013	$.0^{3}33$
	Md	.5760	.4157	.3280	.2722	.1508

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Orthogonal polynomials

✓ Coefficients $c_{r,i}$ of orthogonal polynomials (table 1)

✓ Coefficients $p_{i,r}$ and constants λ_r used for polynomial conversion (table 2)

✓ Reading off the table

✓ Full examples

✓ Computing algorithms

Testing polynomial components

Converting back to a regular polynomial equation

✓ Mathematical presentation

Theory and preparation of the tables

Some useful formulae

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Coefficients $c_{r,i}$ of orthogonal polynomials (N = 3 to 9) (table 1)

N	r	$c_{r,i}$									$\sum c_{r,i}^2$
3	1	-1	0	1							2
	2	1	-2	1							6
4	1	-3	-1	1	3						20
	2	1	-1	-1	1						4
	3	-1	3	-3	1						20
5	1	-2	-1	0	1	2					10
	2	2	-1	-2	-1	2					14
	3	-1	2	0	-2	1					10
	4	1	-4	6	-4	1					70
6	1	-5	-3	-1	1	3	5				70
	2	5	-1	-4	-4	-1	5				84
	3	-5	7	4	-4	-7	5				180
	4	1	-3	2	2	-3	1				28
	5	-1	5	-10	10	-5	1				252
7	1	-3	-2	-1	0	1	2	3			28
	2	5	0	-3	-4	-3	0	5			84
	3	-1	1	1	0	-1	-1	1			6
	4	3	-7	1	6	1	-7	3			154
	5	-1	4	-5	0	5	-4	1			84
	6	1	-6	15	-20	15	-6	1			924
8	1	-7	-5	-3	-1	1	3	5	7		168
	2	7	1	-3	-5	-5	-3	1	7		168
	3	-7	5	7	3	-3	-7	-5	7		264
	4	7	-13	-3	9	9	-3	-13	7		616
	5	-7	23	-17	-15	15	17	-23	7		2184
	6	1	-5	9	-5	-5	9	-5	1		264
	7	-1	7	-21	35	-35	21	-7	1		3432
9	1	-4	-3	-2	-1	0	1	2	3	4	60
	2	28	7	-8	-17	-20	-17	-8	7	28	2772
	3	-14	7	13	9	0	-9	-13	-7	14	990
	4	14	-21	-11	9	18	9	-11	-21	14	2002
	5	-4	11	-4	-9	0	9	4	-11	4	468
	6	4	-17	22	1	-20	1	22	-17	4	1980
	7	-1	6	-14	14	0	-14	14	-6	1	858
	8	1	-8	28	-56	70	-56	28	-8	1	12870

Coefficients $c_{r,i}$ of orthogonal polynomials (N = 10 to 13) (table 1, cont.)

N	r	$c_{r,i}$													$\sum c_{r,i}^2$
10	1	-9	-7	-5	-3	-1	1	3	5	7	9				330
	2	6	2	-1	-3	-4	-4	-3	-1	2	6				132
	3	-42	14	35	31	12	-12	-31	-35	-14	42				8580
	4	18	-22	-17	3	18	18	3	-17	-22	18				2860
	5	-6	14	-1	-11	-6	6	11	1	-14	6				780
	6	3	-11	10	6	-8	-8	6	10	-11	3				660
	7	-9	47	-86	42	56	-56	-42	86	-47	9				29172
	8	1	-7	20	-28	14	14	-28	20	-7	1				2860
	9	-1	9	-36	84	-126	126	-84	36	-9	1				48620
11	1	-5	-4	-3	-2	-1	0	1	2	3	4	5			110
	2	15	6	-1	-6	-9	-10	-9	-6	-1	6	15			858
	3	-30	6	22	23	14	0	-14	-23	-22	-6	30			4290
	4	6	-6	-6	-1	4	6	4	-1	-6	-6	6			286
	5	-3	6	1	-4	-4	0	4	4	-1	-6	3			156
	6	15	-48	29	36	-12	-40	-12	36	29	-48	15			11220
	7	-5	23	-33	2	28	0	-28	-2	33	-23	5			4862
	8	5	-31	73	-68	-14	70	-14	-68	73	-31	5			27170
	9	-1	8	-27	48	-42	0	42	-48	27	-8	1			9724
	10	1	-10	45	-120	210	-252	210	-120	45	-10	1			184756
12	1	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11		572
	2	55	25	1	-17	-29	-35	-35	-29	-17	1	25	55		12012
	3	-33	3	21	25	19	7	-7	-19	-25	-21	-3	33		5148
	4	33	-27	-33	-13	12	28	28	12	-13	-33	-27	33		8008
	5	-33	57	21	-29	-44	-20	20	44	29	-21	-57	33		15912
	6	11	-31	11	25	4	-20	-20	4	25	11	-31	11		4488
	7	-55	225	-251	-83	204	140	-140	-204	83	251	-225	55		369512
	8	11	-61	119	-65	-74	70	70	-74	-65	119	-61	11		65208
	9	-11	79	-227	303	-102	-210	210	102	-303	227	-79	11		408408
	10	1	-9	35	-75	90	-42	-42	90	-75	35	-9	1		33592
	11	-1	11	-55	165	-330	462	-462	330	-165	55	-11	1		705432
13	1	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	182
	2	22	11	2	-5	-10	-13	-14	-13	-10	-5	2	11	22	2002
	3	-11	0	6	8	7	4	0	-4	-7	-8	-6	0	11	572
	4	99	-66	-96	-54	11	64	84	64	11	-54	-96	-66	99	68068
	5	-22	33	18	-11	-26	-20	0	20	26	11	-18	-33	22	6188
	6	22	-55	8	43	22	-20	-40	-20	22	43	8	-55	22	14212
	7	-33	121	-103	-75	65	100	0	-100	-65	75	103	-121	33	92378
	8	11	-55	89	-19	-71	10	70	10	-71	-19	89	-55	11	38038
	9	-2	13	-32	31	6	-30	0	30	-6	-31	32	-13	2	6188
	10	6	-49	166	-285	210	78	-252	78	210	-285	166	-49	6	386308
	11	-1	10	-44	110	-165	132	0	-132	165	-110	44	-10	1	117572
	12	1	-12	66	-220	495	-792	924	-792	495	-220	66	-12	1	2704156

Coefficients $c_{r,i}$ of orthogonal polynomials (N = 14 and 15) (table 1, cont.)

N	*r*	$c_{r,i}$															$\sum c_{r,i}^2$
14	1	-13	-11	-9	-7	-5	-3	-1	1	3	5	7	9	11	13		910
	2	13	7	2	-2	-5	-7	-8	-8	-7	-5	-2	2	7	13		728
	3	-143	-11	66	98	95	67	24	-24	-67	-95	-98	-66	11	143		97240
	4	143	-77	-132	-92	-13	63	108	108	63	-13	-92	-132	-77	143		136136
	5	-143	187	132	-28	-139	-145	-60	60	145	139	28	-132	-187	143		235144
	6	143	-319	-11	227	185	-25	-200	-200	-25	185	227	-11	-319	143		497420
	7	-143	473	-297	-353	95	375	200	-200	-375	-95	353	297	-473	143		1293292
	8	13	-59	79	7	-65	-25	50	50	-25	-65	7	79	-59	13		34580
	9	-13	77	-163	107	89	-105	-90	90	105	-89	-107	163	-77	13		142324
	10	13	-97	288	-392	125	279	-216	-216	279	125	-392	288	-97	13		772616
	11	-13	119	-464	968	-1045	231	792	-792	-231	1045	-968	464	-119	13		5878600
	12	1	-11	54	-154	275	-297	132	132	-297	275	-154	54	-11	1		416024
	13	-1	13	-78	286	-715	1287	-1716	1716	-1287	715	-286	78	-13	1		10400600
15	1	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	280
	2	91	52	19	-8	-29	-44	-53	-56	-53	-44	-29	-8	19	52	91	37128
	3	-91	-13	35	58	61	49	27	0	-27	-49	-61	-58	-35	13	91	39780
	4	1001	-429	-869	-704	-249	251	621	756	621	251	-249	-704	-869	-429	1001	6466460
	5	-1001	1144	979	44	-751	-1000	-675	0	675	1000	751	-44	-979	-1144	1001	10581480
	6	143	-286	-55	176	197	50	-125	-200	-125	50	197	176	-55	-286	143	426360
	7	-13	39	-17	-31	-3	25	25	0	-25	-25	3	31	17	-39	13	8398
	8	91	-377	415	157	-311	-275	125	350	125	-275	-311	157	415	-377	91	1193010
	9	-91	494	-901	344	659	-250	-675	0	675	250	-659	-344	901	-494	91	4269720
	10	91	-624	1631	-1724	-159	1568	-27	-1512	-27	1568	-159	-1724	1631	-624	91	19315400
	11	-7	59	-205	356	-253	-121	297	0	-297	121	253	-356	205	-59	7	678300
	12	7	-71	313	-766	1067	-649	-363	924	-363	-649	1067	-766	313	-71	7	5616324
	13	-1	12	-65	208	-429	572	-429	0	429	-572	429	-208	65	-12	1	1485800
	14	1	-14	91	-364	1001	-2002	3003	-3432	3003	-2002	1001	-364	91	-14	1	40116600

Coefficients $p_{i,r}$ and constants λ_r used for polynomial conversion (N = 3 to 9) (table 2)

N	i	λ	$p_{r,i}$
3	1	1	
	2	3	$p_{2,0}=-2/3$
4	1	2	
	2	1	$p_{2,0}=-5/4$
	3	10/3	$p_{3,1}=-41/20$
5	1	1	
	2	1	$p_{2,0}=-2$
	3	5/6	$p_{3,1}=-17/5$
	4	35/12	$p_{4,0}=72/35$ $p_{4,2}=-31/7$
6	1	2	
	2	3/2	$p_{2,0}=-35/12$
	3	5/3	$p_{3,1}=-101/20$
	4	7/12	$p_{4,0}=81/16$ $p_{4,2}=-95/4$
	5	21/10	$p_{5,1}=11567/1008$ $p_{5,3}=-145/18$
7	1	1	
	2	1	$p_{2,0}=-4$
	3	1/6	$p_{3,1}=-7$
	4	7/12	$p_{4,0}=72/7$ $p_{4,2}=-67/7$
	5	7/20	$p_{5,1}=524/21$ $p_{5,3}=-35/3$
	6	77/60	$p_{6,0}=-1200/77$ $p_{6,2}=434/11$ $p_{6,4}=-145/11$
8	1	2	
	2	1	$p_{2,0}=-21/4$
	3	2/3	$p_{3,1}=-37/4$
	4	7/12	$p_{4,0}=297/16$ $p_{4,2}=-179/14$
	5	7/10	$p_{5,1}=15709/336$ $p_{5,3}=-95/6$
	6	11/60	$p_{6,0}=-2925/64$ $p_{6,2}=13769/176$ $p_{6,4}=-805/44$
	7	143/210	$p_{7,1}=-1172307/9152$ $p_{7,3}=242837/2288$ $p_{7,5}=-3129/156$
9	1	1	
	2	3	$p_{2,0}=-20/3$
	3	5/6	$p_{3,1}=-59/5$
	4	7/12	$p_{4,0}=216/7$ $p_{4,2}=-115/7$
	5	3/20	$p_{5,1}=716/9$ $p_{5,3}=-185/9$
	6	11/60	$p_{6,0}=-1200/11$ $p_{6,2}=1514/11$ $p_{6,4}=-265/11$
	7	143/1680	$p_{7,1}=-47820/143$ $p_{7,3}=28007/143$ $p_{7,5}=-350/13$
	8	143/448	$p_{8,0}=31360/143$ $p_{8,2}=-1310788/2145$ $p_{8,4}=9527/39$ $p_{8,6}=-434/15$

Coefficients $p_{i,r}$ and constants λ_r used for polynomial conversion (N = 10 to 15) (table 2, cont.)

N	*i*	λ	$p_{r,i}$
10	1	2	
	2	1/2	$p_{2,0}=-33/4$
	3	5/3	$p_{3,1}=-293/20$
	4	5/12	$p_{4,0}=3861/80$ $p_{4,2}=-41/2$
	5	1/10	$p_{5,1}=6067/48$ $p_{5,3}=-155/6$
	6	11/240	$p_{6,0}=-14625/64$ $p_{6,2}=39329/176$ $p_{6,4}=-1354/44$
	7	143/840	$p_{7,1}=-6750831/9152$ $p_{7,3}=749357/2288$ $p_{7,5}=-1799/52$
	8	143/4032	$p_{8,0}=187425/256$ $p_{8,2}=-16628803/11440$ $p_{8,4}=44247/104$ $p_{8,6}=-189/5$
	9	2431/18144	$p_{9,1}=7462789083/3111680$ $p_{9,3}=-7755749/3536$ $p_{9,5}=342447/680$ $p_{9,7}=-681/17$
11	1	1	
	2	1	$p_{2,0}=-10$
	3	5/6	$p_{3,1}=-89/5$
	4	1/12	$p_{4,0}=72$ $p_{4,2}=-25$
	5	1/40	$p_{5,1}=572/3$ $p_{5,3}=-95/3$
12	1	2	
	2	3	$p_{2,0}=-143/12$
	3	2/3	$p_{3,1}=-85/4$
	4	7/24	$p_{4,0}=11583/112$ $p_{4,2}=-419/14$
	5	3/20	$p_{5,1}=39761/144$ $p_{5,3}=-685/18$
13	1	1	
	2	1	$p_{2,0}=-14$
	3	1/6	$p_{3,1}=-25$
	4	7/12	$p_{4,0}=144$ $p_{4,2}=-247/7$
	5	7/120	$p_{5,1}=2708/7$ $p_{5,3}=-45$
14	1	2	
	2	1/2	$p_{2,0}=-65/4$
	3	5/3	$p_{3,1}=-581/20$
	4	7/12	$p_{4,0}=21879/112$ $p_{4,2}=-575/14$
	5	7/30	$p_{5,1}=59063/112$ $p_{5,3}=-105/2$
15	1	1	
	2	3	$p_{2,0}=-56/3$
	3	5/6	$p_{3,1}=-167/5$
	4	35/12	$p_{4,0}=1296/5$ $p_{4,2}=-331/7$
	5	21/20	$p_{5,1}=44252/63$ $p_{5,3}=-545/9$

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Reading off the tables

Tables 1 and 2 offer sets of coefficients and constants relative to orthogonal polynomial regression analysis. Table 1 furnishes, for a predictor variable with N levels, coefficients $c_{r,i}$ appropriate for degrees (or powers) $r = 1$ to $N-1$, for each of N = 3 to 15. Also, to help converting a linear orthogonal polynomial into a regular polynomial, table 2 gives transfer constants $p_{i,r}$. The reader will find more information at the end of this chapter together with formulae to enable him to extend parts of our tables.

The interested reader will find, following the illustrations and examples, a computing algorithm for testing the significance of individual orthogonal components (of regression), and another one for reconstructing a polynomial regression equation from its orthogonal components.

Illustration 1. Find coefficients $c_{r,i}$ for the cubic (*i.e.* 3rd degree) component in a series of eight data. In table 1, for N = 8, the coefficients for the cubic orthogonal component are $c_{3,i} = \{ -7, 5, 7, 3, -3, -7, -5, 7 \}$.

Illustration 2. What are the transfer constants $p_{i,r}$ for establishing coefficient β_3 in a series of eight data? Section N = 8 in table 2, for $r = 3$ (relative to the 3rd degree polynomial coefficient), presents $p_{5,3} = -95/6$ and $p_{7,3} = 242837/2288$, whence we may write: $\beta_3 = \alpha_3 + (-95/6)\alpha_5 + (242837/2288)\alpha_7$.

Full examples

Example 1 [Data from G.V. Glass & K.D. Hopkins: *Statistical methods in education and psychology*, 3rd ed., Allyn and Bacon 1996, p. 181]. Scores were obtained for the motor performance of samples of $n = 4$ individuals, at each of N = 7 age levels: 10, 20, 30, ···, 70 years. Averages for the successive samples form the series $Y_i = \{ 8.5, 9.5, 10.5, 11.5, 10.0, 9.0, 8.5 \}$. The error mean square, computed elsewhere, is $MS_{error} = MS_{within\text{-}groups} = 1.1905$, with $\nu_e = 7\times(4-1) = 21$ *df.* Which are the significant polynomial powers in the regression of motor performance scores with age? May we build up a (regular) regression equation using these significant components? *Solution*: There are two questions, and we shall answer them in turn, by applying the computing algorithms described later.

– *Identification of significant orthogonal components*

Step 1. The total variance to be accounted for is indicated here by the variance among the N = 7 means, $s^2(Y_i)$ = 1.22619 or, more adequately, by the "between-levels" mean square, $MS_{Cond} = n \times s^2(Y_i)$ = 4×1.22619 = 4.90476. Also, $\hat{\eta}^2 = (7-1)\times 4.90476 / [(7-1)\times 4.90476 + 21\times 1.1905] \approx 0.5407$; the η^2 statistic is quite helpful for building up regression models as it represents the maximum value that the coefficient of determination (R^2) can attain for a given set of data, whatever the chosen model.

Step 2 (For component 1). In table 1, section N = 7, line r = 1, we find the 1st degree coefficient $c_{1,i}$ = { −3, −2, −1, 0, 1, 2, 3 } ans $\sum c_{1,i}^2 = 28$, from which $v(\alpha_1) = (-3\times 8.5 - 2\times 9.5 - \cdots + 3\times 8.5)^2/28 = (-1.5)^2/28 = 0.08036$.

Step 3 (For component 1). The quotient $n\cdot v(\alpha_1)/MS_{error}$, equal to 4×0.08036/1.1905 ≈ 0.270, is to be compared with the $F_{1,21}$ distribution. At the 5 % significance threshold, the critical value is $F_{1,21[.95]} \approx 4.326$ (obtained through linear interpolation). The observed value being too low, component 1 is not distinctly present and can be deemed null.

Step 2 (For component 2). After substracting component 1, the amount of variance remaining is $R_1 = (N-1)MS_{Cond} - n\cdot v(\alpha_1) = 64.90476 - 4\times 0.08036 \approx 29.107$. Now, $F = R_1/MS_{error} \approx 24.449$ exceeds the critical value $F_{1,21[.95]}$ found earlier, which indicates that there is some variance still to be assigned. Looking at the next line in table 1, section N = 7, we read off coefficients $c_{2,i}$ and calculate $v(\alpha_2) = (-22,5)^2/84 = 6.02679$.

Step 3 (For component 2). The quotient $F(\alpha_2)$, equal to 4×6.02679/1.1905 ≈ 20.249, is now significant and justifies the retention of orthogonal component 2 ("quadradic component") in the regression model.

Step 2 (For component 3). The remainder of variance, after substracting components 1 and 2, is $R_2 = R_1 - n\cdot v(\alpha_2) = 29.107 - 4\times 6.02679 \approx 5.000$. The test of this, $F = R_2/MS_{error}$ = 5.000/1.1905 ≈ 4.200, does not reach 4.326 (= $F_{1,21[.95]}$). Thus, it would be unjustified, and useless, to pursue our identification of more components.

To recapitulate, only component 2 was proven significant, with $v(\alpha_2)$ = 6.02679. The proportion of variance accounted for, $PV = \sum v(\alpha_r)/[(N-1)s^2(Y)]$, equals 6.02679 / (6×1.22619) ≈ 0.819 for our model. This quantity (PV) is proportional to the coefficient of determination (R^2), here equal to 0.443, as divided by its theoretical maximal value, $\hat{\eta}^2$ = 0.541.

– *Converting back to a regular polynomial equation*

Step 1. The successive tests above suggest that we retain only component 2 (the "quadratic component"), based on coefficient α_2. In table 2, section N = 7, we find $\lambda_2 = 1$; using the other quantities already obtained, we compute $\alpha_2 = 1\times(-22.5)/84 \approx -0,2679$[1]. And, $\alpha_0 = \bar{Y} \approx 9.643$, $M = X_{mp} = 40$, $\Delta = X_{i+1} - X_i = 10$. The other α_r coefficients are reputed null, being non-significant.

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[1] Coefficient α_r may also be obtained from the corresponding variance component $v(\alpha_r)$, using the relation: $\alpha_r = \pm\lambda_r\sqrt{[v(\alpha_r)/\sum c_{r,i}^2]}$, the sign of α_r being that of the numerator of $v(\alpha_r)$ before its squaring.

Step 2. Using the constants $p_{i,r}$ in table 2, section N = 7, we calculate:

$$\beta_0 = \alpha_0 + p_{2,0}\alpha_2 = 9.643 + (-4)(-0.2679) \approx 10.7146$$

$$\beta_1 = 0$$

$$\beta_2 = \alpha_2 = -0.2679 \,.$$

The resulting equation is: $\hat{Y} = \beta_0 + \beta_2[(X-M)/\Delta]^2 = 10.7146 - 0.2679\times[(X-40)/10]^2$.

Step 3. Expansion of the term $[(X-40)/10]^2$ leads to $0.01X^2 - 0.8X + 16$. After grouping the terms having the same power and simplifying, we finally obtain the following regular polynomial equation :

$$\hat{Y} = 6.4282 + 0.21432X - 0.002679X^2 \,,$$

a regression equation whose coefficient of determination (mentioned above) is $R^2 \approx 0.443$. If we had proceeded directly with a second degree regression analysis, thus incorporating the non-significant α_1 (= $\beta_1 = -0.0536$), the predictive function would have been :

$$\hat{Y} = 6.6426 + 0.2089X - 0.002679X^2 \,.$$

Example 2 [Simple linear regression]. After choosing the successive values $X_i = 10$, 13, 16, 19, 22 and 25 for his independent variable, a researcher obtained the respective measures Y_i = 3.02, 3.89, 4.99, 5.77, 7.05, 7.42. Find the simple (1st degree) linear regression equation and the associated coefficient of determination for these data, using orthogonal polynomials. *Solution*: In the sections "N = 6" of tables 1 and 2, we find the required constants $c_{1,i}$, Σc^2 and λ_1 : here, $c_{1,i} = \{-5, -3, -1, 1, 3, 5\}$, $\Sigma c^2 = 70$, $\lambda_1 = 2$. The amount of variance associated with simple linear regression, *i.e.* the 1st degree component, is $v(\alpha_1) = \Sigma c_{1,i}Y_i / \Sigma c^2 = (-5\times 3.02 - 3\times 3.389 - \cdots + 5\times 7.42)^2/70 = (32.26)^2/70 \approx 14.867$. The total variance available being $(N-1)s^2(Y_i) = 5\times 3.008 \approx 15.040$, the proportion of variance accounted for in the regression model is PV = 14.867/15.040 = 0.9885, which in this case is identical to the coefficient of determination $(R^2)^2$. The correlation between X_i's and Y_i's is thus $r = R = \sqrt{0.9885} \approx 0.9942$. Establishing $\bar{Y}$ (= α_0) ≈ 5.3567, X_{mp} (= M) = 17.50, Δ (= $X_{i+1} - X_i$) = 3 and $\alpha_1 = \lambda_1 \Sigma c_{1,i}Y_i / \Sigma c^2 = 2\times 32.26/70 \approx 0.9217$, we can directly calculate:

$$b_0 = \bar{Y} - \alpha_1 X_{mp}/\Delta \;;\;\; b_1 = \alpha_1/\Delta \,,$$

obtaining $b_0 \approx -0.020$ and $b_1 \approx 0.307$. The desired regression equation is thus $\hat{Y} = -0.020 + 0.307\,X$.

[2] The general relation between PV and R^2 is: $R^2 = PV\times(N-1)\cdot s^2(Y)/[(N-1)\cdot s^2(Y) + N\sigma_e^2]$, where σ_e^2 is the error variance for each observation. Here, because we have individual observations (instead of averaged values of Y for each value X_i) and in ignorance of σ_e^2, we stipulate $\sigma_e^2 = 0$, which entails R^2 = PV.

Example 3. Determine the coefficients $c_{r,i}$ appropriate for N = 20 pairs { X_i,Y_i } and polynomial components (degrees, powers) $r = 1$ and $r = 2$. *Solution*: Table 1 stops after N = 15; therefore, we must resort to the formulae given in the next section, that is $\varphi_1(x) = \lambda_1 x$ for $r = 1$, and $\varphi_2(x) = \lambda_2[x^2-(N^2-1)/12]$ for $r = 2$. The N predictors (or levels) X_i, in their standard form $(X_i-M)/\Delta$, are: { −9.5, −8.5, ···, 9.5 }, with $\lambda_1 = 1$, or { −19, 17, ···, 19 } using $\lambda_1 = 2$; these are also coefficients $c_{1,i}$. Applying now formula $\varphi_2(x)$ for N = 20, we have, for instance, $c_{2,1} = \lambda_2[-9.5^2-(20^2-1)/12] = \lambda_2[57]$, then $c_{2,2} = \lambda_2[39]$, etc.; finally, fixing $\lambda_2 = 1$, we generate $c_{2,i}$ = { 57, 39, 23, 9, −3, −13, −21, −27, −31, −33, −31, −27, −21, −13, −3, 9, 23, 39, 57 }.

Computing algorithms

Linear polynomial regression, which is expressed as:

$$\hat{Y} = b_0 + b_1X + b_2X^2 + \cdots + b_kX^k , \tag{1}$$

can be rewritten as:

$$\hat{Y} = \beta_0 + \beta_1\left(\frac{X-M}{\Delta}\right) + \cdots + \beta_k\left(\frac{X-M}{\Delta}\right)^k \tag{2}$$

and again, as:

$$\hat{Y}_i = \alpha_0 + \alpha_1(p_{1,0}+x_i) + \alpha_2(p_{2,0}+p_{2,1}x_i+x_i^2) + \cdots + \alpha_k(p_{k,0}+p_{k,1}x_i+\cdots+x_i^k) . \tag{3}$$

This last expression (3) enables us to isolate each individual r^{th} degree component of regression and to measure its importance in terms of the amount of variance accounted for in dependent variable Y, through the associated regression coefficient α_r. Suppose that we have obtained N values Y_i corresponding to N predetermined levels X_i acting as predictors. The variation among the Y_i's attributable to the X_i's, as it is represented in equation (1), is now split up into $k \le N-1$ powers of X, each component power being statistically independent (= orthogonal) of each other and allowing a test of statistical significance. In the context of ANOVA, the Y_i are usually averages, each one based on n observations, with which is associated a global error mean square (MS_{error}) having ν_e *df*.

Orthogonal polynomial regression analysis allows to test the significance of any r^{th} degree component of regression on the basis of the amount of variance, $v(\alpha_r)$, associated with it: the appropriate test is $F_{comp.\,r} = n \cdot v(\alpha_r)/MS_{error}$, with 1 and ν_e *df* (*see* section on *F* distribution). One may also test the significance of an aggregate of k ($k \ge 1$) components, with

$F_{\text{aggr.}} = n[\sum v(\alpha_j)/k]/MS_{\text{error}}$, with k and ν_e *df.* Furthermore, this same analysis enables one to reconstruct a regular polynomial equation, such as equation (1), using only wanted (or significant) components[3].

Testing polynomial components

The data to be analysed can be laid out in the following manner:

$$X: \{ X_1 \; X_2 \; \cdots \; X_N \};$$

$$Y: \{ Y_1 \; Y_2 \; \cdots \; Y_N \}.$$

The X_i's are the successive values of the predictor (or independent) variable, with common interval Δ (= $X_{i+1} - X_i$). The Y_i's are the measured observations; each value Y_i usually denotes the average of n individual data points. An appropriate measure of error variability, the error mean square (MS_{error}), depends entirely on each particular data design. In a design using N independent groups, or samples, the error term would be the within-groups mean square (MS_{Within}), obtained as the average of every group's variance and having $\nu_e = N(n-1)$ *df*; in one other simple design, one with one N-levels factor and repeated measures, the error term would be $MS_{\text{Cond}\times\text{Subjects}}$, and $\nu_e = (N-1)(n-1)$ *df.*

Step 1 [Computing MS_{Cond}]. Each value Y_i being the average of n observations[4], the global measure of variance among Y_i's is simply $MS_{\text{Cond}} = n \cdot s^2(Y_i)$, with $N-1$ *df.*

Step 2 [Computing the variance component associated with polynomial degree r]. Using table 1, in the section relative to the number (N) of predictors, one obtains coefficients $c_{r,i}$ on line r, then one computes $v(\alpha_r)$ using formula:

$$v(\alpha_r) = \left(\sum_{i=1}^{N} c_{r,i} Y_i \right)^2 / \sum_{i=1}^{N} c_{r,i}^2 \; ; \qquad (4)$$

the sum $\sum c_{r,i}^2$ is furnished in table 1.

[3] It is interesting to note that the variance components associated with each of the $N-1$ polynomial powers are exhaustive and that they contain together the total available variance, *i.e.* $n[v(\alpha_1)+v(\alpha_2)+ \cdots +v(\alpha_{N-1})] = (N-1)ns^2(Y_i) = (N-1)MS_{\text{Cond}}$. As a corollary, if equation (1) is built using all $k = N-1$ polynomial components, it will pass through every point (X_i, Y_i).

[4] It may happen, due to subjects' defection or other causes or reasons, that the Y_i's be based on unequal numbers of observations (n_i). As it is the case for general ANOVA procedures, the calculations and probabilistic interpretations can only be approximate in that situation. For the present calculations, we recommend the "harmonic mean solution", i.e. to use $n_h = N/(n_1^{-1}+n_2^{-1}+\cdots+n_N^{-1})$ instead of n in the formulae.

Step 3 [Testing the significance of component *r*]. Each quantity Y_i is based on n individual, normally distributed observations. Then, the quotient $n \cdot v(\alpha_r)/MS_{error}$ is distributed as F_{1,ν_e}; the associated polynomial component can be declared significant at level α if the obtained quotient exceeds $F_{1,\nu_e[1-\alpha]}$. This would indicate that the r^{th} power of predictor X really modifies dependent variable Y.

The usual way to proceed is to begin with the component of first degree, $v(\alpha_1)$. This being done, the remaining variance is obtained with $R_1 = (N-1)MS_{Cond} - n \cdot v(\alpha_1)$, and has $N-2$ *df*. If the test $F = R_1/MS_{error}$, using 1 and ν_e *df*, is significant at some threshold, it suggests that at least one other significant polynomial component may be fished out: we thus proceed to component 2, obtaining $v(\alpha_2)$. The remaining variance is again $R_2 = R_1 - n \cdot v(\alpha_2)$ with $N-3$ *df*, and we may decide to proceed further on using $F = R_2/MS_{error}$ with 1 and ν_e *df*, and so on, until the remainder of variance is no more significant.

The portion of variance available for modelling, in the array $\{ Y_i \}$, is denoted $\hat{\eta}^2$ ("η" is pronounced "eta") and called the non-linear coefficient of determination. It is given by:

$$\hat{\eta}^2 = (N-1)MS_{Cond} \; / \; [(N-1)MS_{Cond} + \nu_e MS_{error}] \; .$$

As for R^2, the coefficient of determination, it designates the portion of variance actually accounted for (or predicted) by our model. This coefficient depends on the polynomial components retained, as shown by:

$$R^2 = n\sum v(\alpha_r) \; / \; [(N-1)MS_{Cond} + \nu_e MS_{error}] \; .$$

Observe that $R^2 \le \hat{\eta}^2$; equality ($R^2 = \hat{\eta}^2$) obtains only if we include indiscriminately all the $N-1$ possible components. Lastly, the proportion of variance (PV) expressed in a specific polynomial model, by which we measure to what point the model succeeds in conveying all the information in our data, is furnished by:

$$PV = R^2 \; / \; \hat{\eta}^2 \; .$$

This last quantity may serve as a guide in our process of model development.

Converting back to a regular polynomial equation

Our purpose here is to build up a regular polynomial equation like (1) using the results of orthogonal regression analysis, such as the one outlined in the preceding section. For example, with a set of N data points $\{ X_i, Y_i \}$, if only components α_1, α_3 and α_6 proved to be significant, it is judicious and legitimate to build up a of equation of type (1) using only these components.

Step 1 [Preliminary calculations]. One may compute the value of α_r directly, with the formula:

$$\alpha_r = \lambda_r \sum_{i=1}^{N} c_{r,i} Y_i \; / \sum_{i=1}^{N} c_{r,i}^2 \, , \tag{5}$$

which is analogous to formula (4) for the associated variance component; constant λ_r is furnished in table 2, in the appropriate section for N at line r. Three more quantities must be defined:

$$\alpha_0 = \bar{Y} \; ; \quad M = X_{mp} \; ; \quad \Delta = X_{i+1} - X_i \; ;$$

X_{mp} is the midpoint of the X_i's. The researcher must also decide which components α_r he will keep and which he will eliminate (by crushing them to zero or ignoring them); he can base his decision either by contemplating a target (stipulated or theoretical) model, or by leaning upon the respective statistical significance of components $v(\alpha_r)$.

> Suppose that the researcher decides in favor of a 3rd degree polynomial, by reason of a theoretical argument or based on the statistical significance of the relevant component. Apart from establishing coefficients β_3 and the origin β_0, he may decide to include or not the orthogonal contributions of degree 1 (*i.e.* α_1) and 2 (α_2), the usual course being to incorporate every lower degree: the ensuing polynomial equation then coincides with that obtained from the technique of polynomial regression analysis, a special case of multiple regression analysis.

Step 2 [Determining coefficients β_r for equation (2)]. Coefficients β_r derive from coefficients α_r by means of the following transformation:

$$\begin{aligned}
\beta_0 &= \alpha_0 + p_{2,0}\alpha_2 + p_{4,0}\alpha_4 + \cdots \\
\beta_1 &= \alpha_1 + p_{3,1}\alpha_3 + p_{5,1}\alpha_5 + \cdots \\
\beta_2 &= \alpha_2 + p_{4,2}\alpha_4 + \cdots \\
&\cdots \\
\beta_k &= \alpha_k \, .
\end{aligned} \tag{6}$$

Constants $p_{i,r}$ can also be found in table 2.

Step 3 [Expanding and simplifying equation (2)]. Equation (2) may be completely written out using coefficients β_r; it uses the transformed quantity $x = (X-M)/\Delta$. We need now to expand the expressions $[(X-M)/\Delta]^r$, group all terms of power X^r, and simplify, re-substituting X_{mp} (= M) and $X_{i+1}-X_i$ (= Δ). The resulting expression is of the form (1), *i.e.* a regular polynomial regression equation. Example 1 above illustrates the process.

Mathematical presentation

The technique of orthogonal polynomial regression analysis (OPRA) was developed by the same R.A. Fisher who also created the technique of analysis of variance (ANOVA), the F (or variance ratio) distribution and several other concepts and techniques for statistical analysis. OPRA extends and supplements polynomial regression analysis, a special case of multiple regression analysis. Consider a bivariate { X,Y } series of N data points:

$$\text{X: } \{ X_1 \ X_2 \ \cdots \ X_N \};$$
$$\text{Y: } \{ Y_1 \ Y_2 \ \cdots \ Y_N \}.$$

The X_i's usually denote the successive (increasing) values of the "independent variable", and the Y_i's are the corresponding observations from the "dependent variable"; the Y_i's represent habitually averages based on n observations. Any model that predicts Y_i from X_i can be written generally as "$\hat{Y} = f(X)$". One such model, called "linear polynomial regression", is written as:

$$\hat{Y} = b_0 + b_1X + b_2X^2 + \cdots + b_kX^k \,. \tag{7}$$

The order k of this general model may vary from $k = 0$, in which case $\hat{Y} = b_0 = \bar{Y}$ for every X_i, up to $k = N-1$, in which case the fit between the model and observed data is "perfect", in the sense that $\hat{Y}_i = f_{N-1}(X_i) = Y_i$[5]. Estimation of parameters b_r $(0 \le r \le k)$ is done for polynomial regression in the same way as for multiple regression (where the independent, or predictor variables, X_j are all distinct) by the method of least squares, which can be found in most mathematically oriented textbooks.

Be it for elaborating a simple model $f(X)$ or for assessing the presence of one power X^r of predictor X, the researcher is occasionnaly interested in the power components, particularly the first few components of the predictive model. For instance, he could wish to determine the smallest subset of power components of X sufficient for accounting for the significant variance in the Y_i's or simply to check for the presence of a 2[nd] degree component, which would defeat the hypothesis of linear proportionality in the data.

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[5] Such a model is said to be "saturated" and is generally not recommended, for at least two reasons. Firstly, a model of order $(N-1)$ forces the regression function to pass through each data point, ignoring the presence of error variance in the observed data and the model's ability to generalize to new, independent, observations. Secondly, saturation of the model runs contrary to one major principle of scientific enquiry and model-building, parsimony.

Now, it turns out that the direct determination of coefficients b_r for equation (7) has three potentially serious drawbacks. The first is that the statistical significance of a b_r coefficient may be ambiguous: even if $b_r \neq 0$, the r^{th} power may nonetheless be nonexistent in the predictor X, the higher powers in model (7) propagating their variational effects on the lower ones. Second, the computations required, based on linear (matrix) algebra, are a thrifle laborious. Third, the estimated coefficients $\{ b_0, b_1, \cdots, b_k \}$ are relative to the k^{th}-order model retained, in such a way that, for example, b_0 and b_1 in a 2^{nd}-order model (with b_0, b_1, b_2) will differ from b_0 and b_1 in a 3^{rd}-order one (with b_0, b_1, b_2, b_3). Such is not the case with orthogonal polynomial regression.

The regression model using orthogonal polynomials, which may be viewed as a transformation of equation (7), can be written:

$$\hat{Y} = \alpha_0 + \alpha_1\varphi_1(X) + \alpha_2\varphi_2(X) + \cdots + \alpha_k\varphi_k(X) \ , \tag{8}$$

where each $\varphi_r(X)$ denotes a function of X of degree r, and α_r is its coefficient. The functions φ_r are orthonormal, *i.e.* for $i = 1$ to N, their sums of squares must obey $\sum_i\varphi_r^2(1) = 1$, and their sums of simple products, $\sum_i\varphi_r(1)\varphi_s(1) = 0$. It is this last property which gives orthogonal regression analysis its worth, for the reason that it allows to deal with the total variance in Y (s_Y^2), based on $N-1$ *df*, and to split it in $N-1$ distinct and additively independent power components of variance associated with the X_i's. The statistical literature on analysis of variance covers these applications.

The predictor (or independent) variable X presents N values which may be successively equidistant or not, and each Y_i value of the dependent variable is endowed with an error variance which may be particular[6] or common to all Y_i. A general method of orthogonal analysis exists for each possible combination. Here, we shall keep to the simplest case, the one having equidistant predictor values and a common (and equal) error variance for all Y_i.

Tables 1 and 2, at the beginning of this chapter, present the coefficients $c_{r,i}$ needed to compute the amount of variance associated with a given power r in model (8), for series of N = 3 to 15 data points, and coefficients λ_r and $p_{i,r}$ that are used to reconstruct a reguler polynomial equation of type (7) from its equivalent orthogonal form (8).

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[6] In ANOVA, for instance, a sample of n_j observations of the dependent variable Y is associated with each value (X_j) of the independent variable, and their average (= Y_i, in the notation of this section) has a variance of σ^2/n_j, which implies *unequal variances* when the sample sizes n_j are unequal.

ANOVA with orthogonal polynomials. The observed variance $s^2(\mathrm{Y})$ among the N values Y_i rests upon N−1 *df* and, consequently, its numerator $(\mathrm{N}-1)s^2(\mathrm{Y})$ may be analysed (*i.e.* broken down) in N−1 parts, each part having 1 *df* and pertaining to degree r, the r^{th} power of predictor X. The variance associated with degree r of X is obtained by:

$$v_r(\mathrm{Y}) = \left(\sum_{i=1}^{\mathrm{N}} \mathrm{c}_{r,i}\mathrm{Y}_i\right)^2 / \sum_{i=1}^{\mathrm{N}} \mathrm{c}_{r,i}^2 ,$$

using coefficients $\mathrm{c}_{r,i}$ in table 1. The reader may check that $v_1 + v_2 + \cdots + v_{\mathrm{N}-1} = (\mathrm{N}-1)s^2(\mathrm{Y})$.

Reconverting to a regular polynomial equation. Coefficients α_r of the orthogonal polynomial equation (8) are obtained with:

$$\alpha_r = \lambda_r \sum_{i=1}^{\mathrm{N}} \mathrm{c}_{r,i}\mathrm{Y}_i / \sum_{i=1}^{\mathrm{N}} \mathrm{c}_{r,i}^2 ,$$

and $\alpha_0 = \bar{\mathrm{Y}}$; constants λ_r are to be found in table 2. Using these values, we may construct a regular polynomial equation of any degree. Note, in passing, that v_r, the portion of variance in Y associated with the r^{th} power of X, equals $\alpha_r^2\Sigma \mathrm{c}_{r,i}^2 / \lambda_r^2$. To construct a regular polynomial equation of degree k, we must determine coefficients $\beta_0, \beta_1, \cdots, \beta_k$ in the following manner:

$$\begin{aligned}
\beta_0 &= \alpha_0 + \mathrm{p}_{1,0}\alpha_1 + \mathrm{p}_{2,0}\alpha_2 + \cdots + \mathrm{p}_{k,0}\alpha_k \\
\beta_1 &= \alpha_1 + \mathrm{p}_{2,1}\alpha_2 + \cdots + \mathrm{p}_{k,1}\alpha_k \\
&\cdots \\
\beta_k &= \alpha_k .
\end{aligned} \tag{9}$$

The system of tables proposed here rests on a recoding of predictor (or independent variable) X around its midpoint ($\mathrm{M} = \mathrm{X}_{\mathrm{mp}}$), and the common interval ($\Delta_{\mathrm{X}} = \mathrm{X}_{i+1} - \mathrm{X}_i$) between consecutive values. This system gives rise to a simple structure of coefficients, in which $\mathrm{c}_{r,i} = 0$ if $r + i$ is odd. Thus, using our system, coefficient matrix (9) becomes:

$$\begin{aligned}
\beta_0 &= \alpha_0 + \mathrm{p}_{2,0}\alpha_2 + \mathrm{p}_{4,0}\alpha_4 + \cdots \\
\beta_1 &= \alpha_1 + \mathrm{p}_{3,1}\alpha_3 + \mathrm{p}_{5,1}\alpha_5 + \cdots \\
\beta_2 &= \alpha_2 + \mathrm{p}_{4,2}\alpha_4 + \cdots \\
&\cdots \\
\beta_k &= \alpha_k ;
\end{aligned} \tag{9'}$$

no term in each summation goes beyond coefficients α_k. These β_r coefficients express the least squares regression of degree k of Y on the predictor $x = (X-M)/\Delta$, *i.e.*

$$\hat{Y} = \beta_0 + \beta_1\left(\frac{X-M}{\Delta}\right) + \cdots + \beta_k\left(\frac{X-M}{\Delta}\right)^k .$$

Finally, to obtain coefficients b_r of the regular least squares polynomial equation (7), it remains to expand the parenthesized expressions and group terms of the same power X^r. Thus, for b_0, we may compute $b_0 = \beta_0 - \beta_1 M/\Delta + \beta_2(M/\Delta)^2 - \beta_3(M/\Delta)^3 + \cdots$, etc. In the case of simple linear regression, $\hat{Y} = b_0 + b_1X$, we have $b_0 = \bar{Y}$ and $b_1 = \alpha_1/\Delta$.

Theory and preparation of the tables

If we develop each polynomial $\varphi_r(X)$ and evaluate it for each pair (X_i, Y_i), the application of equation (8) gives rise to the following system of equations:

$$\hat{Y}_i = \alpha_0 + \alpha_1(p_{1,0}+X_i) + \alpha_2(p_{2,0}+p_{2,1}X_i+X_i^2) + \cdots + \alpha_k(p_{k,0}+p_{k,1}X_i+\cdots+X_i^k) .$$

The condition of *normalization*, *i.e.* $\Sigma_i[\varphi_r(1)]^2 = \Sigma[c'_{r,i}]^2 = 1$, may be implemented afterwards, for instance by dividing each obtained coefficient with the square root of the sum of squares of all coefficients, $c'_{r,i} = c_{r,i}/\sqrt{\Sigma c^2_{r,i}}$. The condition of *orthogonality* is obtained by insuring that $\Sigma_i\varphi_r(1)\varphi_s(1) = \Sigma_i c_{r,i}c_{s,i} = 0$ for every $s < r$. Since, at degree 0, the coefficients are $(1, 1, \cdots, 1)$, this condition also implies that $\Sigma_i c_{r,i} = 0$ for $r > 0$.

Choice of variable $x = (X-M)/\Delta$. In order to set up tables of coefficients, the predictor variable X must present regular intervals (Δ) and, above all, be standardized in some way. We have opted for a system of recoding centered on the midpoint X_{mp} of X, *i.e.* $x = (X-X_{mp})/\Delta$, which generates values x_i running from $-½(N-1)$ to $½(N-1)$ with unit intervals. This system, less laborious to compute, produces also a sparser table of coefficients; indeed, as the sum of odd powers of x_i is null, $\Sigma_i x_i^{2r+1} = 0$, half of the needed coefficients vanish!

An example of computation for N = 5. The values of predictor X (X_1, X_2, X_3, X_4, X_5) are recoded to $(-2, -1, 0, 1, 2)$ through the transformation $(X_i - M)/\Delta$, where $M = X_{mp} = X_3$ et $\Delta = X_{i+1} - X_i$.

For $r = 0$, coefficients $c_{0,i}$ of $\varphi_0(x)$ are (1, 1, 1, 1, 1).

For $r = 1$, it is sufficient that $\Sigma_i[p_{1,0}+x_i] = Np_{1,0} + \Sigma_i x_i = 0$. Here, $5p_{1,0} + 0 = 0$ implies $p_{1,0} = 0$. Thus, coefficients $c_{1,i}$ are respectively $(0 + x_i) = (-2, -1, 0, 1, 2)$.

For $r = 2$, we need to insure $\Sigma_i\varphi_2(x_i) = 0$ (orthogonality with $r = 0$) and $\Sigma_i\varphi_2(x_i)\varphi_1(x_i) = 0$ (the same with $r = 1$). The first equation gives $\Sigma_i(p_{2,0} + p_{2,1}x_i + x_i^2) = Np_{2,0} + p_{2,1}\Sigma_i x_i + \Sigma_i x_i^2 = 5p_{2,0} + 0 + 10$, from which $p_{2,0} = -2$. Then, using coefficients $c_{1,i}$ already computed, $\Sigma_i[c_{1,i}(p_{2,0} + p_{2,1}x_i + x_i^2)] = 0 + 10p_{2,1} + 0 = 0$, from which $p_{2,1} = 0$. This solution generates coefficients $(-2 + 0 + x_i^2) = (2, -1, -2, -1, 2)$.

For $r = 3$, the three "orthogonality" equations: $\Sigma_i\varphi_3(x_i) = 0$, $\Sigma_i\varphi_3(x_i)\varphi_2(x_i) = 0$ and $\Sigma_i\varphi_3(x_i)\varphi_1(x_i) = 0$, must be insured, giving rise to $5p_{3,0} + 10p_{3,2} = 0$, $10p_{3,1} + 34 = 0$, and $14p_{3,2} = 0$, from which $p_{3,0} = p_{3,2} = 0$, and $p_{3,1} = -3{,}4 = -17/5$. Coefficients $c_{3,i}$ are $(0 - 3.4x_i + 0 + x_i^3) = (-1.2, 2.4, 0, -2.4, 1.2)$, or $(-1, 2, 0, -2, 1)/\lambda_3$, where $\lambda_3 = \frac{5}{6}$.

The same process, applied **for $r = 4$,** produces equations $5p_{4,0} + 10p_{4,2} + 34 = 0$, $10p_{4,1} + 34p_{4,3} = 0$, $14p_{4,2} + 62 = 0$, and $14{,}4p_{4,3} = 0$, which entail $p_{4,0} = 72/35$, $p_{4,2} = -31/7$, $p_{4,1} = p_{4,3} = 0$, and, finally, coefficients $(1, -4, 6, -4, 1)/\lambda_4$, where $\lambda_4 = 35/12$.

Some useful formulae

For the first few powers of predictor X, there exist formulae with which it is easy to derive coefficients $c_{r,i}$ and $p_{i,r}$. All these formulae pertain to the centered, "mid-point", recoding system adopted here, that is: $x_i = (X_i - X_{mp})/\Delta$.

Coefficients $c_{r,i}$. We present hereafter the generic formulae for the coefficients of $\varphi_r(x_i)$, $1 \le r \le 5$. Note that the constants λ_r, required for producing integer-valued coefficients, depend immediately on N and must be determined.

$$\varphi_1(x) = \lambda_1 x \;;$$

$$\varphi_2(x) = \lambda_2[x^2 - (N^2 - 1)/12] \;;$$

$$\varphi_3(x) = \lambda_3[x^3 - x(3N^2 - 7)/20] \;;$$

$$\varphi_4(x) = \lambda_4[x^4 - x^2(2N^2 - 13)/14 + 3(N^2 - 1)(N^2 - 9)/560] \;;$$

$$\varphi_5(x) = \lambda_5[x^5 - 5x^3(N^2 - 7)/18 + x(15N^4 - 230N^2 + 407)/1008] \;.$$

There also exists a recursive algorithm which computes the above coefficients. Let a temporary variable $X = 0, 1, 2, \cdots, N-1$ and base functions: $\Phi_0(X) = 1$, $\Phi_1(X) = 1-(2X)/(N-1)$. Then,

$$\Phi_{r+1}(X) = \frac{(2r+1)(N-2X-1)\,\Phi_r(X) - r(N+r)\,\Phi_{r-1}(X)}{(r+1)(N-r-1)}$$

and $\varphi_r(x) = (-1)^r \lambda_r \Phi_r(X)$, with new constants λ_r to be determined.

Coefficients $p_{i,r}$. Coefficients $p_{i,r}$, which make possible the transition from an orthogonal to a regular polynomial equation, depend entirely on the recoding of predictor variable X. A recoding (or standardization) such as $x = (X-X_1)/\Delta$, which would give $x_i = 0, 1, \cdots, N-1$, would generate the same $c_{r,i}$ but an altogether different, and more furnished, set of $p_{i,r}$. For the centered variable $(X-X_{mp})/\Delta$ adopted here, note at once that $p_{i,r} = 0$ when $r+i$ is odd or $i \geq N-1$. Here are some formulae:

$$p_{2,0} = -(N^2-1)/12 \; ;$$

$$p_{3,1} = -(3N^2-7)/20 \; ;$$

$$p_{4,0} = 3(N^4-10N^2+9)/560 \; ; \; p_{4,2} = -(3N^2-13)/14 \; ;$$

$$p_{5,1} = (15N^4-230N^2+407)/1008 \; ; \; p_{5,3} = -5(N^2-7)/18 \; .$$

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Binomial distribution

✓ Graphical representations

✓ Binomial sum of probabilities for some values of π (table 1)

✓ Maximum size n such that $\Pr(r \geq x) \leq \alpha$ (table 2a) and minimum size n such that $\Pr(r \leq x) \leq \alpha$ (table 2b)

✓ Maximum value of π such that $\Pr(r \geq x \mid n, \pi) \leq \alpha$ (table 3a) and minimum value of π such that $\Pr(r \leq x \mid n, \pi) \leq \alpha$ (table 3b)

✓ Reading off the tables

✓ Full examples

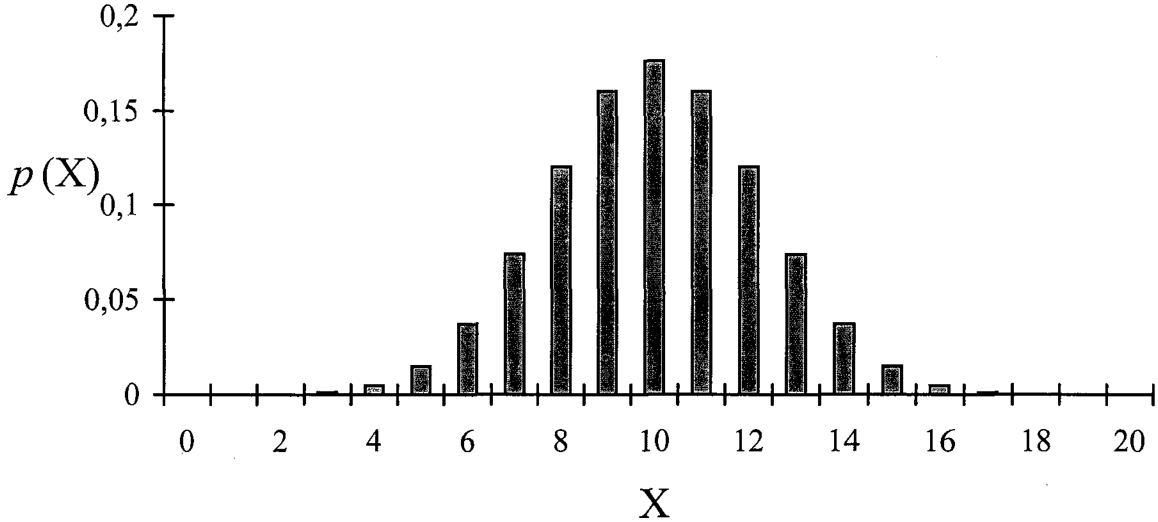
✓ Mathematical presentation

Calculation and moments

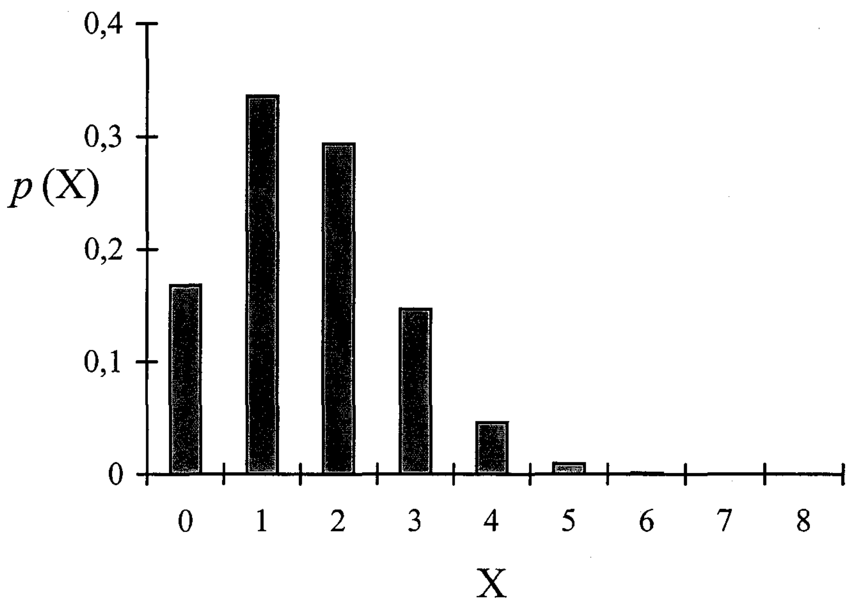
Approximations to $P(x)$

Generation of pseudo random variates

Binomial distribution $B\,(20;\,0{,}5)$



Binomial distribution $B\,(8;\,0{,}2)$



Binomial sum (integral) of probabilities for some values of π (table 1)

n	x	$\pi = 1/2$	1/3	1/4	1/5	1/10
2	0	25000	44444	56250	64000	81000
	1	75000	88889	93750	96000	99000
3	0	12500	29630	42188	51200	72900
	1	50000	74074	84375	89600	97200
	2	87500	96296	98438	99200	99900
4	0	06250	19753	31641	40960	65610
	1	31250	59259	73828	81920	94770
	2	68750	88889	94922	97280	99630
	3	93750	98765	99609	99840	99990
5	0	03125	13169	23730	32768	59049
	1	18750	46091	63281	73728	91854
	2	50000	79012	89648	94208	99144
	3	81250	95473	98438	99328	99954
	4	96875	99588	99902	99968	99999
6	0	01563	08779	17798	26214	53144
	1	10938	35117	53394	65536	88573
	2	34375	68038	83057	90112	98415
	3	65625	89986	96240	98304	99873
	4	89063	98217	99536	99840	99994
	5	98438	99863	99976	99994	1^-
7	0	00781	05853	13348	20972	47830
	1	06250	26337	44495	57672	85031
	2	22656	57064	75641	85197	97431
	3	50000	82670	92944	96666	99727
	4	77344	95473	98712	99533	99982
	5	93750	99314	99866	99963	99999
	6	99219	99954	99994	99999	1^-
8	0	00391	03902	10011	16777	43047
	1	03516	19509	36708	50332	81310
	2	14453	46822	67854	79692	96191
	3	36328	74135	88618	94372	99498
	4	63672	91206	97270	98959	99957
	5	85547	98034	99577	99877	99998
	6	96484	99741	99962	99992	1^-
	7	99609	99985	99998	1^-	1^-
9	0	00195	02601	07508	13422	38742
	1	01953	14307	30034	43621	77484
	2	08984	37718	60068	73820	94703
	3	25391	65031	83427	91436	99167
	4	50000	85515	95107	98042	99911
	5	74609	95758	99000	99693	99994
	6	91016	99172	99866	99969	1^-
	7	98047	99903	99988	99998	1^-
	8	99805	99995	1^-	1^-	1^-

n	x	$\pi = 1/2$	1/3	1/4	1/5	1/10
10	0	00098	01734	05631	10737	34868
	1	01074	10405	24403	37581	73610
	2	05469	29914	52559	67780	92981
	3	17188	55926	77588	87913	98720
	4	37695	78687	92187	96721	99837
	5	62305	92344	98027	99363	99985
	6	82813	98034	99649	99914	99999
	7	94531	99660	99958	99992	1^-
	8	98926	99964	99997	1^-	1^-
	9	99902	99998	1^-	1^-	1^-
11	0	00049	01156	04224	08590	31381
	1	00586	07515	19710	32212	69736
	2	03271	23411	45520	61739	91044
	3	11328	47256	71330	83886	98147
	4	27441	71100	88537	94959	99725
	5	50000	87791	96567	98835	99970
	6	72559	96136	99244	99803	99998
	7	88672	99117	99881	99976	1^-
	8	96729	99863	99987	99998	1^-
	9	99414	99987	99999	1^-	1^-
	10	99951	99999	1^-	1^-	1^-
12	0	00024	00771	03168	06872	28243
	1	00317	05395	15838	27488	65900
	2	01929	18112	39068	55835	88913
	3	07300	39307	64878	79457	97436
	4	19385	63152	84236	92744	99567
	5	38721	82228	94560	98059	99946
	6	61278	93355	98575	99610	99995
	7	80615	98124	99722	99942	1^-
	8	92700	99614	99961	99994	1^-
	9	98071	99946	99996	1^-	1^-
	10	99683	99995	1^-	1^-	1^-
	11	99976	1^-	1^-	1^-	1^-
13	0	00012	00514	02376	05498	25419
	1	00171	03854	12671	23365	62134
	2	01123	13873	33260	50165	86612
	3	04614	32242	58425	74732	96584
	4	13342	55204	79396	90086	99354
	5	29053	75869	91979	96996	99908
	6	50000	89646	97571	99300	99990
	7	70947	96535	99435	99875	99999
	8	86658	99117	99901	99983	1^-
	9	95386	99835	99987	99998	1^-
	10	98877	99979	99999	1^-	1^-
	11	99829	99998	1^-	1^-	1^-
	12	99988	1^-	1^-	1^-	1^-

Binomial sum (integral) of probabilities for some values of π (table 1, cont.)

n	*x*	*π* = 1/2	1/3	1/4	1/5	1/10
14	0	00006	00343	01782	04398	22877
	1	00092	02740	10097	19791	58463
	2	00647	10533	28113	44805	84164
	3	02869	26119	52134	69819	95587
	4	08978	47550	74153	87016	99077
	5	21198	68981	88833	95615	99853
	6	39526	85054	96173	98839	99982
	7	60474	94238	98969	99760	99998
	8	78802	98257	99785	99962	1^-
	9	91022	99596	99966	99995	1^-
	10	97131	99931	99996	1^-	1^-
	11	99353	99992	1^-	1^-	1^-
	12	99908	99999	1^-	1^-	1^-
	13	99994	1^-	1^-	1^-	1^-
15	0	00003	00228	01336	03518	20589
	1	00049	01941	08018	16713	54904
	2	00369	07936	23609	39802	81594
	3	01758	20924	46129	64816	94443
	4	05923	40406	68649	83577	98728
	5	15088	61837	85163	93895	99775
	6	30362	79696	94338	98194	99969
	7	50000	91177	98270	99576	99997
	8	69638	96917	99581	99922	1^-
	9	84912	99150	99921	99988	1^-
	10	94077	99819	99988	99999	1^-
	11	98242	99971	99999	1^-	1^-
	12	99631	99997	1^-	1^-	1^-
	13	99951	1^-	1^-	1^-	1^-
	14	99997	1^-	1^-	1^-	1^-
16	0	00002	00152	01002	02815	18530
	1	00026	01370	06348	14074	51473
	2	00209	05938	19711	35184	78925
	3	01064	16595	40499	59813	93159
	4	03841	33912	63019	79825	98300
	5	10506	54694	81035	91831	99670
	6	22725	73743	92044	97334	99950
	7	40181	87350	97287	99300	99994
	8	59819	95004	99253	99852	99999
	9	77275	98405	99836	99975	1^-
	10	89494	99596	99971	99997	1^-
	11	96158	99921	99996	1^-	1^-
	12	98936	99988	1^-	1^-	1^-
	13	99791	99999	1^-	1^-	1^-
	14	99974	1^-	1^-	1^-	1^-
	15	99998	1^-	1^-	1^-	1^-

n	*x*	*π* = 1/2	1/3	1/4	1/5	1/10
17	0	00001	00101	00752	02252	16677
	1	00014	00964	05011	11822	48179
	2	00117	04415	16370	30962	76180
	3	00636	13042	35302	54888	91736
	4	02452	28140	57389	75822	97786
	5	07173	47767	76531	89430	99533
	6	16615	67393	89292	96234	99922
	7	31453	82814	95975	98907	99988
	8	50000	92452	98762	99742	99999
	9	68547	97272	99690	99951	1^-
	10	83385	99199	99937	99992	1^-
	11	92827	99813	99990	99999	1^-
	12	97548	99966	99999	1^-	1^-
	13	99364	99995	1^-	1^-	1^-
	14	99883	1^-	1^-	1^-	1^-
	15	99986	1^-	1^-	1^-	1^-
	16	99999	1^-	1^-	1^-	1^-
18	0	00000	00068	00564	01801	15009
	1	00007	00677	03946	09908	45028
	2	00066	03265	13531	27134	73380
	3	00377	10167	30569	50103	90180
	4	01544	23107	51867	71635	97181
	5	04813	41224	71745	86708	99358
	6	11894	60851	86102	94873	99883
	7	24034	77674	94305	98372	99983
	8	40726	89240	98065	99575	99998
	9	59274	95665	99458	99909	1^-
	10	75966	98556	99876	99984	1^-
	11	88106	99608	99977	99998	1^-
	12	95187	99915	99997	1^-	1^-
	13	98456	99986	1^-	1^-	1^-
	14	99622	99998	1^-	1^-	1^-
	15	99934	1^-	1^-	1^-	1^-
	16	99993	1^-	1^-	1^-	1^-
	17	1^-	1^-	1^-	1^-	1^-
19	0	00000	00045	00423	01441	13509
	1	00004	00474	03101	08287	42026
	2	00036	02402	11134	23689	70544
	3	00221	07866	26309	45509	88500
	4	00961	18794	46542	67329	96480
	5	03178	35185	66776	83694	99141
	6	08353	54309	82512	93240	99830
	7	17964	72066	92254	97672	99973
	8	32380	85385	97124	99334	99996
	9	50000	93523	99110	99842	1^-
	10	67620	97593	99771	99969	1^-

Binomial sum (integral) of probabilities for some values of π (table 1, cont.)

n	x	π = 1/2	1/3	1/4	1/5	1/10
19	11	82036	99258	99952	99995	1^-
	12	91647	99813	99992	99999	1^-
	13	96822	99962	99999	1^-	1^-
	14	99039	99994	1^-	1^-	1^-
	15	99779	99999	1^-	1^-	1^-
	16	99964	1^-	1^-	1^-	1^-
	17	99996	1^-	1^-	1^-	1^-
	18	1^-	1^-	1^-	1^-	1^-
20	0	00000	00030	00317	01153	12158
	1	00002	00331	02431	06918	39175
	2	00020	01759	09126	20608	67693
	3	00129	06045	22516	41145	86705
	4	00591	15151	41484	62965	95683
	5	02069	29721	61717	80421	98875
	6	05766	47934	78578	91331	99761
	7	13159	66147	89819	96786	99958
	8	25172	80945	95907	99002	99994
	9	41190	90810	98614	99741	99999
	10	58810	96236	99605	99944	1^-
	11	74827	98703	99905	99990	1^-
	12	86841	99628	99982	99998	1^-
	13	94234	99912	99997	1^-	1^-
	14	97931	99983	1^-	1^-	1^-
	15	99409	99997	1^-	1^-	1^-
	16	99871	1^-	1^-	1^-	1^-
	17	99980	1^-	1^-	1^-	1^-
	18	99998	1^-	1^-	1^-	1^-
	19	1^-	1^-	1^-	1^-	1^-
21	0	00000	00020	00238	00922	10942
	1	00001	00231	01903	05765	36473
	2	00011	01283	07452	17870	64841
	3	00074	04616	19168	37038	84803
	4	00360	12116	36742	58601	94785
	5	01330	24865	56659	76930	98555
	6	03918	41863	74363	89149	99673
	7	09462	60076	87009	95695	99939
	8	19166	76012	94385	98559	99990
	9	33181	87522	97937	99593	99999
	10	50000	94428	99358	99903	1^-
	11	66819	97881	99831	99981	1^-
	12	80833	99319	99963	99997	1^-
	13	90538	99817	99993	1^-	1^-
	14	96082	99960	99999	1^-	1^-
	15	98670	99993	1^-	1^-	1^-
	16	99640	99999	1^-	1^-	1^-
	17	99926	1^-	1^-	1^-	1^-

n	x	π = 1/2	1/3	1/4	1/5	1/10
21	18	99988	1^-	1^-	1^-	1^-
	19	99999	1^-	1^-	1^-	1^-
	20	1^-	1^-	1^-	1^-	1^-
22	0	00000	00013	00178	00738	09848
	1	00001	00160	01487	04796	33920
	2	00006	00932	06065	15449	62004
	3	00043	03505	16239	33204	82807
	4	00217	09616	32349	54288	93787
	5	00845	20615	51680	73264	98178
	6	02624	36197	69937	86705	99561
	7	06690	54005	83847	94386	99912
	8	14314	70700	92541	97986	99985
	9	26173	83685	97049	99386	99998
	10	41591	92126	99003	99841	1^-
	11	58409	96730	99713	99965	1^-
	12	73827	98839	99930	99994	1^-
	13	85686	99651	99986	99999	1^-
	14	93310	99912	99998	1^-	1^-
	15	97376	99982	1^-	1^-	1^-
	16	99155	99997	1^-	1^-	1^-
	17	99783	1^-	1^-	1^-	1^-
	18	99957	1^-	1^-	1^-	1^-
	19	99994	1^-	1^-	1^-	1^-
	20	99999	1^-	1^-	1^-	1^-
	21	1^-	1^-	1^-	1^-	1^-
23	0	00000	00009	00134	00590	08863
	1	00000	00111	01159	03984	31513
	2	00003	00675	04920	13319	59196
	3	00024	02648	13696	29653	80727
	4	00130	07579	28321	50071	92689
	5	00531	16949	46847	69469	97739
	6	01734	31003	65373	84017	99422
	7	04657	48069	80370	92849	99877
	8	10502	65135	90368	97266	99978
	9	20244	79357	95922	99106	99997
	10	33882	89312	98514	99750	1^-
	11	50000	95195	99535	99940	1^-
	12	66118	98136	99876	99988	1^-
	13	79756	99381	99972	99998	1^-
	14	89498	99825	99995	1^-	1^-
	15	95343	99958	99999	1^-	1^-
	16	98266	99992	1^-	1^-	1^-
	17	99469	99999	1^-	1^-	1^-
	18	99870	1^-	1^-	1^-	1^-
	19	99976	1^-	1^-	1^-	1^-
	20	99997	1^-	1^-	1^-	1^-

Binomial sum (integral) of probabilities for some values of π (table 1, cont.)

n	*x*	π = 1/2	1/3	1/4	1/5	1/10
23	21	1^-	1^-	1^-	1^-	1^-
	22	1^-	1^-	1^-	1^-	1^-
24	0	00000	00006	00100	00472	07977
	1	00000	00077	00903	03306	29248
	2	00002	00487	03980	11452	56427
	3	00014	01990	11502	26386	78574
	4	00077	05935	24665	45988	91493
	5	00331	13825	42216	65589	97234
	6	01133	26318	60741	81107	99254
	7	03196	42380	76620	91083	99832
	8	07579	59447	87868	96383	99968
	9	15373	74617	94534	98738	99995
	10	27063	85994	97866	99621	99999
	11	41941	93233	99280	99902	1^-
	12	58059	97156	99791	99978	1^-
	13	72937	98966	99948	99996	1^-
	14	84627	99677	99988	99999	1^-
	15	92421	99914	99998	1^-	1^-
	16	96804	99981	1^-	1^-	1^-
	17	98867	99996	1^-	1^-	1^-
	18	99669	99999	1^-	1^-	1^-
	19	99923	1^-	1^-	1^-	1^-
	20	99986	1^-	1^-	1^-	1^-
	21	99998	1^-	1^-	1^-	1^-
	22	1^-	1^-	1^-	1^-	1^-
25	0	00000	00004	00075	00378	07179
	1	00000	00053	00702	02739	27121
	2	00001	00350	03211	09823	53709
	3	00008	01489	09621	23399	76359
	4	00046	04620	21374	42067	90201
	5	00204	11195	37828	61669	96660
	6	00732	22154	56110	78004	99052
	7	02164	37026	72651	89088	99774
	8	05388	53757	85056	95323	99954
	9	11476	69560	92867	98267	99992
	10	21218	82201	97033	99444	99999
	11	34502	90821	98927	99846	1^-
	12	50000	95849	99663	99963	1^-
	13	65498	98363	99908	99992	1^-
	14	78782	99440	99979	99999	1^-
	15	88524	99835	99996	1^-	1^-
	16	94612	99958	99999	1^-	1^-
	17	97836	99991	1^-	1^-	1^-
	18	99268	99998	1^-	1^-	1^-
	19	99796	1^-	1^-	1^-	1^-
	20	99954	1^-	1^-	1^-	1^-
25	21	99992	1^-	1^-	1^-	1^-
	22	99999	1^-	1^-	1^-	1^-
	23	1^-	1^-	1^-	1^-	1^-
26	0	00000	00003	00056	00302	06461
	1	00000	00037	00546	02267	25126
	2	00001	00251	02584	08406	51051
	3	00004	01110	08019	20684	74094
	4	00027	03576	18436	38334	88815
	5	00125	09004	33714	57748	96014
	6	00468	18501	51539	74737	98813
	7	01448	32069	68515	86871	99702
	8	03776	48181	81955	94076	99936
	9	08432	64293	90913	97678	99988
	10	16347	77988	95992	99209	99998
	11	27860	87948	98453	99766	1^-
	12	42251	94173	99479	99940	1^-
	13	57748	97525	99847	99987	1^-
	14	72140	99081	99961	99997	1^-
	15	83653	99703	99991	1^-	1^-
	16	91568	99917	99998	1^-	1^-
	17	96224	99980	1^-	1^-	1^-
	18	98552	99996	1^-	1^-	1^-
	19	99532	99999	1^-	1^-	1^-
	20	99875	1^-	1^-	1^-	1^-
	21	99973	1^-	1^-	1^-	1^-
	22	99996	1^-	1^-	1^-	1^-
	23	99999	1^-	1^-	1^-	1^-
	24	1^-	1^-	1^-	1^-	1^-
27	0	00000	00002	00042	00242	05815
	1	00000	00026	00423	01874	23260
	2	00000	00180	02074	07178	48458
	3	00002	00824	06660	18228	71790
	4	00016	02754	15832	34804	87344
	5	00076	07194	29895	53866	95294
	6	00296	15335	47083	71339	98533
	7	00958	27546	64271	84444	99613
	8	02612	42810	78595	92635	99913
	9	06104	58922	88675	96957	99983
	10	12389	73423	94722	98903	99997
	11	22103	84628	97838	99654	1^-
	12	35055	92098	99222	99905	1^-
	13	50000	96407	99755	99977	1^-
	14	64945	98562	99933	99995	1^-
	15	77897	99496	99984	99999	1^-
	16	87611	99846	99997	1^-	1^-
	17	93896	99959	99999	1^-	1^-

Binomial sum (integral) of probabilities for some values of π (table 1, cont.)

n	x	$\pi = 1/2$	1/3	1/4	1/5	1/10
27	18	97388	99991	1^-	1^-	1^-
	19	99042	99998	1^-	1^-	1^-
	20	99704	1^-	1^-	1^-	1^-
	21	99924	1^-	1^-	1^-	1^-
	22	99984	1^-	1^-	1^-	1^-
	23	99998	1^-	1^-	1^-	1^-
	24	1^-	1^-	1^-	1^-	1^-
28	0	00000	00001	00032	00193	05233
	1	00000	00018	00328	01547	21515
	2	00000	00128	01661	06117	45938
	3	00001	00609	05514	16018	69457
	4	00009	02111	13539	31489	85789
	5	00046	05714	26379	50053	94499
	6	00186	12622	42786	67844	98209
	7	00627	23476	59974	81823	99505
	8	01785	37722	75014	90996	99883
	9	04358	53551	86155	96093	99976
	10	09247	68589	93210	98514	99996
	11	17246	80893	97059	99504	99999
	12	28579	89608	98876	99855	1^-
	13	42528	94971	99622	99963	1^-
	14	57472	97844	99888	99992	1^-
	15	71421	99185	99971	99998	1^-
	16	82754	99729	99993	1^-	1^-
	17	90752	99922	99999	1^-	1^-
	18	95642	99980	1^-	1^-	1^-
	19	98215	99996	1^-	1^-	1^-
	20	99373	99999	1^-	1^-	1^-
	21	99814	1^-	1^-	1^-	1^-
	22	99954	1^-	1^-	1^-	1^-
	23	99991	1^-	1^-	1^-	1^-
	24	99999	1^-	1^-	1^-	1^-
	25	1^-	1^-	1^-	1^-	1^-
29	0	00000	00001	00024	00155	04710
	1	00000	00012	00254	01277	19887
	2	00000	00092	01328	05203	43496
	3	00001	00449	04551	14038	67105
	4	00005	01610	11532	28395	84156
	5	00027	04513	23169	46340	93628
	6	00116	10319	38684	64286	97838
	7	00407	19858	55677	79027	99375
	8	01206	32973	71254	89162	99844
	9	03071	48275	83370	95074	99966
	10	06802	63576	91446	98029	99994
	11	13247	76791	96097	99306	99999
	12	22913	86703	98422	99785	1^-
29	13	35554	93183	99435	99941	1^-
	14	50000	96886	99822	99986	1^-
	15	64446	98738	99950	99997	1^-
	16	77087	99548	99988	99999	1^-
	17	86753	99857	99997	1^-	1^-
	18	93198	99961	1^-	1^-	1^-
	19	96929	99991	1^-	1^-	1^-
	20	98794	99998	1^-	1^-	1^-
	21	99593	1^-	1^-	1^-	1^-
	22	99884	1^-	1^-	1^-	1^-
	23	99973	1^-	1^-	1^-	1^-
	24	99995	1^-	1^-	1^-	1^-
	25	99999	1^-	1^-	1^-	1^-
	26	1^-	1^-	1^-	1^-	1^-
30	0	00000	00001	00018	00124	04239
	1	00000	00008	00196	01052	18370
	2	00000	00065	01060	04418	41135
	3	00000	00330	03745	12271	64744
	4	00003	01223	09787	25523	82451
	5	00016	03545	20260	42751	92681
	6	00072	08384	34805	60697	97417
	7	00261	16678	51429	76079	99222
	8	00806	28602	67360	87135	99798
	9	02139	43174	80341	93891	99955
	10	04937	58475	89427	97438	99991
	11	10024	72385	94934	99051	99998
	12	18080	83399	97841	99689	1^-
	13	29233	91023	99182	99910	1^-
	14	42777	95652	99725	99977	1^-
	15	57223	98120	99918	99995	1^-
	16	70767	99278	99978	99999	1^-
	17	81920	99754	99995	1^-	1^-
	18	89976	99926	99999	1^-	1^-
	19	95063	99981	1^-	1^-	1^-
	20	97861	99996	1^-	1^-	1^-
	21	99194	99999	1^-	1^-	1^-
	22	99739	1^-	1^-	1^-	1^-
	23	99927	1^-	1^-	1^-	1^-
	24	99984	1^-	1^-	1^-	1^-
	25	99997	1^-	1^-	1^-	1^-
	26	1^-	1^-	1^-	1^-	1^-

Maximum size n such that $\Pr(r \geq x \mid n,\pi) \leq \alpha$ in a binomial distribution (table 2a)

x	$\pi = 1/2$				$\pi = 1/3$				$\pi = 1/4$				$\pi = 1/5$				$\pi = 1/10$				x
α =	.05	.025	.01	.005	.05	.025	.01	.005	.05	.025	.01	.005	.05	.025	.01	.005	.05	.025	.01	.005	
1	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1
2	-	-	-	-	-	-	-	-	-	-	-	-	2	-	-	-	3	2	-	-	2
3	-	-	-	-	3	-	-	-	3	3	-	-	4	3	3	-	8	6	5	4	3
4	-	-	-	-	5	4	-	-	6	5	4	4	7	6	5	4	14	11	9	7	4
5	5	-	-	-	7	6	5	5	9	7	6	6	10	9	7	7	20	17	14	12	5
6	6	6	-	-	9	8	7	6	11	10	9	8	14	12	10	9	27	23	19	17	6
7	8	7	7	-	11	10	9	8	14	13	11	10	17	15	13	12	34	29	25	22	7
8	9	9	8	8	13	12	11	10	17	15	13	12	21	19	16	15	41	36	31	28	8
9	11	10	9	9	16	14	13	12	20	18	16	15	25	22	20	18	48	43	37	33	9
10	13	12	11	10	18	16	15	14	23	21	19	17	29	26	23	21	56	50	43	39	10
11	14	13	12	12	20	19	17	16	26	24	22	20	32	29	26	24	63	57	50	46	11
12	16	15	14	13	23	21	19	18	29	27	24	23	36	33	30	28	71	64	57	52	12
13	18	17	15	15	25	23	21	20	33	30	27	25	40	37	33	31	79	71	64	59	13
14	19	18	17	16	27	26	23	22	36	33	30	28	44	41	37	34	86	79	71	65	14
15	21	20	19	18	30	28	26	24	39	36	33	31	48	45	40	38	94	86	78	72	15
16	23	21	20	19	32	30	28	26	42	39	36	34	52	48	44	41	102	94	85	79	16
17	24	23	22	21	35	33	30	29	46	42	39	37	56	52	48	45	110	102	92	86	17
18	26	25	23	22	37	35	33	31	49	46	42	40	61	56	52	49	119	109	99	93	18
19	28	26	25	24	40	38	35	33	52	49	45	43	65	60	55	52	127	117	107	100	19
20	30	28	27	26	43	40	37	36	56	52	48	46	69	64	59	56	135	125	114	108	20
21	31	30	28	27	45	42	40	38	59	55	51	49	73	68	63	60	143	133	122	115	21
22	33	31	30	29	48	45	42	40	62	59	54	52	77	72	67	63	152	141	130	122	22
23	35	33	31	30	50	47	44	42	66	62	58	55	81	76	71	67	160	149	137	130	23
24	37	35	33	32	53	50	47	45	69	65	61	58	86	80	75	71	168	157	145	137	24
25	38	37	35	34	55	52	49	47	73	68	64	61	90	85	79	75	177	165	153	145	25
26	40	38	36	35	58	55	52	50	76	72	67	64	94	89	83	79	185	173	161	152	26
27	42	40	38	37	61	58	54	52	80	75	70	67	99	93	87	83	194	182	168	160	27
28	44	42	40	39	63	60	57	54	83	79	74	70	103	97	91	87	202	190	176	168	28
29	45	44	42	40	66	63	59	57	86	82	77	74	107	101	95	90	211	198	184	175	29
30	47	45	43	42	68	65	62	59	90	85	80	77	111	105	99	94	219	206	192	183	30
31	49	47	45	44	71	68	64	62	93	89	83	80	116	110	103	98	228	215	200	191	31
32	51	49	47	45	74	70	66	64	97	92	87	83	120	114	107	102	236	223	208	199	32
33	53	51	48	47	76	73	69	66	100	95	90	86	124	118	111	106	245	231	216	206	33
34	54	52	50	49	79	75	72	69	104	99	93	90	129	122	115	110	254	240	224	214	34
35	56	54	52	50	82	78	74	71	107	102	97	93	133	127	119	114	262	248	233	222	35
36	58	56	54	52	84	81	77	74	111	106	100	96	138	131	123	118	271	257	241	230	36
37	60	58	55	54	87	83	79	76	114	109	103	99	142	135	127	122	280	265	249	238	37
38	62	59	57	55	90	86	82	79	118	113	107	103	146	139	132	126	289	274	257	246	38
39	63	61	59	57	92	88	84	81	122	116	110	106	151	144	136	131	297	282	265	254	39
40	65	63	60	59	95	91	87	84	125	119	113	109	155	148	140	135	306	291	274	262	40
41	67	65	62	61	98	94	89	86	129	123	117	112	160	152	144	139	315	299	282	270	41
42	69	67	64	62	100	96	92	89	132	126	120	116	164	157	148	143	324	308	290	278	42
43	71	68	66	64	103	99	94	91	136	130	123	119	168	161	152	147	332	316	298	287	43
44	72	70	67	66	106	102	97	94	139	133	127	122	173	165	157	151	341	325	307	295	44
45	74	72	69	67	108	104	99	96	143	137	130	126	177	170	161	155	350	333	315	303	45
46	76	74	71	69	111	107	102	99	146	140	134	129	182	174	165	159	359	342	323	311	46
47	78	75	73	71	114	110	105	101	150	144	137	132	186	178	169	163	368	351	332	319	47
48	80	77	74	73	117	112	107	104	154	147	140	136	191	183	174	168	377	359	340	328	48
49	82	79	76	74	119	115	110	107	157	151	144	139	195	187	178	172	385	368	349	336	49
50	83	81	78	76	122	117	112	109	161	154	147	142	200	191	182	176	394	377	357	344	50

Minimum size n such that $\Pr(r \le x | n, \pi) \le \alpha$ in a binomial distribution (table 2b)

x	$\pi = 1/2$				$\pi = 1/3$				$\pi = 1/4$				$\pi = 1/5$				$\pi = 1/10$				x
α =	.05	.025	.01	.005	.05	.025	.01	.005	.05	.025	.01	.005	.05	.025	.01	.005	.05	.025	.01	.005	
1	8	9	11	12	13	15	17	19	18	20	24	27	22	26	31	34	46	54	64	72	1
2	11	12	14	15	17	19	22	24	23	27	31	34	30	34	39	43	61	70	81	90	2
3	13	15	17	18	21	24	27	29	29	33	37	40	37	41	47	51	76	85	97	106	3
4	16	17	19	21	25	28	31	34	34	38	43	46	44	48	55	59	89	100	113	122	4
5	18	20	22	24	29	32	35	38	40	44	49	52	50	55	62	66	103	114	127	137	5
6	21	23	25	26	33	36	40	42	45	49	54	58	57	62	69	74	116	127	142	152	6
7	23	25	27	29	37	40	44	46	50	54	60	64	63	69	76	81	129	141	156	167	7
8	26	28	30	32	40	44	48	51	55	59	65	69	69	75	83	88	142	154	170	181	8
9	28	30	33	34	44	47	52	55	60	65	70	75	76	82	89	95	154	167	183	195	9
10	30	33	35	37	48	51	55	59	65	70	76	80	82	88	96	102	167	180	197	209	10
11	33	35	38	39	51	55	59	62	70	75	81	85	88	94	102	108	179	193	210	222	11
12	35	37	40	42	55	59	63	66	74	80	86	91	94	101	109	115	191	206	223	236	12
13	37	40	42	44	58	62	67	70	79	85	91	96	100	107	115	121	203	218	236	249	13
14	40	42	45	47	62	66	71	74	84	89	96	101	106	113	122	128	215	231	249	262	14
15	42	44	47	49	65	69	74	78	89	94	101	106	112	119	128	134	227	243	262	275	15
16	44	47	50	52	69	73	78	82	93	99	106	111	118	125	134	141	239	255	275	288	16
17	47	49	52	54	72	77	82	85	98	104	111	116	124	131	141	147	251	267	287	301	17
18	49	51	54	57	76	80	85	89	103	109	116	121	129	137	147	154	263	280	300	314	18
19	51	54	57	59	79	84	89	93	107	114	121	126	135	143	153	160	275	292	312	327	19
20	53	56	59	61	83	87	93	97	112	118	126	131	141	149	159	166	286	304	325	340	20
21	56	58	62	64	86	91	96	100	117	123	131	136	147	155	165	172	298	316	337	352	21
22	58	61	64	66	90	94	100	104	121	128	136	141	153	161	171	179	310	328	350	365	22
23	60	63	66	69	93	98	104	108	126	132	141	146	158	167	177	185	321	340	362	377	23
24	62	65	69	71	96	101	107	111	130	137	145	151	164	173	184	191	333	352	374	390	24
25	65	67	71	73	100	105	111	115	135	142	150	156	170	179	190	197	345	363	386	402	25
26	67	70	73	76	103	108	114	118	139	146	155	161	176	185	196	203	356	375	398	415	26
27	69	72	76	78	107	112	118	122	144	151	160	166	181	190	202	209	368	387	411	427	27
28	71	74	78	80	110	115	121	126	149	156	164	171	187	196	208	215	379	399	423	439	28
29	74	77	80	83	113	119	125	129	153	160	169	175	193	202	213	221	391	411	435	452	29
30	76	79	82	85	117	122	128	133	158	165	174	180	198	208	219	228	402	422	447	464	30
31	78	81	85	87	120	125	132	136	162	170	179	185	204	214	225	234	413	434	459	476	31
32	80	83	87	90	123	129	135	140	167	174	183	190	210	219	231	240	425	446	471	488	32
33	82	86	89	92	127	132	139	143	171	179	188	195	215	225	237	246	436	457	482	500	33
34	85	88	92	94	130	136	142	147	176	183	193	199	221	231	243	252	447	469	494	512	34
35	87	90	94	97	134	139	146	151	180	188	197	204	227	237	249	258	459	480	506	524	35
36	89	92	96	99	137	142	149	154	185	193	202	209	232	242	255	263	470	492	518	536	36
37	91	94	98	101	140	146	153	158	189	197	207	214	238	248	261	269	481	503	530	548	37
38	93	97	101	104	143	149	156	161	193	202	211	218	243	254	266	275	493	515	542	560	38
39	96	99	103	106	147	153	160	165	198	206	216	223	249	260	272	281	504	526	553	572	39
40	98	101	105	108	150	156	163	168	202	211	221	228	255	265	278	287	515	538	565	584	40
41	100	103	107	110	153	159	167	172	207	215	225	232	260	271	284	293	526	549	577	596	41
42	102	106	110	113	157	163	170	175	211	220	230	237	266	277	290	299	537	561	589	608	42
43	104	108	112	115	160	166	173	179	216	224	235	242	271	282	296	305	549	572	600	620	43
44	106	110	114	117	163	169	177	182	220	229	239	246	277	288	301	311	560	583	612	632	44
45	109	112	116	119	167	173	180	185	225	233	244	251	282	294	307	317	571	595	623	644	45
46	111	114	119	122	170	176	184	189	229	238	248	256	288	299	313	322	582	606	635	655	46
47	113	117	121	124	173	180	187	192	233	242	253	260	293	305	319	328	593	618	647	667	47
48	115	119	123	126	177	183	191	196	238	247	257	265	299	310	324	334	604	629	658	679	48
49	117	121	125	129	180	186	194	199	242	251	262	270	304	316	330	340	615	640	670	691	49
50	119	123	128	131	183	190	197	203	247	256	267	274	310	322	336	346	627	652	681	702	50

Maximum value of π such that $\Pr(r \geq x|n,\pi) \leq \alpha$ (table 3a)

n	*x*	α=.05	.025	.01	.005
1	1	.0500	.0250	.0100	.0050
2	1	.0253	.0125	.0050	.0025
	2	.2236	.1581	.1000	.0707
3	1	.0169	.0084	.0033	.0016
	2	.1353	.0942	.0589	.0414
	3	.3683	.2924	.2154	.1709
4	1	.0127	.0063	.0025	.0012
	2	.0976	.0675	.0419	.0294
	3	.2486	.1941	.1408	.1108
	4	.4728	.3976	.3162	.2659
5	1	.0102	.0050	.0020	.0010
	2	.0764	.0527	.0326	.0228
	3	.1892	.1466	.1056	.0828
	4	.3425	.2835	.2220	.1850
	5	.5492	.4781	.3981	.3465
6	1	.0085	.0042	.0016	$.0^{3}83$
	2	.0628	.0432	.0267	.0187
	3	.1531	.1181	.0847	.0662
	4	.2713	.2227	.1730	.1435
	5	.4181	.3587	.2943	.2539
	6	.6069	.5407	.4641	.4135
7	1	.0073	.0036	.0014	$.0^{3}71$
	2	.0533	.0366	.0226	.0158
	3	.1287	.0989	.0708	.0552
	4	.2253	.1840	.1422	.1177
	5	.3412	.2904	.2363	.2029
	6	.4792	.4212	.3566	.3150
	7	.6518	.5903	.5179	.4691
8	1	.0063	.0031	.0012	$.0^{3}62$
	2	.0463	.0318	.0196	.0137
	3	.1111	.0852	.0608	.0474
	4	.1929	.1570	.1209	.0998
	5	.2892	.2448	.1982	.1697
	6	.4003	.3491	.2932	.2578
	7	.5293	.4734	.4100	.3684
	8	.6876	.6305	.5623	.5156
9	1	.0056	.0028	.0011	$.0^{3}55$
	2	.0410	.0281	.0173	.0121
	3	.0977	.0748	.0533	.0415
	4	.1687	.1369	.1052	.0867
	5	.2513	.2120	.1709	.1460
	6	.3449	.2992	.2500	.2191
	7	.4503	.3999	.3436	.3073
	8	.5708	.5175	.4559	.4150
	9	.7168	.6637	.5994	.5550
10	1	.0051	.0025	.0010	.0005
	2	.0367	.0252	.0155	.0108
	3	.0872	.0667	.0475	.0370
	4	.1500	.1215	.0932	.0767
	5	.2224	.1870	.1504	.1283
	6	.3035	.2623	.2183	.1908
	7	.3933	.3475	.2970	.2648
10	8	.4930	.4439	.3882	.3517
	9	.6058	.5549	.4956	.4557
	10	.7411	.6915	.6309	.5887
11	1	.0046	.0022	$.0^{3}91$	$.0^{3}45$
	2	.0333	.0228	.0140	.0098
	3	.0788	.0602	.0428	.0333
	4	.1350	.1092	.0836	.0688
	5	.1995	.1674	.1343	.1144
	6	.2712	.2337	.1939	.1693
	7	.3498	.3079	.2621	.2332
	8	.4356	.3902	.3395	.3067
	9	.5299	.4822	.4276	.3915
	10	.6356	.5872	.5301	.4914
	11	.7615	.7150	.6579	.6177
12	1	.0042	.0021	$.0^{3}83$	$.0^{3}41$
	2	.0304	.0208	.0128	.0089
	3	.0718	.0548	.0389	.0303
	4	.1228	.0992	.0758	.0624
	5	.1810	.1516	.1214	.1033
	6	.2452	.2109	.1746	.1521
	7	.3152	.2766	.2348	.2085
	8	.3908	.3488	.3024	.2724
	9	.4726	.4281	.3777	.3447
	10	.5618	.5158	.4626	.4270
	11	.6613	.6152	.5604	.5229
	12	.7790	.7353	.6812	.6430
13	1	.0039	.0019	$.0^{3}77$	$.0^{3}38$
	2	.0280	.0192	.0118	.0082
	3	.0660	.0503	.0357	.0278
	4	.1126	.0909	.0694	.0570
	5	.1656	.1385	.1108	.0942
	6	.2239	.1922	.1588	.1382
	7	.2870	.2513	.2128	.1886
	8	.3547	.3157	.2728	.2454
	9	.4273	.3857	.3390	.3087
	10	.5053	.4618	.4122	.3793
	11	.5899	.5455	.4938	.4589
	12	.6836	.6397	.5871	.5509
	13	.7941	.7529	.7017	.6652
14	1	.0036	.0018	$.0^{3}71$	$.0^{3}35$
	2	.0259	.0177	.0109	.0076
	3	.0611	.0465	.0330	.0257
	4	.1040	.0838	.0640	.0525
	5	.1527	.1275	.1019	.0865
	6	.2060	.1766	.1456	.1267
	7	.2635	.2303	.1947	.1723
	8	.3250	.2886	.2487	.2234
	9	.3904	.3513	.3079	.2798
	10	.4599	.4189	.3725	.3420
	11	.5343	.4920	.4433	.4108
	12	.6146	.5718	.5217	.4876
	13	.7032	.6613	.6109	.5759
	14	.8073	.7683	.7196	.6849

Maximum value of π such that $\Pr(r \geq x \mid n,\pi) \leq \alpha$ (table 3a, cont.)

n	x	α=.05	.025	.01	.005
15	1	.0033	.0016	$.0^{3}66$	$.0^{3}33$
	2	.0242	.0165	.0101	.0071
	3	.0568	.0433	.0307	.0238
	4	.0966	.0778	.0593	.0487
	5	.1416	.1182	.0943	.0801
	6	.1908	.1633	.1345	.1169
	7	.2437	.2126	.1794	.1587
	8	.2999	.2658	.2287	.2051
	9	.3595	.3228	.2822	.2561
	10	.4225	.3838	.3402	.3118
	11	.4892	.4489	.4031	.3726
	12	.5602	.5191	.4714	.4394
	13	.6365	.5953	.5468	.5136
	14	.7206	.6805	.6321	.5984
	15	.8189	.7819	.7356	.7024
16	1	.0032	.0015	$.0^{3}62$	$.0^{3}31$
	2	.0226	.0155	.0095	.0066
	3	.0531	.0404	.0286	.0223
	4	.0902	.0726	.0553	.0454
	5	.1321	.1101	.0878	.0745
	6	.1777	.1519	.1250	.1086
	7	.2266	.1975	.1664	.1471
	8	.2786	.2465	.2117	.1896
	9	.3333	.2987	.2606	.2362
	10	.3910	.3543	.3134	.2867
	11	.4516	.4133	.3700	.3415
	12	.5156	.4762	.4310	.4008
	13	.5834	.5435	.4970	.4656
	14	.6561	.6165	.5695	.5372
	15	.7360	.6976	.6511	.6186
	16	.8292	.7940	.7498	.7181
17	1	.0030	.0014	$.0^{3}59$	$.0^{3}29$
	2	.0213	.0145	.0089	.0062
	3	.0498	.0379	.0269	.0209
	4	.0846	.0680	.0518	.0425
	5	.1237	.1031	.0821	.0696
	6	.1663	.1420	.1167	.1013
	7	.2118	.1844	.1552	.1370
	8	.2601	.2298	.1971	.1764
	9	.3108	.2781	.2422	.2192
	10	.3639	.3292	.2906	.2655
	11	.4197	.3832	.3422	.3154
	12	.4780	.4404	.3974	.3689
	13	.5394	.5010	.4566	.4268
	14	.6043	.5656	.5203	.4895
	15	.6738	.6355	.5900	.5586
	16	.7498	.7131	.6683	.6369
	17	.8384	.8049	.7626	.7322
18	1	.0028	.0014	$.0^{3}55$	$.0^{3}27$
	2	.0201	.0137	.0084	.0058
	3	.0470	.0357	.0253	.0197
	4	.0796	.0640	.0487	.0400
18	5	.1164	.0969	.0771	.0654
	6	.1563	.1334	.1095	.0950
	7	.1989	.1729	.1454	.1283
	8	.2439	.2153	.1844	.1649
	9	.2912	.2601	.2262	.2046
	10	.3405	.3075	.2710	.2473
	11	.3921	.3574	.3185	.2931
	12	.4459	.4099	.3690	.3421
	13	.5021	.4651	.4228	.3945
	14	.5611	.5236	.4801	.4507
	15	.6233	.5858	.5417	.5115
	16	.6897	.6528	.6088	.5783
	17	.7623	.7270	.6839	.6536
	18	.8466	.8146	.7742	.7450
19	1	.0026	.0013	$.0^{3}52$	$.0^{3}26$
	2	.0190	.0130	.0080	.0055
	3	.0444	.0338	.0239	.0186
	4	.0752	.0605	.0460	.0377
	5	.1099	.0914	.0727	.0616
	6	.1474	.1257	.1032	.0895
	7	.1875	.1628	.1368	.1206
	8	.2297	.2025	.1732	.1548
	9	.2739	.2444	.2123	.1918
	10	.3200	.2886	.2539	.2316
	11	.3681	.3349	.2980	.2739
	12	.4180	.3835	.3446	.3191
	13	.4700	.4344	.3939	.3670
	14	.5242	.4879	.4462	.4181
	15	.5808	.5443	.5017	.4728
	16	.6405	.6042	.5612	.5318
	17	.7041	.6686	.6259	.5963
	18	.7736	.7397	.6981	.6688
	19	.8541	.8235	.7847	.7566
20	1	.0025	.0012	.0005	$.0^{3}25$
	2	.0180	.0123	.0075	.0052
	3	.0421	.0320	.0227	.0176
	4	.0713	.0573	.0436	.0357
	5	.1040	.0865	.0688	.0583
	6	.1395	.1189	.0975	.0845
	7	.1773	.1539	.1291	.1138
	8	.2170	.1911	.1634	.1459
	9	.2586	.2305	.2000	.1806
	10	.3019	.2719	.2389	.2177
	11	.3469	.3152	.2800	.2572
	12	.3935	.3605	.3234	.2990
	13	.4419	.4078	.3690	.3434
	14	.4921	.4572	.4171	.3903
	15	.5444	.5089	.4678	.4402
	16	.5989	.5633	.5217	.4933
	17	.6563	.6210	.5792	.5505
	18	.7173	.6830	.6416	.6128
	19	.7838	.7512	.7112	.6828
	20	.8608	.8315	.7943	.7672

Minimum value of π such that $\Pr(r \leq x \mid n,\pi) \leq \alpha$ (table 3b)

n	x	α=.05	.025	.01	.005
1	0	.9500	.9750	.9900	.9950
2	0	.7764	.8419	.9000	.9293
	1	.9747	.9875	.9950	.9975
3	0	.6316	.7076	.7846	.8291
	1	.8647	.9058	.9411	.9586
	2	.9831	.9916	.9967	.9984
4	0	.5272	.6024	.6838	.7341
	1	.7514	.8059	.8592	.8892
	2	.9024	.9325	.9581	.9706
	3	.9873	.9937	.9975	.9988
5	0	.4508	.5219	.6019	.6535
	1	.6575	.7165	.7780	.8150
	2	.8108	.8534	.8944	.9172
	3	.9236	.9473	.9674	.9772
	4	.9898	.9950	.9980	.9990
6	0	.3931	.4593	.5359	.5865
	1	.5819	.6413	.7057	.7461
	2	.7287	.7773	.8270	.8565
	3	.8469	.8819	.9153	.9338
	4	.9372	.9568	.9733	.9813
	5	.9915	.9958	.9984	$.9^{3}17$
7	0	.3482	.4097	.4821	.5309
	1	.5208	.5788	.6434	.6850
	2	.6588	.7096	.7637	.7971
	3	.7747	.8160	.8578	.8823
	4	.8713	.9011	.9292	.9448
	5	.9467	.9634	.9774	.9842
	6	.9927	.9964	.9986	$.9^{3}29$
8	0	.3124	.3695	.4377	.4844
	1	.4707	.5266	.5900	.6316
	2	.5997	.6509	.7068	.7422
	3	.7108	.7552	.8018	.8303
	4	.8071	.8430	.8791	.9002
	5	.8889	.9148	.9392	.9526
	6	.9537	.9682	.9804	.9863
	7	.9937	.9969	.9988	$.9^{3}38$
9	0	.2832	.3363	.4006	.4450
	1	.4292	.4825	.5441	.5850
	2	.5497	.6001	.6564	.6927
	3	.6551	.7008	.7500	.7809
	4	.7487	.7880	.8291	.8540
	5	.8313	.8631	.8948	.9133
	6	.9023	.9252	.9467	.9585
	7	.9590	.9719	.9827	.9879
	8	.9944	.9972	.9989	$.9^{3}45$
10	0	.2589	.3085	.3691	.4113
	1	.3942	.4451	.5044	.5443
	2	.5070	.5561	.6118	.6483
	3	.6067	.6525	.7029	.7352
	4	.6965	.7377	.7817	.8091
	5	.7776	.8130	.8496	.8717
	6	.8500	.8785	.9068	.9233
10	7	.9128	.9333	.9525	.9630
	8	.9633	.9748	.9845	.9892
	9	.9949	.9975	.9990	.9995
11	0	.2385	.2850	.3421	.3823
	1	.3644	.4128	.4699	.5086
	2	.4701	.5178	.5724	.6085
	3	.5644	.6098	.6605	.6933
	4	.6502	.6921	.7378	.7668
	5	.7288	.7663	.8061	.8307
	6	.8005	.8326	.8657	.8856
	7	.8650	.8908	.9164	.9312
	8	.9212	.9398	.9572	.9667
	9	.9667	.9772	.9860	.9902
	10	.9954	.9978	$.9^{3}09$	$.9^{3}55$
12	0	.2210	.2647	.3188	.3570
	1	.3387	.3848	.4396	.4771
	2	.4382	.4842	.5374	.5730
	3	.5274	.5719	.6222	.6553
	4	.6092	.6512	.6976	.7276
	5	.6848	.7234	.7652	.7915
	6	.7548	.7891	.8254	.8479
	7	.8190	.8484	.8786	.8967
	8	.8772	.9008	.9242	.9376
	9	.9282	.9452	.9611	.9697
	10	.9696	.9792	.9872	.9911
	11	.9958	.9979	$.9^{3}17$	$.9^{3}59$
13	0	.2059	.2471	.2983	.3348
	1	.3164	.3603	.4129	.4491
	2	.4101	.4545	.5062	.5411
	3	.4947	.5382	.5878	.6207
	4	.5727	.6143	.6609	.6913
	5	.6453	.6843	.7272	.7546
	6	.7130	.7487	.7872	.8113
	7	.7761	.8078	.8412	.8618
	8	.8344	.8615	.8892	.9058
	9	.8874	.9091	.9306	.9430
	10	.9340	.9497	.9643	.9722
	11	.9720	.9808	.9882	.9918
	12	.9961	.9981	$.9^{3}23$	$.9^{3}62$
14	0	.1927	.2317	.2804	.3151
	1	.2968	.3387	.3891	.4241
	2	.3854	.4282	.4783	.5124
	3	.4657	.5080	.5567	.5892
	4	.5401	.5811	.6275	.6580
	5	.6096	.6487	.6921	.7202
	6	.6750	.7114	.7513	.7766
	7	.7365	.7697	.8053	.8277
	8	.7940	.8234	.8544	.8733
	9	.8473	.8725	.8981	.9135
	10	.8960	.9162	.9360	.9475
	11	.9389	.9535	.9670	.9743
	12	.9741	.9823	.9891	.9924
	13	.9964	.9982	$.9^{3}29$	$.9^{3}65$

Minimum value of π such that $\Pr(r \leq x|n,\pi) \leq \alpha$ (table 3b, cont.)

n	x	α=.05	.025	.01	.005
15	0	.1811	.2181	.2644	.2976
	1	.2794	.3195	.3679	.4016
	2	.3635	.4047	.4532	.4864
	3	.4398	.4809	.5286	.5606
	4	.5108	.5511	.5969	.6274
	5	.5775	.6162	.6598	.6882
	6	.6405	.6772	.7178	.7439
	7	.7001	.7342	.7713	.7949
	8	.7563	.7874	.8206	.8413
	9	.8092	.8367	.8655	.8831
	10	.8584	.8818	.9057	.9199
	11	.9034	.9222	.9407	.9513
	12	.9432	.9567	.9693	.9762
	13	.9758	.9835	.9899	.9929
	14	.9966	.9984	$.9^{3}33$	$.9^{3}67$
16	0	.1708	.2060	.2502	.2819
	1	.2640	.3024	.3489	.3814
	2	.3439	.3835	.4305	.4628
	3	.4166	.4565	.5030	.5344
	4	.4844	.5238	.5690	.5992
	5	.5484	.5867	.6300	.6585
	6	.6090	.6457	.6866	.7133
	7	.6667	.7013	.7394	.7638
	8	.7214	.7535	.7883	.8104
	9	.7734	.8025	.8336	.8529
	10	.8223	.8481	.8750	.8914
	11	.8679	.8899	.9122	.9255
	12	.9098	.9274	.9447	.9546
	13	.9469	.9596	.9714	.9777
	14	.9774	.9845	.9905	.9934
	15	.9968	.9985	$.9^{3}38$	$.9^{3}69$
17	0	.1616	.1951	.2374	.2678
	1	.2502	.2869	.3317	.3631
	2	.3262	.3645	.4100	.4413
	3	.3957	.4344	.4797	.5105
	4	.4606	.4990	.5434	.5732
	5	.5220	.5596	.6026	.6310
	6	.5803	.6168	.6578	.6846
	7	.6360	.6708	.7094	.7345
	8	.6892	.7219	.7578	.7808
	9	.7399	.7702	.8029	.8236
	10	.7881	.8156	.8448	.8630
	11	.8337	.8580	.8832	.8987
	12	.8763	.8969	.9179	.9304
	13	.9154	.9319	.9482	.9575
	14	.9502	.9621	.9731	.9791
	15	.9787	.9855	.9911	.9938
	16	.9970	.9986	$.9^{3}41$	$.9^{3}71$
18	0	.1534	.1854	.2258	.2550
	1	.2377	.2730	.3161	.3464
	2	.3103	.3472	.3912	.4217
	3	.3767	.4142	.4583	.4885

n	x	α=.05	.025	.01	.005
18	4	.4389	.4764	.5199	.5493
	5	.4979	.5349	.5772	.6055
	6	.5541	.5901	.6310	.6579
	7	.6079	.6426	.6815	.7069
	8	.6595	.6925	.7290	.7527
	9	.7088	.7399	.7738	.7954
	10	.7561	.7847	.8156	.8351
	11	.8011	.8271	.8546	.8717
	12	.8437	.8666	.8905	.9050
	13	.8836	.9031	.9229	.9346
	14	.9204	.9360	.9513	.9600
	15	.9530	.9643	.9747	.9803
	16	.9799	.9863	.9916	.9942
	17	.9972	.9986	$.9^{3}45$	$.9^{3}73$
19	0	.1459	.1765	.2153	.2434
	1	.2264	.2603	.3019	.3312
	2	.2959	.3314	.3741	.4037
	3	.3595	.3958	.4388	.4682
	4	.4192	.4557	.4983	.5272
	5	.4758	.5121	.5538	.5818
	6	.5300	.5656	.6061	.6330
	7	.5820	.6165	.6554	.6809
	8	.6319	.6651	.7020	.7261
	9	.6800	.7114	.7461	.7684
	10	.7261	.7556	.7877	.8082
	11	.7703	.7975	.8268	.8452
	12	.8125	.8372	.8632	.8794
	13	.8526	.8743	.8968	.9105
	14	.8901	.9086	.9273	.9384
	15	.9248	.9395	.9540	.9623
	16	.9556	.9662	.9761	.9814
	17	.9810	.9870	.9920	.9945
	18	.9974	.9987	$.9^{3}48$	$.9^{3}74$
20	0	.1392	.1685	.2057	.2328
	1	.2162	.2488	.2888	.3172
	2	.2827	.3170	.3584	.3872
	3	.3437	.3790	.4208	.4495
	4	.4011	.4367	.4783	.5067
	5	.4556	.4911	.5322	.5598
	6	.5079	.5428	.5829	.6097
	7	.5581	.5922	.6310	.6566
	8	.6065	.6395	.6766	.7010
	9	.6531	.6848	.7200	.7428
	10	.6981	.7281	.7611	.7823
	11	.7414	.7695	.8000	.8194
	12	.7830	.8089	.8366	.8541
	13	.8227	.8461	.8709	.8862
	14	.8605	.8811	.9025	.9155
	15	.8960	.9135	.9312	.9417
	16	.9287	.9427	.9564	.9643
	17	.9579	.9680	.9773	.9824
	18	.9820	.9877	.9925	.9948
	19	.9975	.9988	.9995	$.9^{3}75$

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Reading off the tables

Table 1 enumerates the binomial sum (or integral) of probabilities, $\Pr(r \leq x \mid n,\pi)$, where π ("pi") is used here to denote the probability of success in each trial, for $n = 2$ to 30 trials, $x = 0$ to $n-1$ successes, and parametric values π = 1/2, 1/3, 1/4, 1/5, and 1/10. Each quantity is preceded with a hidden decimal point.

Tables 2a and 2b, for the same range of π values, indicate the limit sample size n^* at which the given number of successes attains significance, at each of the (one-tailed) significance levels 0.05, 0.025, 0.01 and 0.005. Tabe 2a pertains to relatively high numbers of successes ($x/n > \pi$), and table 2b, to relatively low numbers, ($x/n < \pi$). For example, with $\pi = ⅓$, $\alpha = 0.01$ and $x = 10$, table 2a furnishes $n^* = 15$, indicating that $\Pr(r \geq 10 \mid n,⅓) \leq 0.01$, *i.e.* that $x = 10$ is a remarkable number of successes in $n = 15$ trials, and would still be more so in n = 14, 13, 12, 11, and 10 (= x) trials. Similarly, again with $\pi = ⅓$, $\alpha = 0.01$ and $x = 10$, table 2b furnishes $n^* = 55$, indicating that 10 successes out of 55 trials, or 10 out of 56, 10 out of 57, etc., are significantly few.

In table 3a, for the same one-tailed significance levels (α = 0.05, 0.025, 0.01, 0.005) and for given values of n and x, we find the *maximum* possible value of π (noted π^*) such that the occurrence of r successes or more in n trials, $r \geq x$, is significant, *i.e.* is such that $\Pr(r \geq x \mid n,\pi) \leq \alpha$ for $\pi \in (0, \pi^*)$; table 3b gives the *minimum* π^* such that the occurrence of at most r successes in n trials, where $r \leq x$, be significant, *i.e.* $\Pr(r \leq x \mid n,\pi) \leq \alpha$ for $\pi \in (\pi^*, 1)$. This table 3, which is slightly more complex than the preceding two, allows for binomial testing while freeing the user from a set of predefined π values.

Illustration 1. Find $P(4) = \Pr(x \leq 4 \mid 10,⅓)$, the probability of observing four successes or less in ten trials, each success having probability ⅓ of occurring. Table 1, at line $n = 10$, $x = 4$ and column $\pi = ⅓$, gives directly $P(4) = 0.78687$.

Illustration 2. If each trial has probability ¼ of success, what is the probability of obtaining ten successes *or more* in 20 trials? And that of obtaining exactly ten successes? Table 1 again gives us the required information. Actually, the probability of obtaining ten successes or more is the complement (or remainder, from a total of 1) of the probability of obtaining nine successes or less, *i.e.* $\Pr(r \geq 10 \mid 20,¼) = p(10)+p(11)+\cdots+p(20) = 1 - \Pr(r \leq 9 \mid 20,¼)$. Now, we find $\Pr(r \leq 9 \mid 20,¼) = 0.98614$, so that $\Pr(r \geq 10) = 1 - 0.98614 = 0.01386$, which represents the probability of observing ten successes or more in 20 trials. As for the probability of observing exactly ten successes, we may obtain it by substraction. Referring to the same section in table 1,

$\Pr(x=10) = \Pr(r\leq 10) - \Pr(r\leq 9) = 0.99605 - 0.98614 = 0.00991$; we may also compute directly this probability using formula $p(x)$ given below, in the Calculation and moments section.

Illustration 3. At the game of "heads or tails", what is the maximum number of guesses (or trials) allowed so that ten good guesses would be deemed remarkable, at the 5 % significance threshold? In this situation, $\pi = ½$ (indicating the probability of a correct, random guess at heads or tails), $x = 10$ and $\alpha = 0.05$. Table 2a furnishes the desired quantity, namely the maximum number of trials n^* such that $\Pr(r\geq 10\,|\,n,½)$ for every $n \leq n^*$. At line $x = 10$ and column $\pi = ½$, $\alpha = 0.05$, we read $n^* = 13$; therefore, for any number n in the interval $x \leq n \leq n^*$, or $n = 10$, 11, 12, or 13, it would be "remarkable" (at a 5 % threshold) to register ten correct guesses; such would not be the case if those 10 good guesses appeared in more ($n > 13$) trials.

Illustration 4. Suppose that, in a given situation, the individual probability of success per trial was ⅛. Now, someone, in ten trials, succeeded five times. Was the observed rate of 50 % abnormally high, at the 0.01 significance level? To answer that question, one could perform the few necessary calculations, otherwise table 3a can help us out. With $n = 10$, $x = 5$ and $\alpha = 0.01$, table 3a indicates $\pi^* = 0.1504$ as the maximum (individual) probability such that the occurrence of five or more successes in ten trials is significant. Thus, any value π in the interval $(0, \pi^*)$ would meet the significance criterion. Our supposed value π (= ⅛) does meet it since $0 < \pi < 0.1504$. So, if π = ⅛, the obtained percentage of successes of 50 % can be deemed abnormally high (at the 0.01 threshold), or else the true probability of success (π) should be revised upwards.

Full examples

Example 1. A student comes to take a test, but being totally ignorant of the subject matter, relies solely on luck. The test comprises ten multiple-choice questions: for each question, four answers are proposed, only one of which is correct. What is the probability of guessing seven right answers? If a score of 60 % is required to pass the test, what are the odds of obtaining it? *Solution*: The probability of producing seven correct answers at random out of ten 4-choice questions can easily be calculated as a binomial probability. Let $n = 10$, $x = 7$, and $\pi = 0.25$ if the student picks each answer at random. Then,

$$\begin{aligned} p(x) &= \tbinom{n}{x}\pi^x(1-\pi)^{n-x} \\ &= \tbinom{10}{7}0.25^7 0.75^3 \\ &\approx 120\times 0.000061\times 0.421875 \approx 0.0031\ . \end{aligned}$$

Thanks to table 1, which gives the sum $P(x) = p(0)+p(1)+ \cdots + p(x)$, one can calculate $p(x) = P(x)-P(x-1)$. Here, with $n = 10$ and $\pi = ¼$, $p(7) = P(7)-P(6) \approx 0.99958-0.99649 = 0.0031$, as before. Next, the probability of passing the test amounts to the probability of guessing the right answers to at least six questions out of ten, *i.e.* $\Pr(x\geq 6) = p(6)+p(7)+ \cdots +p(10)$. This sum of upward probabilities is the complement of the sum given in table 1, *i.e.* $\Pr(x\geq 6) = 1-\Pr(x<6) = 1-\Pr(x\leq 5) \approx 1 - 0.98027 = 0.01973$. It seems that our scatterbrained student has less than 2 % of chances to pass the test, and should have spent some time studying!

Example 2. Out of the first 20 customers coming to the perfume counter of a department store, 70 % were female, *i.e.* 14 women. Determine the confidence interval for that percentage, with a confidence rate of 95 %. *Solution*: Tables 3a and 3b contain together the needed information. The confidence rate is complementary to the significance level, so that $\alpha = 1 - 0.95 = 0.05$ for a unilateral (one-tailed) interval, 0.025 (= 0.05/2) for a bilateral (two-tailed) interval. With $n = 20$, $x = 14$ and $\alpha = 0.025$, tables 3a and 3b indicate respectively $\pi = 0.4572$ and $\pi = 0.8811$, from which the searched-for interval is approximately (45.7 %, 88.1 %). Moreover, since that interval embraces the value 50 %, we can not assure, from available data, that there are really more women than men that come to the perfume counter.

Example 3. At the admissions desk of the emergency service in a large hospital, for a span of seven days, a total of 178 men and 237 women were registered. Are there really more admissions of one gender than of the other? *Solution*: The hypothesis to be verified is that admission of a woman has the same probability as that of a man, *i.e.* $\pi = ½$; also, $n = 178+237 = 415$. The test should be bidirectional (two-tailed), applying a signifiance level of, say, 1 %. Now, our tables do not reach sample sizes as high as $n = 415$. Nevertheless, the normal approximation, $z = \{|2x-n| - 1\}/\sqrt{n}$, appropriate with $\pi = ½$, can be used. We thus have $x = 178$ (or 237), $n = 415$, and $z = \{|2\times 178-415|-1\}/\sqrt{415} \approx 2.847$. For a two-tailed 1 % test, the critical normal value is $\pm z_{[0.995]} = \pm 2.576$ (*see* section on normal distribution, table 2). Since our computed z value exceeds the critical value, it follows that there are really (significantly) more women than men who are admitted to the emergency service.

Example 4. Let us take up again illustration 4 above in order to demonstrate various types of computation, exact or approximate, that can be applied to that case. Using $\pi = ⅛$, $x = 5$ and $n = 10$, we can compute exactly $\Pr(r\geq 5\,|\,10,⅛) = p(5)+p(6)+ \cdots + p(10) \approx 0.00394+0.00047 +0.00004+0.00000+0.00000+0.00000 = 0.00445$; for instance, $p(6) = \binom{10}{6}(⅛)^6(⅞)^4 = 210\times 0.000003815\times 0.5861816 \approx 0.00047$. The global probability of 0.00445 is assuredly less than the threshold probability $\alpha = 0.01$.

The simple normal approximation, as in example 3, is a special case of a general formula for any π : $z = [x\pm½ - n\pi]/\sqrt{[n\pi(1-\pi)]}$. The "correction for continuity" (*i.e.* "+½" or

"$-\frac{1}{2}$") in the numerator is made *to reduce the absolute value* of the expression and it generally brings the resulting probability nearer to its true value. Applying now this formula to the data in illustration 4, we substract "½" from the numerator and obtain $z \approx 3.1076$. In table 1 (Normal distribution) of the normal integral, we read $P(3.1076) \approx 1^-$, whence $\Pr(Z \geq 3.1076) \approx 0$, a highly significant value. However, for $\pi \neq \frac{1}{2}$, the proposed approximation can be very coarse, and two other formulae, developped in the next section, may be preferable. Using the so-called "angular" normal approximation (cf $\hat{P}_2$), we calculate $z^* = \sqrt{(4n)}\{\sin^{-1}\sqrt{[(x+\frac{3}{8}\pm\frac{1}{2})/(n+\frac{3}{4})]} - \sin^{-1}\sqrt{\pi}\} = \sqrt{40}\{\sin^{-1}\sqrt{[(5+\frac{3}{8}-\frac{1}{2})/(10+\frac{3}{4})]} - \sin^{-1}\sqrt{\frac{1}{8}}\} \approx \sqrt{40}\{0.7388 - 0.3614\} \approx 2.387$; note that angles, that result from the arcsin ($\sin^{-1}$) function, are obtained in radians. In table 1 (Normal distribution), $P(2.387) \approx 0.99151$, the complementary probability, $1-0.99151 = 0.00849$, being once more conclusive. [For the inquisitive reader, approximation $\hat{P}_3$, named after Gram-Charlier, is nearly equal to 0.99576, or 0.00424 for its complementary value, coming quite close to the exact, binomial probability of 0.00445.]

Mathematical presentation

The number x of favorable outcomes (or "successes") after n trials (or "events") is a binomial r.v. if, on the one hand, the successive trials are statistically independent and, on the other hand, the probability π $(0 < \pi < 1)$ of a favorable outcome is a constant. Be it for games of chance or lottery, Mendelian genetics, the analysis of test scores in multiple-choice questionnaires, etc., the binomial distribution (or law) is a statistical model as famous and every bit as useful as the normal distribution. James Bernoulli introduced it formally in 1713; then, in 1733, De Moivre, in his efforts to approximate binomial probabilities for very large numbers of events, used it as a gateway to the normal distribution. Many years earlier (~1650), binomial probabilities had already appeared, albeit namelessly, in the letters of Pascal and Fermat.

What is the probability of obtaining x successes in n trials, or events? If π is the individual probability of a success, how many successes do we need in 10 trials in order to consider the observed sequence remarkable, at a given threshold of probability? Such are some of the questions which the binomial distribution enables us to answer.

Calculation and moments

The expansion of the binomial $([1-\pi] + \pi)^n$ gives rise to a sequence of terms: $(1-\pi)^n$, $n(1-\pi)^{n-1}\pi$, $\frac{1}{2}n(n-1)(1-\pi)^{n-2}\pi^2$, etc., which constitute the binomial distribution of probabilities. If π $(0 < \pi < 1)$ is the probability of obtaining a success in any one trial, then each term gives the probability of obtaining $x = 0, 1, 2 \cdots$ successes in n trials. The individual probabilities, simple to obtain, and their sum are:

$$p(x) = \binom{n}{x}\pi^x(1-\pi)^{n-x} ,$$

$$P(x) = \sum_{r=0}^{x} p(x) = \sum_{r=0}^{x} \binom{n}{r}\pi^r(1-\pi)^{n-r} .$$

The binomial sum (or integral) $P(x)$ is usefully related to the F (*see* section on F) and *Beta* distributions. Later, we consider a few approximations to this sum.

Moments. The expectation (mean), variance, and shape coefficients of a binomial variable, with parameters n and π, are:

$$\mathrm{E}(x) = \mu = n\pi$$

$$\mathrm{var}(x) = \sigma^2 = n\pi(1-\pi)$$

$$\gamma_1 = (1 - 2\pi)/\sigma$$

$$\gamma_2 = [1 - 6\pi(1-\pi)]/\sigma^2 .$$

Note that, for the case where $\pi = ½$, $\mu = n/2$, $\sigma^2 = n/4$, $\gamma_1 = 0$ and $\gamma_2 = -2/n$.

The mode of x (Mo) is such that:

$$(n + 1)\pi - 1 < \mathrm{Mo} \le (n + 1)\pi .$$

Let us add that, when $(n+1)\pi$ is an integer, the mode is duplicated, sitting on each of the above limits.

Approximations to *P*(*x*)

In cases where the desired combination of parameter values (n,π,x) is not to be found in available tables and n is large, the detailed calculations of binomial probabilities can prove fastidious, even with mechanical aids. That is mostly why some approximations to the binomial sum of probabilities were worked out.

The simple normal approximation, $z = (x-\mu)/\sigma \sim N(0,1)$, suffers from the discreteness of variable x and is not efficient. A correction for continuity, of the form "$x \pm ½$", greatly improves the situation, especially for the case where $\pi = ½$. For that precise case, the following equivalence:

$$z = \frac{|2x - n| - 1}{\sqrt{n}} \sim N(0,1)$$

proves to be excellent, even with very small n's.

In the general case where $\pi \neq ½$, the normal approximation is still preferred since, for large values of n, the binomial distribution tends toward the normal. However, for practical situations characterized by a small n and a value of π straying from ½ – which generates a skewed distribution –, other approximations are available in the literature. Here are some that we deem most suitable:

Simple normal : $\hat{P}_1(x) = \Phi(z)$, $z = (x \pm ½ - \mu)/\sigma$;

"Angular" normal[1] : $\hat{P}_2(x) = \Phi(z^*)$, $z^* = \sqrt{(4n)}\{\sin^{-1}\sqrt{[(x+3/8 \pm 1/2)/(n+3/4)]} - \sin^{-1}\sqrt{\pi}\}$;

Normal G-C : $\hat{P}_3(x) = \Phi(z) - (1-2\pi)\varphi(z)(z^2-1)/(6\sigma)$, $z = (x \pm ½ - \mu)/\sigma$;

Poisson : $\hat{P}_4(x) = e^{-\mu} \sum_{r=0}^{x} \frac{\mu^r}{r!}$.

In the first three approximations, Φ denotes the standard normal d.f. and φ, its relevant p.d.f. The second approximation uses the famous "angular" transformation proposed by Fisher, $x' = \sin^{-1}\sqrt{(x/n)}$, more precisely $\sin^{-1}\sqrt{[(x+3/8)/(n+3/4)]}$, with approximate expectation $\sin^{-1}\sqrt{\pi}$ and variance $1/(4n)$; to this, we have added the recommended continuity correction. The expression for $\hat{P}_3$ contains the first terms of a series expansion (attributed to Gram-Charlier) which embodies a skewness correction, with factor $1-2\pi$. As for $\hat{P}_4$, this approximation rests upon another well-known relation, namely that, with $\mu = n\pi$ being constant, if π decreases as n increases, the binomial distribution tends to a Poisson distribution with parameter μ: such an approximation should be confined to situations wherein π approaches 0.

To guide the potential user, we gathered in a table (next page) some indications on the accuracy of each approximation for different values of π, as a function of n. For $\hat{P}_1$, $\hat{P}_2$ and $\hat{P}_3$, the table indicates the size n at which and above which the approximation reaches the given accuracy of 0.01 or 0.005, *i.e.* the absolute difference between the approximate $\hat{P}_j(x)$ and the exact binomial $P(x)$ is limited by 0.01 or 0.005 for any value x. With regard to Poisson's $\hat{P}_4$, the first

[1] Angles are given in radian units; if degrees are used instead of radians, the variance would be equal to $(90\pi^{-1})^2/n \approx 820.7/n$ instead of $1/(4n)$, where the symbol π used here denotes the area of the unit circle, $\pi \approx 3.1416$.

Data on the precision of four approximations to the binomial sum *P*(*x*) (*see* text)

	Accuracy	$\pi = ½$	$\pi = ⅓$	$\pi = ¼$	$\pi = 1/5$	$\pi = 1/10$
$\hat{P}_1$	0.01	4	22	60	100	310
	0.005	6	87	232	390	1230
$\hat{P}_2$	0.01	14	43	95	158	486
	0.005	28	145	366	613	1925
$\hat{P}_3$	0.01	4	4	5	6	12
	0.005	6	10	15	9	42
$\hat{P}_4$	All n	.118	.072	.052	.040	.019
	Large n	.084	.049	.036	.028	.013

line ("all n") indicates the maximum absolute error, which occurs for small values of n[2]. The second line ("large n") gives evidence to the fact that this approximation is more influenced by π than by n and that it becomes advantageous for smaller values of π.

For current usage, we recommend the Gram-Charlier approximation, our $\hat{P}_3$, for $n \geq 10$ and $0.15 < \pi < 0.85$.

Generation of pseudo random variates

The computer program outlined below allows the production of binomial r.v.'s, using a function (designated UNIF) which generates serially r.v.'s from the standard uniform $U(0,1)$ distribution. An important set-up phase, executed once, is required. See Remarks, below.

Preparation : Suppose constants π (the probability of success in any binomial trial), n (the number of trials), and B (= 50, 100, 200, a natural number). Declare the real arrays $p2[0..n]$ et $jp2[0..n]$, and the integer array Table[0..B−1].

$pj \leftarrow (1-\pi)^n$; $j, t, tp2, sp2 \leftarrow 0$; **Goto** P_1 ;
While $j < n$ **do**
    $j \leftarrow j+1$; $pj \leftarrow pja \times (n+1-j) \times \pi/[j \times (1-\pi)]$;

[2] This maximum difference between the Poisson d.f., $e^{-\mu}\sum_{r=0}^{x}\mu^r/r!$, and the binomial d.f. $P(x)$ occurs at the integral value $x = \lfloor \pi^{-1} \rfloor$ (i.e. x is the highest integer such that $x \leq \pi^{-1}$).

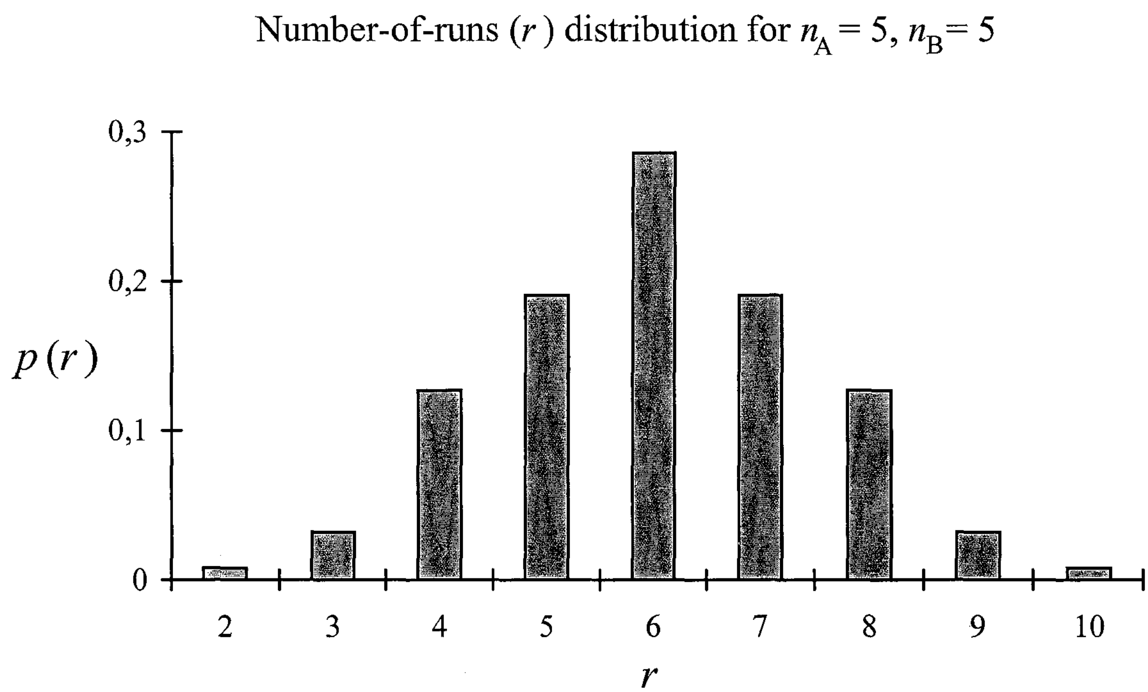
```
P1:  nj ← ⌊B×pj⌋ ; pja ← pj ;
     If nj ≥ 1 then  For i = t to t+nj−1 do Table[i] ← j ;
                     t ← t + nj ;
     r ← pj − nj/B ; tp2 ← tp2 + r ;
     If j = 0 then p2[0] ← r ; jp2[0] ← 0
            else
               k ← j ;
               Repeat  If r ≤ p2[k−1] Goto P2 ;
                       p2[k] ← p2[k−1] ; jp2[k] ← jp2[k−1] ; k ← k−1
               Until k = 0
           P2: p2[k] ← r ; jp2[k] ← j ;
  For j = 0 to n do sp2 ← sp2 + p2[j] , p2[j] ← sp2 / tp2 ;
Production : k ← ⌊B×UNIF⌋ ;
       If k < t then  Return Table[k] → x
               else   u ← UNIF; j ← 0 ;
                      While u > p2[j] do j ← j+1 ;
                      Return jp2[j] → x .
```

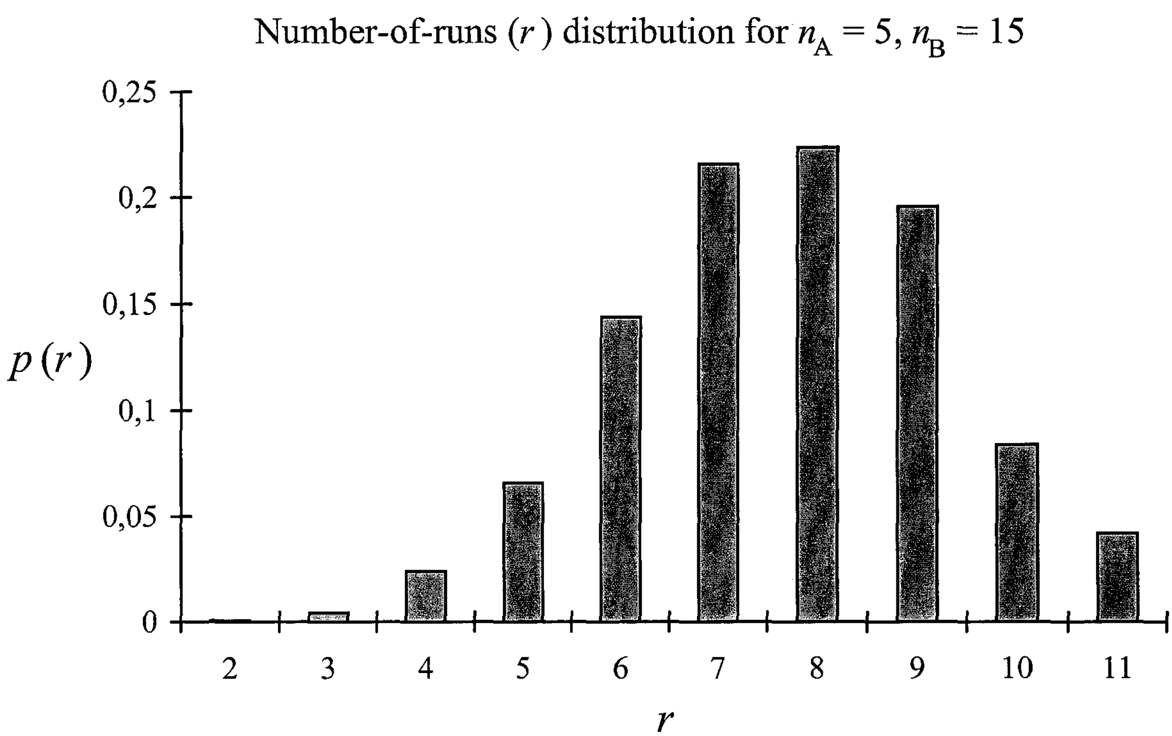
Remarks :

1. Standard temporal cost : 2.6 to 2.8 × t(UNIF)
2. The method of generation, which consists in spreading the d.f. of a discrete variable over an equivalent table of integer values, can easily be adapted to any discrete distribution. Its efficacy depends mainly on constant B, as the expected number of calls to function UNIF is nearly $1 + n/B$.
3. Another generation method, of general interest for discrete distributions, is the so-called "alias method" (Gentle 1998; Laurencelle 2001). It also requires a major set-up phase, is more space-efficient and slightly less time-efficient than the "table method" used here.
4. For small values of n, one may use the following naive algorithm which simulates the underlying Bernoulli process: "For $j = 1$ to n do If UNIF $< \pi$ then $x \leftarrow x+1$", or, again, "For $j = 1$ to n do $x \leftarrow x + \lfloor \text{UNIF}+\pi \rfloor$".

Number-of-runs distribution

- ✓ Graphical representations
- ✓ Critical values of number of runs (r) when $n_A = n_B$ at extreme percentage points (table 1)
- ✓ Critical values of number of runs (r) when $n_A \neq n_B$ at percentage points 5, 2.5, 1 and 0.5 (lower triangle) and 95, 97.5, 99 and 99.5 (upper triangle) (table 2)
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation
  - Calculation and moments


Number-of-runs (r) distribution for $n_A = 5$, $n_B = 5$
0,3
0,2
0,1
0
p(r)
2
3
4
5
6
7
8
9
10
r


Number-of-runs (r) distribution for $n_A = 5$, $n_B = 15$
0,25
0,2
0,15
0,1
0,05
0
p(r)
2
3
4
5
6
7
8
9
10
11
r

Critical values of number of runs (r) when $n_A = n_B$ at extreme percentage points (table 1)

$n_A = n_B$	$P \leq .005$	.01	.025	.05	$P \geq .95$	.975	.99	.995
3	-	-	-	-	-	-	-	-
4	-	-	-	2	8	-	-	-
5	-	2	2	3	9	10	10	-
6	2	2	3	3	11	11	12	12
7	3	3	3	4	12	13	13	13
8	3	4	4	5	13	14	14	15
9	4	4	5	6	14	15	16	16
10	5	5	6	6	16	16	17	17
11	5	6	7	7	17	17	18	19
12	6	7	7	8	18	19	19	20
13	7	7	8	9	19	20	21	21
14	7	8	9	10	20	21	22	23
15	8	9	10	11	21	22	23	24
16	9	10	11	11	23	23	24	25
17	10	10	11	12	24	25	26	26
18	11	11	12	13	25	26	27	27
19	11	12	13	14	26	27	28	29
20	12	13	14	15	27	28	29	30
21	13	14	15	16	28	29	30	31
22	14	14	16	17	29	30	32	32
23	14	15	16	17	31	32	33	34
24	15	16	17	18	32	33	34	35
25	16	17	18	19	33	34	35	36
26	17	18	19	20	34	35	36	37
27	18	19	20	21	35	36	37	38
28	19	19	21	22	36	37	39	39
29	19	20	22	23	37	38	40	41
30	20	21	23	24	38	39	41	42
31	21	22	23	25	39	41	42	43
32	22	23	24	25	41	42	43	44
33	23	24	25	26	42	43	44	45
34	24	25	26	27	43	44	45	46
35	24	25	27	28	44	45	47	48
36	25	26	28	29	45	46	48	49
37	26	27	29	30	46	47	49	50
38	27	28	30	31	47	48	50	51
39	28	29	30	32	48	50	51	52
40	29	30	31	33	49	51	52	53
41	29	31	32	34	50	52	53	55
42	30	31	33	35	51	53	55	56
43	31	32	34	35	53	54	56	57
44	32	33	35	36	54	55	57	58
45	33	34	36	37	55	56	58	59
46	34	35	37	38	56	57	59	60
47	35	36	38	39	57	58	60	61
48	36	37	38	40	58	60	61	62
49	36	38	39	41	59	61	62	64
50	37	38	40	42	60	62	64	65

Critical values of number of runs (r) when $n_A \neq n_B$ at percentage points 5 (lower triangle) and 95 (upper triangle) (table 2)

(n_A, n_B)	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	-																							
3		-	7																					
4			-	9	9	9																		
5		2	2	-	10	10	11	11	11															
6		2	3	3	-	11	12	12	12	13	13	13	13											
7		2	3	3	4	-	13	13	13	14	14	14	14	15	15	15	15	15						
8	2	2	3	3	4	4	-	14	14	15	15	15	16	16	16	16	16	16	17	17	17	17	17	
9	2	2	3	4	4	5	5	-	15	15	16	16	17	17	17	17	18	18	18	18	18	18	18	19
10	2	3	3	4	5	5	6	6	-	16	17	17	17	18	18	18	19	19	19	19	20	20	20	20
11	2	3	3	4	5	5	6	6	7	-	17	18	18	19	19	19	20	20	20	20	21	21	21	21
12	2	3	4	4	5	6	6	7	7	8	-	18	19	19	20	20	21	21	21	22	22	22	22	22
13	2	3	4	4	5	6	6	7	8	8	9	-	20	20	21	21	21	22	22	22	23	23	23	24
14	2	3	4	5	5	6	7	7	8	8	9	9	-	21	21	22	22	23	23	23	24	24	24	24
15	2	3	4	5	6	6	7	8	8	9	9	10	10	-	22	22	23	23	4	24	25	25	25	26
16	2	3	4	5	6	6	7	8	8	9	10	10	11	11	-	23	24	24	25	25	25	26	26	26
17	2	3	4	5	6	7	7	8	9	9	10	10	11	11	12	-	24	25	25	26	26	27	27	27
18	2	3	4	5	6	7	8	8	9	10	10	11	11	12	12	13	-	25	26	26	27	27	28	28
19	2	3	4	5	6	7	8	8	9	10	10	11	12	12	13	13	14	-	27	27	28	28	28	29
20	2	3	4	5	6	7	8	9	9	10	11	11	12	12	13	13	14	14	-	28	28	29	29	30
21	2	3	4	5	6	7	8	9	10	10	11	12	12	13	13	14	14	15	15	-	29	29	30	30
22	2	4	4	6	6	7	8	9	10	10	11	12	12	13	14	14	15	15	16	16	-	30	30	31
23	2	4	4	6	6	8	8	9	10	11	11	12	13	13	14	15	15	16	16	17	17	-	31	32
24	2	4	5	6	7	8	8	9	10	11	12	12	13	14	14	15	15	16	16	17	17	18	-	32
25	2	4	5	6	7	8	9	10	10	11	12	13	13	14	15	15	16	16	17	17	18	18	19	-
26	2	4	5	6	7	8	9	10	10	11	12	13	14	14	15	15	16	17	17	18	18	19	19	20
27	2	4	5	6	7	8	9	10	11	11	12	13	14	14	15	16	16	17	18	18	19	19	20	20
28	2	4	5	6	7	8	9	10	11	12	12	13	14	15	15	16	17	17	18	18	19	20	20	21
29	2	4	5	6	7	8	9	10	11	12	13	13	14	15	16	16	17	18	18	19	19	20	20	21
30	2	4	5	6	7	8	9	10	11	12	13	14	14	15	16	17	17	18	18	19	20	20	21	21
31	2	4	5	6	7	8	9	10	11	12	13	14	15	15	16	17	17	18	19	19	20	21	21	22
32	2	4	5	6	7	8	9	10	11	12	13	14	15	16	16	17	18	18	19	20	20	21	21	22
33	2	4	5	6	7	8	10	10	12	12	13	14	15	16	16	17	18	19	19	20	21	21	22	22
34	2	4	5	6	8	8	10	11	12	12	13	14	15	16	17	17	18	19	20	20	21	22	22	23
35	2	4	5	6	8	8	10	11	12	13	14	14	15	16	17	18	18	19	20	21	21	22	22	23
36	2	4	5	6	8	9	10	11	12	13	14	15	16	16	17	18	19	19	20	21	21	22	23	23
37	2	4	5	6	8	9	10	11	12	13	14	15	16	16	17	18	19	20	20	21	22	22	23	24
38	2	4	5	6	8	9	10	11	12	13	14	15	16	17	18	18	19	20	21	21	22	23	23	24
39	3	4	5	6	8	9	10	11	12	13	14	15	16	17	18	18	19	20	21	22	22	23	24	24
40	3	4	5	6	8	9	10	11	12	13	14	15	16	17	18	19	20	20	21	22	22	23	24	25
41	3	4	6	6	8	9	10	11	12	13	14	15	16	17	18	19	20	20	21	22	23	23	24	25
42	3	4	6	6	8	9	10	11	12	14	14	15	16	17	18	19	20	21	21	22	23	24	24	25
43	3	4	6	6	8	9	10	11	12	14	14	16	16	17	18	19	20	21	22	22	23	24	25	25
44	3	4	6	7	8	9	10	12	12	14	15	16	17	18	18	19	20	21	22	23	23	24	25	26
45	3	4	6	7	8	9	10	12	13	14	15	16	17	18	19	20	20	21	22	23	24	24	25	26
46	3	4	6	7	8	9	10	12	13	14	15	16	17	18	19	20	20	21	22	23	24	25	25	26
47	3	4	6	7	8	9	10	12	13	14	15	16	17	18	19	20	21	22	22	23	24	25	26	26
48	3	4	6	7	8	10	10	12	13	14	15	16	17	18	19	20	21	22	23	23	24	25	26	27
49	3	4	6	7	8	10	10	12	13	14	15	16	17	18	19	20	21	22	23	24	24	25	26	27
50	3	4	6	7	8	10	10	12	13	14	15	16	17	18	19	20	21	22	23	24	25	25	26	27

Critical values of number of runs (r) when $n_A \neq n_B$ at percentage points 5 (lower triangle) and 95 (upper triangle) (table 2, cont.)

(n_A,n_B)	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
2																									
3																									
4																									
5																									
6																									
7																									
8																									
9	19	19	19	19	19																				
10	20	20	20	20	20	21	21	21	21	21	21	21													
11	22	22	22	22	22	22	22	22	22	22	22	22	23	23	23	23	23	23	23						
12	23	23	23	23	23	24	24	24	24	24	24	24	24	24	24	24	24	24	24	25	25	25	25	25	25
13	24	24	24	24	24	25	25	25	25	25	26	26	26	26	26	26	26	26	26	26	26	26	26	26	26
14	25	25	25	26	26	26	26	26	26	26	27	27	27	27	27	27	28	28	28	28	28	28	28	28	28
15	26	26	26	26	27	27	27	27	28	28	28	28	28	28	28	28	29	29	29	29	29	29	30	30	30
16	27	27	27	28	28	28	28	28	29	29	29	29	30	30	30	30	30	30	30	30	30	30	31	31	31
17	28	28	28	28	29	29	29	30	30	30	30	30	30	31	31	31	31	31	32	32	32	32	32	32	32
18	28	29	29	29	30	30	30	30	31	31	31	32	32	32	32	32	32	32	33	33	33	33	33	34	34
19	29	30	30	30	31	31	31	32	32	32	32	32	33	33	33	33	34	34	34	34	34	34	34	35	35
20	30	30	31	31	32	32	32	32	33	33	33	34	34	34	34	34	35	35	35	35	35	36	36	36	36
21	31	31	32	32	32	33	33	33	34	34	34	34	35	35	35	36	36	36	36	36	36	37	37	37	37
22	31	32	32	33	33	33	34	34	34	35	35	35	36	36	36	36	37	37	37	37	38	38	38	38	38
23	32	33	33	33	34	34	35	35	35	36	36	36	37	37	37	37	38	38	38	38	39	39	39	39	40
24	33	33	34	34	35	35	35	36	36	36	37	37	37	38	38	38	39	39	39	39	40	40	40	40	40
25	33	34	34	35	35	36	36	37	37	37	38	38	38	39	39	39	40	40	40	40	41	41	41	41	42
26	-	34	35	35	36	36	37	37	38	38	38	39	39	39	40	40	40	41	41	41	42	42	42	42	42
27	21	-	36	36	37	37	37	38	38	39	39	40	40	40	41	41	41	42	42	42	42	43	43	43	44
28	21	21	-	37	37	38	38	39	39	39	40	40	41	41	41	42	42	42	43	43	43	44	44	44	44
29	21	22	22	-	38	38	39	39	40	40	41	41	41	42	42	43	43	43	44	44	44	45	45	45	45
30	22	22	23	23	-	39	39	40	40	41	41	42	42	43	43	43	44	44	44	45	45	45	46	46	46
31	22	23	23	24	24	-	40	40	41	41	42	42	43	43	44	44	44	45	45	46	46	46	47	47	47
32	23	23	24	24	25	25	-	41	42	42	43	43	44	44	44	45	45	46	46	46	47	47	47	48	48
33	23	23	24	24	25	25	26	-	42	43	43	44	44	45	45	45	46	46	47	47	47	48	48	48	49
34	23	24	24	25	25	26	26	27	-	43	44	44	45	45	46	46	47	47	47	48	48	49	49	49	50
35	24	24	25	25	26	26	27	27	28	-	44	45	45	46	46	47	47	48	48	49	49	49	50	50	50
36	24	25	25	26	26	27	27	28	28	29	-	45	46	46	47	47	48	48	49	49	50	50	50	51	51
37	24	25	25	26	27	27	28	28	29	29	30	-	47	47	48	48	49	49	49	50	50	51	51	52	52
38	25	25	26	26	27	27	28	28	29	29	30	30	-	48	48	49	49	50	50	51	51	51	52	52	53
39	25	26	26	27	27	28	28	29	29	30	30	31	31	-	49	49	50	50	51	51	52	52	53	53	53
40	25	26	26	27	28	28	29	29	30	30	31	31	32	32	-	50	50	51	51	52	52	53	53	54	54
41	25	26	27	27	28	29	29	30	30	31	31	32	32	33	33	-	51	51	52	52	53	53	54	54	55
42	26	26	27	28	28	29	29	30	31	31	32	32	33	33	34	34	-	52	53	53	54	54	55	55	55
43	26	27	27	28	29	29	30	30	31	31	32	33	33	34	34	35	35	-	53	54	54	55	55	56	56
44	26	27	28	28	29	30	30	31	31	32	32	33	33	34	34	35	35	36	-	54	55	55	56	56	57
45	27	27	28	29	29	30	30	31	32	32	33	33	34	34	35	35	36	36	37	-	55	56	56	57	57
46	27	28	28	29	30	30	31	31	32	33	33	34	34	35	35	36	36	37	37	38	-	56	57	57	58
47	27	28	28	29	30	30	31	32	32	33	33	34	35	35	36	36	37	37	38	38	39	-	57	58	58
48	27	28	29	29	30	31	31	32	33	33	34	34	35	35	36	37	37	38	38	39	39	40	-	59	59
49	28	28	29	30	30	31	32	32	33	34	34	35	35	36	36	37	37	38	39	39	39	40	40	-	60
50	28	28	29	30	31	31	32	33	33	34	34	35	36	36	37	37	38	38	39	39	40	40	41	41	-

Critical values of number of runs (*r*) when $n_A \neq n_B$
at percentage points 2.5 (lower triangle) and 97.5 (upper triangle) (table 2, cont.)

(n_A, n_B)	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	-																							
3		-																						
4			-	9	9																			
5			2	-	10	11	11																	
6		2	2	3	-	12	12	13	13	13	13													
7		2	2	3	3	-	13	14	14	14	14	15	15	15										
8		2	3	3	3	4	-	14	15	15	16	16	16	16	17	17	17	17	17					
9		2	3	3	4	4	5	-	16	16	16	17	17	18	18	18	18	18	18	19	19	19	19	19
10		2	3	3	4	5	5	5	-	17	17	18	18	18	19	19	19	20	20	20	20	20	20	20
11		2	3	4	4	5	5	6	6	-	18	19	19	19	20	20	20	21	21	21	22	22	22	22
12	2	2	3	4	4	5	6	6	7	7	-	19	20	20	21	21	21	22	22	22	22	23	23	23
13	2	2	3	4	5	5	6	6	7	7	8	-	20	21	21	22	22	23	23	23	24	24	24	24
14	2	3	3	4	5	5	6	7	7	8	8	9	-	22	22	23	23	23	24	24	24	25	25	25
15	2	3	3	4	5	6	6	7	7	8	8	9	9	-	23	23	24	24	25	25	25	26	26	26
16	2	3	4	4	5	6	6	7	8	8	9	9	10	10	-	24	25	25	25	26	26	27	27	27
17	2	3	4	4	5	6	7	7	8	9	9	10	10	11	11	-	25	26	26	27	27	27	28	28
18	2	3	4	5	5	6	7	8	8	9	9	10	10	11	11	12	-	26	27	27	28	28	29	29
19	2	3	4	5	6	6	7	8	8	9	10	10	11	11	12	12	13	-	27	28	29	29	29	30
20	2	3	4	5	6	6	7	8	9	9	10	10	11	12	12	13	13	13	-	29	29	30	30	31
21	2	3	4	5	6	7	7	8	9	10	10	11	11	12	12	13	13	14	14	-	30	30	31	31
22	2	3	4	5	6	7	8	8	9	10	10	11	12	12	13	13	14	14	15	15	-	31	31	32
23	2	3	4	5	6	7	8	8	9	10	11	11	12	12	13	14	14	15	15	16	16	-	32	33
24	2	3	4	5	6	7	8	9	9	10	11	11	12	13	13	14	14	15	15	16	16	17	-	33
25	2	3	4	5	6	7	8	9	10	10	11	12	12	13	14	14	15	15	16	16	17	17	18	-
26	2	3	4	5	6	7	8	9	10	10	11	12	13	13	14	14	15	16	16	17	17	18	18	19
27	2	3	4	5	6	7	8	9	10	11	11	12	13	14	14	15	15	16	17	17	18	18	19	19
28	2	3	4	5	6	7	8	9	10	11	12	12	13	14	14	15	16	16	17	17	18	18	19	19
29	2	3	4	6	6	8	8	9	10	11	12	13	13	14	15	15	16	17	17	18	18	19	19	20
30	2	3	4	6	6	8	8	9	10	11	12	13	13	14	15	16	16	17	17	18	19	19	20	20
31	2	3	4	6	7	8	9	10	10	11	12	13	14	14	15	16	16	17	18	18	19	19	20	21
32	2	4	4	6	7	8	9	10	10	11	12	13	14	15	15	16	17	17	18	19	19	20	20	21
33	2	4	4	6	7	8	9	10	11	12	12	13	14	15	16	16	17	18	18	19	19	20	21	21
34	2	4	4	6	7	8	9	10	11	12	12	13	14	15	16	16	17	18	19	19	20	20	21	22
35	2	4	5	6	7	8	9	10	11	12	13	14	14	15	16	17	17	18	19	19	20	21	21	22
36	2	4	5	6	7	8	9	10	11	12	13	14	14	15	16	17	18	18	19	20	20	21	22	22
37	2	4	5	6	7	8	9	10	11	12	13	14	15	16	16	17	18	19	19	20	21	21	22	22
38	2	4	5	6	7	8	9	10	11	12	13	14	15	16	16	17	18	19	20	20	21	22	22	23
39	2	4	5	6	7	8	9	10	11	12	13	14	15	16	17	17	18	19	20	20	21	22	22	23
40	2	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	18	19	20	21	21	22	23	23
41	2	4	5	6	7	8	10	10	12	12	14	14	15	16	17	18	19	19	20	21	22	22	23	24
42	2	4	5	6	7	8	10	10	12	12	14	14	15	16	17	18	19	20	20	21	22	23	23	24
43	2	4	5	6	7	8	10	11	12	13	14	15	16	16	17	18	19	20	21	21	22	23	23	24
44	2	4	5	6	7	8	10	11	12	13	14	15	16	17	18	18	19	20	21	22	22	23	24	24
45	2	4	5	6	8	8	10	11	12	13	14	15	16	17	18	18	19	20	21	22	22	23	24	25
46	2	4	5	6	8	8	10	11	12	13	14	15	16	17	18	19	20	20	21	22	23	23	24	25
47	2	4	5	6	8	9	10	11	12	13	14	15	16	17	18	19	20	20	21	22	23	24	24	25
48	2	4	5	6	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	22	23	24	25	25
49	2	4	5	6	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	22	23	24	25	26
50	2	4	5	6	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	24	25	26

Critical values of number of runs (r) when $n_A \neq n_B$
at percentage points 2.5 (lower triangle) and 97.5 (upper triangle) (table 2, cont.)

(n_A,n_B)	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
2																									
3																									
4																									
5																									
6																									
7																									
8																									
9																									
10	21	21	21	21	21																				
11	22	22	22	22	22	23	23	23	23	23	23														
12	23	24	24	24	24	24	24	24	24	24	24	25	25	25	25	25	25								
13	24	25	25	25	25	26	26	26	26	26	26	26	26	26	26	26	26	27	27	27	27	27	27	27	
14	26	26	26	26	26	27	27	27	27	27	28	28	28	28	28	28	28	28	28	28	28	28	28	28	29
15	27	27	27	27	28	28	28	28	28	28	29	29	29	29	29	30	30	30	30	30	30	30	30	30	30
16	28	28	28	28	29	29	29	29	30	30	30	30	30	30	30	31	31	31	31	31	31	32	32	32	32
17	29	29	29	29	30	30	30	30	31	31	31	31	32	32	32	32	32	32	32	32	32	33	33	33	33
18	29	30	30	30	31	31	31	32	32	32	32	32	32	33	33	33	33	34	34	34	34	34	34	34	34
19	30	31	31	31	32	32	32	32	33	33	33	34	34	34	34	34	34	35	35	35	35	35	36	36	36
20	31	31	32	32	32	33	33	33	34	34	34	34	35	35	35	35	36	36	36	36	36	36	37	37	37
21	32	32	33	33	33	34	34	34	35	35	35	36	36	36	36	36	37	37	37	37	38	38	38	38	38
22	32	33	33	34	34	35	35	35	36	36	36	36	37	37	37	38	38	38	38	38	39	39	39	39	39
23	33	34	34	35	35	35	36	36	36	37	37	37	38	38	38	38	39	39	39	40	40	40	40	40	40
24	34	34	35	35	36	36	36	37	37	38	38	38	39	39	39	40	40	40	40	40	41	41	41	41	42
25	34	35	35	36	36	37	37	38	38	38	39	39	39	40	40	40	41	41	41	42	42	42	42	42	43
26	-	36	36	37	37	38	38	38	39	39	40	40	40	41	41	41	42	42	42	42	43	43	43	44	44
27	19	-	37	37	38	38	39	39	40	40	40	41	41	42	42	42	42	43	43	43	44	44	44	44	45
28	20	20	-	38	38	39	39	40	40	41	41	42	42	42	43	43	43	44	44	44	45	45	45	45	46
29	20	21	21	-	39	39	40	40	41	41	42	42	43	43	43	44	44	44	45	45	46	46	46	46	47
30	21	21	22	22	-	40	41	41	42	42	43	43	43	44	44	45	45	45	46	46	46	47	47	47	48
31	21	22	22	22	23	-	41	42	42	43	43	44	44	45	45	45	46	46	46	47	47	48	48	48	48
32	21	22	22	23	23	24	-	42	43	43	44	44	45	45	46	46	46	47	47	48	48	48	49	49	49
33	22	22	23	23	24	24	25	-	43	44	44	45	45	46	46	47	47	48	48	48	49	49	50	50	50
34	22	23	23	24	24	25	25	26	-	45	45	46	46	47	47	47	48	48	49	49	50	50	50	51	51
35	22	23	24	24	25	25	26	26	26	-	46	46	47	47	48	48	49	49	49	50	50	51	51	51	52
36	23	23	24	24	25	25	26	26	27	27	-	47	47	48	48	49	49	50	50	51	51	51	52	52	53
37	23	24	24	25	25	26	26	27	27	28	28	-	48	48	49	49	50	50	51	51	52	52	53	53	53
38	23	24	25	25	26	26	27	27	28	28	29	29	-	49	50	50	51	51	52	52	52	53	53	54	54
39	24	24	25	25	26	27	27	28	28	29	29	30	30	-	50	51	51	52	52	53	53	54	54	54	55
40	24	25	25	26	26	27	27	28	28	29	29	30	30	31	-	51	52	52	53	53	54	54	55	55	56
41	24	25	25	26	27	27	28	28	29	29	30	30	31	31	32	-	52	53	53	54	54	55	55	56	56
42	25	25	26	26	27	28	28	29	29	30	30	31	31	32	32	33	-	53	54	55	55	55	56	56	57
43	25	25	26	27	27	28	28	29	30	30	31	31	32	32	33	33	34	-	55	55	56	56	57	57	58
44	25	26	26	27	28	28	29	29	30	30	31	32	32	33	33	33	34	34	-	56	56	57	57	58	58
45	25	26	27	27	28	29	29	30	30	31	31	32	32	33	33	34	34	35	35	-	57	57	58	58	59
46	26	26	27	28	28	29	29	30	31	31	32	32	33	33	34	34	35	35	36	36	-	58	58	59	59
47	26	26	27	28	28	29	30	30	31	32	32	33	33	34	34	35	35	36	36	37	37	-	59	59	60
48	26	27	27	28	29	29	30	31	31	32	32	33	34	34	35	35	36	36	37	37	38	38	-	60	61
49	26	27	28	28	29	30	30	31	32	32	33	33	34	34	35	35	36	37	37	37	38	38	39	-	61
50	26	27	28	29	29	30	31	31	32	32	33	34	34	35	35	36	36	37	37	38	38	39	39	40	-

Critical values of number of runs (r) when $n_A \neq n_B$ at percentage points 1 (lower triangle) and 99 (upper triangle) (table 2, cont.)

(n_A, n_B)	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	-																							
3		-																						
4			-	9																				
5				-	11	11																		
6			2	2	-	12	13	13																
7			2	2	3	-	14	14	15	15	15													
8			2	2	3	3	-	15	15	16	16	17	17	17	17									
9		2	2	3	3	4	4	-	16	17	17	18	18	18	18	19	19	19	19					
10		2	2	3	3	4	4	5	-	18	18	19	19	19	20	20	20	20	20	21	21	21	21	
11		2	2	3	4	4	5	5	5	-	19	19	20	20	21	21	21	22	22	22	22	22	22	23
12		2	3	3	4	4	5	5	6	6	-	20	21	21	22	22	22	23	23	23	23	24	24	24
13		2	3	3	4	5	5	6	6	6	7	-	21	22	22	23	23	24	24	24	24	25	25	25
14		2	3	3	4	5	5	6	6	7	7	8	-	23	23	24	24	24	25	25	26	26	26	26
15		2	3	4	4	5	5	6	7	7	8	8	8	-	24	24	25	25	26	26	26	27	27	27
16		2	3	4	4	5	6	6	7	7	8	8	9	9	-	25	26	26	26	27	27	28	28	28
17		2	3	4	5	5	6	7	7	8	8	9	9	10	10	-	26	27	27	28	28	29	29	29
18		2	3	4	5	5	6	7	7	8	8	9	9	10	10	11	-	27	28	28	29	29	30	30
19	2	2	3	4	5	6	6	7	8	8	9	9	10	10	11	11	12	-	29	29	30	30	31	31
20	2	2	3	4	5	6	6	7	8	8	9	10	10	11	11	11	12	12	-	30	30	31	31	32
21	2	2	3	4	5	6	7	7	8	9	9	10	10	11	11	12	12	13	13	-	31	31	32	32
22	2	2	3	4	5	6	7	7	8	9	9	10	11	11	12	12	13	13	14	14	-	32	33	33
23	2	2	4	4	5	6	7	8	8	9	10	10	11	11	12	13	13	13	14	14	15	-	33	34
24	2	3	4	4	5	6	7	8	8	9	10	10	11	12	12	13	13	14	14	15	15	16	-	34
25	2	3	4	4	5	6	7	8	9	9	10	11	11	12	13	13	14	14	15	15	16	16	17	-
26	2	3	4	5	6	6	7	8	9	10	10	11	12	12	13	13	14	14	15	15	16	16	17	17
27	2	3	4	5	6	6	7	8	9	10	10	11	12	12	13	14	14	15	15	16	16	17	17	18
28	2	3	4	5	6	7	8	8	9	10	11	11	12	13	13	14	14	15	16	16	17	17	18	18
29	2	3	4	5	6	7	8	8	9	10	11	12	12	13	14	14	15	15	16	16	17	17	18	18
30	2	3	4	5	6	7	8	8	9	10	11	12	12	13	14	14	15	16	16	17	17	18	18	19
31	2	3	4	5	6	7	8	9	10	10	11	12	13	13	14	15	15	16	16	17	18	18	19	19
32	2	3	4	5	6	7	8	9	10	10	11	12	13	14	14	15	16	16	17	17	18	18	19	20
33	2	3	4	5	6	7	8	9	10	11	11	12	13	14	14	15	16	16	17	18	18	19	19	20
34	2	3	4	5	6	7	8	9	10	11	12	12	13	14	15	15	16	17	17	18	18	19	20	20
35	2	3	4	5	6	7	8	9	10	11	12	12	13	14	15	16	16	17	18	18	19	19	20	21
36	2	3	4	5	6	7	8	9	10	11	12	13	13	14	15	16	16	17	18	18	19	20	20	21
37	2	3	4	5	6	7	8	9	10	11	12	13	14	14	15	16	17	17	18	19	19	20	21	21
38	2	3	4	5	6	7	8	9	10	11	12	13	14	15	15	16	17	18	18	19	20	20	21	21
39	2	3	4	5	6	8	8	10	10	11	12	13	14	15	16	16	17	18	18	19	20	20	21	22
40	2	3	4	5	6	8	8	10	10	11	12	13	14	15	16	16	17	18	19	19	20	21	21	22
41	2	3	4	6	6	8	8	10	10	12	12	13	14	15	16	17	17	18	19	20	20	21	22	22
42	2	3	4	6	6	8	9	10	11	12	12	14	14	15	16	17	18	18	19	20	20	21	22	22
43	2	3	4	6	6	8	9	10	11	12	13	14	14	15	16	17	18	18	19	20	21	21	22	23
44	2	3	4	6	7	8	9	10	11	12	13	14	15	16	16	17	18	19	19	20	21	22	22	23
45	2	3	4	6	7	8	9	10	11	12	13	14	15	16	16	17	18	19	20	20	21	22	23	23
46	2	3	4	6	7	8	9	10	11	12	13	14	15	16	17	17	18	19	20	21	21	22	23	23
47	2	3	4	6	7	8	9	10	11	12	13	14	15	16	17	18	18	19	20	21	22	22	23	24
48	2	3	4	6	7	8	9	10	11	12	13	14	15	16	17	18	19	19	20	21	22	22	23	24
49	2	3	4	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	20	21	22	23	23	24
50	2	3	4	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	20	21	22	23	24	24

Critical values of number of runs (r) when $n_A \neq n_B$ at percentage points 1 (lower triangle) and 99 (upper triangle) (table 2, cont.)

(n_A,n_B)	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
2																									
3																									
4																									
5																									
6																									
7																									
8																									
9																									
10																									
11	23	23	23	23																					
12	24	24	24	24	25	25	25	25	25																
13	26	26	26	26	26	26	26	26	26	27	27	27	27	27	27										
14	26	27	27	27	27	28	28	28	28	28	28	28	28	28	28	29	29	29	29	29	29				
15	28	28	28	28	28	29	29	29	29	30	30	30	30	30	30	30	30	30	30	30	30	31	31	31	31
16	29	29	29	30	30	30	30	30	30	31	31	31	31	31	32	32	32	32	32	32	32	32	32	32	32
17	30	30	30	30	31	31	31	32	32	32	32	32	32	32	33	33	33	33	33	34	34	34	34	34	34
18	30	31	31	32	32	32	32	32	33	33	33	34	34	34	34	34	34	34	34	35	35	35	35	35	36
19	31	32	32	32	33	33	33	34	34	34	34	34	35	35	35	35	36	36	36	36	36	36	36	36	37
20	32	33	33	33	34	34	34	35	35	35	35	36	36	36	36	36	37	37	37	37	38	38	38	38	38
21	33	33	34	34	35	35	35	36	36	36	36	37	37	37	38	38	38	38	38	38	39	39	39	39	39
22	34	34	35	35	35	36	36	36	37	37	37	38	38	38	38	39	39	39	39	40	40	40	40	40	40
23	34	35	35	36	36	37	37	37	38	38	38	39	39	39	40	40	40	40	40	41	41	41	41	42	42
24	35	36	36	36	37	37	38	38	39	39	39	40	40	40	40	41	41	41	42	42	42	42	42	43	43
25	36	36	37	37	38	38	39	39	39	40	40	40	41	41	41	42	42	42	42	43	43	43	44	44	44
26	-	37	37	38	38	39	39	40	40	41	41	41	42	42	42	43	43	43	44	44	44	44	44	45	45
27	18	-	38	39	39	40	40	40	41	41	42	42	42	43	43	44	44	44	44	45	45	45	46	46	46
28	19	19	-	39	40	40	41	41	42	42	42	43	43	44	44	44	45	45	45	46	46	46	47	47	47
29	19	19	20	-	40	41	41	42	42	43	43	44	44	44	45	45	46	46	46	47	47	47	48	48	48
30	19	20	20	21	-	41	42	43	43	43	44	44	45	45	46	46	46	47	47	48	48	48	48	49	49
31	20	20	21	21	22	-	43	43	44	44	45	45	46	46	46	47	47	48	48	48	49	49	49	50	50
32	20	21	21	21	22	22	-	44	44	45	45	46	46	47	47	48	48	48	49	49	50	50	50	50	51
33	20	21	21	22	22	23	23	-	45	45	46	46	47	47	48	48	49	49	50	50	50	51	51	51	52
34	21	21	22	22	23	23	24	24	-	46	47	47	48	48	49	49	49	50	50	51	51	52	52	52	53
35	21	22	22	23	23	24	24	25	25	-	47	48	48	49	49	50	50	51	51	51	52	52	53	53	53
36	21	22	22	23	23	24	24	25	25	26	-	48	49	49	50	50	51	51	52	52	53	53	53	54	54
37	22	22	23	23	24	24	25	25	26	26	27	-	49	50	51	51	52	52	52	53	53	54	54	55	55
38	22	23	23	24	24	25	25	26	26	27	27	28	-	51	51	52	52	53	53	54	54	54	55	55	56
39	22	23	23	24	25	25	26	26	27	27	27	28	28	-	52	52	53	53	54	54	55	55	56	56	56
40	23	23	24	24	25	25	26	26	27	27	28	28	29	29	-	53	53	54	54	55	55	56	56	57	57
41	23	23	24	25	25	26	26	27	27	28	28	29	29	30	30	-	54	55	55	56	56	57	57	57	58
42	23	24	24	25	25	26	27	27	28	28	29	29	30	30	31	31	-	55	56	56	57	57	58	58	59
43	23	24	25	25	26	26	27	27	28	29	29	30	30	31	31	31	32	-	56	57	57	58	58	59	59
44	24	24	25	26	26	27	27	28	28	29	29	30	30	31	31	32	32	33	-	57	58	58	59	59	60
45	24	25	25	26	26	27	28	28	29	29	30	30	31	31	32	32	33	33	34	-	58	59	60	60	61
46	24	25	25	26	27	27	28	28	29	30	30	31	31	32	32	33	33	34	34	35	-	60	60	61	61
47	24	25	26	26	27	28	28	29	29	30	30	31	32	32	33	33	34	34	34	35	35	-	61	61	62
48	25	25	26	27	27	28	28	29	30	30	31	31	32	32	33	33	34	34	35	35	36	36	-	62	62
49	25	26	26	27	28	28	29	29	30	31	31	32	32	33	33	34	34	35	35	36	36	37	37	-	63
50	25	26	26	27	28	28	29	30	30	31	31	32	33	33	34	34	35	35	36	36	37	37	38	38	-

Critical values of number of runs (r) when $n_A \neq n_B$ at percentage points 0.5 (lower triangle) and 99.5 (upper triangle) (table 2, cont.)

(n_A, n_B)	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
2	-																							
3		-																						
4			-																					
5				-	11																			
6				2	-	13	13																	
7				2	2	-	14	15	15	15														
8			2	2	3	3	-	15	16	16	17	17	17											
9			2	2	3	3	3	-	17	17	18	18	18	19	19	19								
10			2	3	3	3	4	4	-	18	19	19	19	20	20	20	21	21	21	21				
11			2	3	3	4	4	5	5	-	19	20	20	21	21	22	22	22	22	22	23	23	23	23
12		2	2	3	3	4	4	5	5	6	-	21	21	22	22	22	23	23	23	24	24	24	24	24
13		2	2	3	3	4	5	5	5	6	6	-	22	22	23	23	24	24	24	25	25	25	26	26
14		2	2	3	4	4	5	5	6	6	7	7	-	23	24	24	25	25	25	26	26	26	27	27
15		2	3	3	4	4	5	6	6	7	7	7	8	-	24	25	25	26	26	27	27	27	28	28
16		2	3	3	4	5	5	6	6	7	7	8	8	9	-	26	26	27	27	28	28	28	29	29
17		2	3	3	4	5	5	6	7	7	8	8	8	9	9	-	27	27	28	28	29	29	30	30
18		2	3	4	4	5	6	6	7	7	8	8	9	9	10	10	-	28	29	29	30	30	30	31
19		2	3	4	4	5	6	6	7	8	8	9	9	10	10	10	11	-	29	30	30	31	31	32
20		2	3	4	4	5	6	7	7	8	8	9	9	10	10	11	11	12	-	31	31	32	32	33
21		2	3	4	5	5	6	7	7	8	9	9	10	10	11	11	12	12	12	-	32	32	33	33
22		2	3	4	5	5	6	7	8	8	9	9	10	10	11	11	12	12	13	13	-	33	33	34
23		2	3	4	5	6	6	7	8	8	9	10	10	11	11	12	12	13	13	14	14	-	34	35
24		2	3	4	5	6	6	7	8	9	9	10	10	11	12	12	13	13	14	14	14	15	-	35
25		2	3	4	5	6	7	7	8	9	9	10	11	11	12	12	13	13	14	14	15	15	16	-
26		2	3	4	5	6	7	7	8	9	10	10	11	11	12	13	13	14	14	15	15	16	16	16
27	2	2	3	4	5	6	7	8	8	9	10	10	11	12	12	13	13	14	15	15	15	16	16	17
28	2	2	3	4	5	6	7	8	8	9	10	11	11	12	13	13	14	14	15	15	16	16	17	17
29	2	2	4	4	5	6	7	8	9	9	10	11	12	12	13	13	14	15	15	16	16	17	17	18
30	2	2	4	4	5	6	7	8	9	10	10	11	12	12	13	14	14	15	15	16	16	17	17	18
31	2	2	4	4	6	6	7	8	9	10	10	11	12	13	13	14	14	15	16	16	17	17	18	18
32	2	2	4	4	6	6	7	8	9	10	11	11	12	13	13	14	15	15	16	16	17	18	18	19
33	2	2	4	5	6	6	7	8	9	10	11	12	12	13	14	14	15	16	16	17	17	18	18	19
34	2	3	4	5	6	7	8	8	9	10	11	12	12	13	14	14	15	16	16	17	18	18	19	19
35	2	3	4	5	6	7	8	8	9	10	11	12	13	13	14	15	15	16	17	17	18	18	19	20
36	2	3	4	5	6	7	8	9	10	10	11	12	13	14	14	15	16	16	17	18	18	19	19	20
37	2	3	4	5	6	7	8	9	10	10	11	12	13	14	14	15	16	16	17	18	18	19	20	20
38	2	3	4	5	6	7	8	9	10	10	11	12	13	14	15	15	16	17	17	18	19	19	20	20
39	2	3	4	5	6	7	8	9	10	11	12	12	13	14	15	16	16	17	18	18	19	20	20	21
40	2	3	4	5	6	7	8	9	10	11	12	12	13	14	15	16	16	17	18	18	19	20	20	21
41	2	3	4	5	6	7	8	9	10	11	12	13	14	14	15	16	17	17	18	19	19	20	21	21
42	2	3	4	5	6	7	8	9	10	11	12	13	14	14	15	16	17	18	18	19	20	20	21	22
43	2	3	4	5	6	7	8	9	10	11	12	13	14	15	15	16	17	18	18	19	20	20	21	22
44	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	16	17	18	19	19	20	21	21	22
45	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	16	17	18	19	20	20	21	22	22
46	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	17	18	19	20	20	21	22	22
47	2	3	4	5	6	7	8	10	10	12	12	13	14	15	16	17	18	18	19	20	21	21	22	23
48	2	3	4	5	6	8	8	10	10	12	12	13	14	15	16	17	18	18	19	20	21	22	22	23
49	2	3	4	5	6	8	8	10	11	12	12	14	14	15	16	17	18	19	20	20	21	22	22	23
50	2	3	4	5	6	8	8	10	11	12	13	14	14	16	16	17	18	19	20	20	21	22	23	23

Critical values of number of runs (r) when $n_A \neq n_B$
at percentage points 0.5 (lower triangle) and 99.5 (upper triangle) (table 2, cont.)

(n_A,n_B)	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50
2																									
3																									
4																									
5																									
6																									
7																									
8																									
9																									
10																									
11																									
12	25	25	25	25	25																				
13	26	26	26	26	26	27	27	27	27	27															
14	27	27	28	28	28	28	28	28	28	28	29	29	29	29	29										
15	28	28	29	29	29	30	30	30	30	30	30	30	30	30	30	31	31	31	31	31	31				
16	29	30	30	30	30	30	31	31	31	31	32	32	32	32	32	32	32	32	32	32	32	33	33	33	33
17	30	31	31	31	32	32	32	32	32	32	33	33	33	33	34	34	34	34	34	34	34	34	34	34	34
18	31	32	32	32	32	33	33	33	34	34	34	34	34	34	35	35	35	35	35	36	36	36	36	36	36
19	32	32	33	33	34	34	34	34	35	35	35	35	36	36	36	36	36	36	36	37	37	37	37	37	38
20	33	33	34	34	34	35	35	35	36	36	36	36	37	37	37	37	38	38	38	38	38	38	38	38	39
21	34	34	35	35	35	36	36	36	37	37	37	38	38	38	38	38	39	39	39	39	40	40	40	40	40
22	34	35	35	36	36	37	37	37	38	38	38	38	39	39	39	40	40	40	40	40	40	41	41	41	41
23	35	36	36	37	37	37	38	38	39	39	39	40	40	40	40	41	41	41	41	42	42	42	42	42	42
24	36	36	37	37	38	38	39	39	39	40	40	40	41	41	41	42	42	42	42	43	43	43	43	44	44
25	37	37	38	38	39	39	39	40	40	41	41	41	42	42	42	43	43	43	43	44	44	44	44	45	45
26	-	38	38	39	39	40	40	41	41	41	42	42	43	43	43	44	44	44	44	45	45	45	46	46	46
27	17	-	39	39	40	40	41	41	42	42	43	43	43	44	44	44	45	45	45	46	46	46	46	47	47
28	18	18	-	40	41	41	42	42	43	43	43	44	44	45	45	45	46	46	46	47	47	47	48	48	48
29	18	19	19	-	41	42	42	43	43	44	44	45	45	45	46	46	47	47	47	48	48	48	48	49	49
30	18	19	19	20	-	42	43	43	44	44	45	45	46	46	47	47	47	48	48	48	49	49	49	50	50
31	19	19	20	20	21	-	44	44	45	45	46	46	47	47	47	48	48	49	49	49	50	50	50	51	51
32	19	20	20	21	21	21	-	45	45	46	46	47	47	48	48	49	49	49	50	50	51	51	51	52	52
33	19	20	20	21	21	22	22	-	46	46	47	47	48	48	49	49	50	50	51	51	51	52	52	52	53
34	20	20	21	21	22	22	23	23	-	47	48	48	49	49	50	50	51	51	51	52	52	53	53	53	54
35	20	21	21	22	22	23	23	24	24	-	48	49	49	50	50	51	51	52	52	53	53	53	54	54	54
36	20	21	21	22	23	23	23	24	24	25	-	49	50	50	51	51	52	52	53	53	54	54	55	55	55
37	21	21	22	22	23	23	24	24	25	25	26	-	51	51	52	52	53	53	54	54	54	55	55	56	56
38	21	22	22	23	23	24	24	25	25	26	26	26	-	52	52	53	53	54	54	55	55	56	56	56	57
39	21	22	22	23	24	24	25	25	26	26	26	27	27	-	53	53	54	54	55	55	56	56	57	57	58
40	22	22	23	23	24	24	25	25	26	26	27	27	28	28	-	54	54	55	56	56	57	57	57	58	58
41	22	22	23	24	24	25	25	26	26	27	27	28	28	29	29	-	55	56	56	57	57	58	58	59	59
42	22	23	23	24	24	25	26	26	27	27	28	28	29	29	29	30	-	56	57	57	58	58	59	59	60
43	22	23	24	24	25	25	26	26	27	27	28	28	29	29	30	30	31	-	57	58	58	59	59	60	60
44	23	23	24	25	25	26	26	27	27	28	28	29	29	30	30	31	31	32	-	59	59	60	60	61	61
45	23	24	24	25	25	26	27	27	28	28	29	29	30	30	31	31	32	32	32	-	60	60	61	61	62
46	23	24	24	25	26	26	27	27	28	28	29	30	30	31	31	32	32	32	33	33	-	61	61	62	62
47	23	24	25	25	26	27	27	28	28	29	29	30	30	31	31	32	32	33	33	34	34	-	62	62	63
48	24	24	25	26	26	27	27	28	29	29	30	30	31	31	32	32	33	33	34	34	35	35	-	63	64
49	24	25	25	26	26	27	28	28	29	29	30	31	31	32	32	33	33	34	34	35	35	35	36	-	64
50	24	25	25	26	27	27	28	29	29	30	30	31	31	32	32	33	34	34	35	35	35	36	36	37	-

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Reading off the tables

Tables 1 and 2 allow to test the significance of r, the total number of runs in a series of n observations, n_A being of one type and n_B of the other type, at the significance levels $\alpha = 0.05$ and 0.01, in one-tailed or two-tailed mode. The r statistic, for a binary-coded symbolic series, indicates the number of uninterrupted runs of the same symbol, or equivalently, the number of symbol changes or alternations, plus one. Note that, for $n_A \geq 1$ and $n_B \geq 1$, we have $2 \leq r \leq 2\times\min(n_A,n_B)+1$. A low value of r indicates some *over-grouping* of similar elements whereas a high value of r indicates *over-alternation.*

Table 1, for cases where $n_A = n_B$, and table 2, for $n_A \neq n_B$, furnish values r^* such that $\Pr(r \leq r^*_{[\alpha]}) \leq \alpha$ or $\Pr(r \leq r^*_{[\alpha/2]}) \leq \alpha/2$ for testing over-grouping in the series (relatively to small values of r), and $\Pr(r \geq r^*_{[1-\alpha]}) \geq 1-\alpha$ or $\Pr(r \geq r^*_{[1-\alpha/2]}) \geq 1-\alpha/2$ for testing over-alternation (relatively to large values of r). To keep the tables in compact form, table 2 was arranged by combining the lower and higher quasi centiles r^* respectively in the lower and upper triangles of a square array: to find a given r^*, the user may have to swap arguments n_A and n_B.

Illustration 1. What is the critical value of r at the 5 % significance level for a series comprising $n_A = 7$ and $n_B = 7$? Table 1 furnishes the desired value, at the line designated $n_A = n_B = 7$. For a one-tailed test of over-grouping, we get $r^* = 4$; for over-alternation, it would be $r^* = 12$. For a two-tailed test (having the same 5 % global significance level), the critical values would be 3 and 13.

Illustration 2. Suppose a sequence of $n = 18$ observations in their order of arrival, of which 11 are of one type and 7 of another. Find the critical values for a bilateral (two-tailed) test at the 5 % significance level. In table 2, we first locate the pages for percentiles $P = .025$ and $P = .975$. For $r^*_{[.025]}$, at column 7 and line 11, we have 5; then, at line 7 and column 11, we read 14 for $r^*_{[.975]}$. Thus, any value $r \leq 5$ or $r \geq 14$ can occur randomly at most in 5 % of the cases.

Full examples

Example 1. At a bus stop, we observed the following line of 13 persons, coded according to gender: { ♂ ♀♀♀ ♂♂ ♀♀♀♀ ♂ ♀♀ }. May we think that these persons arrived there in random order (using a 5 % probability threshold)? *Solution*: There are $n_A = 4$ men and

$n_B = 9$ women, and $r = 6$ runs (elements in the same run were grouped together). In table 2, at percentiles 0.025 and 0.975, we read $r^* = 3$ in column 4, line 9, whereas in line 4, column 9, we read... nothing! This last observation conveys that, with so few elements as 4 and 9, over-alternation can never attain a probability as fine as 0.025 (= .05/2); nevertheless, a number of runs r equal to or less than 3 would indicate an over-grouping of elements. In our example, the observed data do not belie the null hypothesis, according to which people arrived there haphazardly.

Example 2. In a regression analysis of general form $\hat{Y} = f(X)$, done on a sample of 20 measured points, the user wishes to ascertain the pertinency of the chosen model $f(X)$ by examining the scatter of residuals $(Y_i - \hat{Y}_i)$ around the regression line. The computed residuals are, in increasing order of the X_i:

{+.06 +.09 −.09 +.05 +.11 −.08 +.14 −.08 −.05 −.09 +.06 +.01 −.07 −.00 +.05 −.08 +.02 +.06 −.06 −.05}

On the basis of the above residuals, does the chosen model seem adequate (using a 1 % significance level)? *Solution*: The number-of-runs test is but one way to tackle the complex undertaking of statistical modelling. Here, the question would be to see if the model fits tightly the data, as opposed to a possible over-grouping of residuals. We count $n_A = n_B = 10$ residuals of each sign (*i.e.* under and above the regression line). At the one-tailed 1 % significance level, table 1 proposes $r^*_{[.01]} = 5$, while the number of runs observed is $r = 12$. The adopted model thus seems adequate, at least with regard to the scatter of residuals.

Mathematical presentation

The "number-of-runs" distribution is somewhat out of place in the body of distributions gathered in the present book: it pertains to a *discrete* rather than a continuous variable (as is the case of the binomial distribution), and it can be appplied to check the irregular, haphazard aspect of a series of observations. Consider a statistical series $\{ x_i, 1 \leq i \leq n \}$, composed of n binary (or dichotomous[1]) observations, say x_i = A or x_i = B: we count n_A and n_B ($n = n_A + n_B$) observations of each type. The series contain r_A runs, or uninterrupted sequences, of "A"'s, r_B runs of "B"'s, and $r = r_A + r_B$ is the total number of runs. When systematic factors intervene in the production process or in the occurrence of observations, the number r will tend to be too high (in an over-alternated series) or too low (in an over-grouped series).

[1] The observed data may be naturally in binary form (like male/female, left/right, yes/no) or they may have been purposely dichotomized by categorization (for instance, $x_i' = 0$ if $x_i \leq 0$ and $x_i' = 1$ if $x_i > 0$).

The test of the total number of runs, which compares the observed number r with critical value r^*, using parameters n_A, n_B and α (the significance level), may be unilateral (one-tailed) "to the left" ($r \leq r^*_{[\alpha]}$), unilateral "to the right" ($r \geq r^*_{[1-\alpha]}$), or bilateral (two-tailed) ($r \leq r^*_{[\alpha/2]}$ or $r \geq r^*_{[1-\alpha/2]}$). A noteworthy application of this test, in its unilateral "to the left" version, is to check the possibility of over-grouping of residuals in a complex (polynomial or non-linear) regression analysis.

Calculation and moments

The distribution of $r = r_A + r_B$, for $n_A \geq 1$, $n_B \geq 1$, $n = n_A + n_B$, has for domain $r = 2$ up to n if $n_A = n_B$ or to $2\min(n_A,n_B)+1$ if $n_A \neq n_B$. The distribution of probabilities alternates between even and odd values of r, as:

$$p(r=2k) = \frac{2\binom{n_A-1}{k-1}\binom{n_B-1}{k-1}}{\binom{n}{n_A}} \quad ; \quad p(r=2k+1) = \frac{\binom{n_A-1}{k-1}\binom{n_B-1}{k} + \binom{n_A-1}{k}\binom{n_B-1}{k-1}}{\binom{n}{n_A}} \; .$$

In these expressions, $\binom{a}{b}$ designates combinations; recall that $\binom{a}{b} = 0$ when $a < b$. Approximation to this distribution is discussed later.

Moments. The expectation, variance and indices of skewness and kurtosis of r, the total number of runs, depend solely on n_A and n_B, and they may be obtained from the factorial moments of r_A and r_B (Laurencelle 1995). They are:

$$\mu = E(r) = \frac{n}{2} + 1 \qquad \text{if } n_A = n_B = n/2 \; ,$$

$$= \frac{2n_An_B}{n} + 1 \qquad \text{otherwise;}$$

$$\sigma^2 = \mu_2 = \frac{n(n-2)}{4(n-1)} \qquad \text{if } n_A = n_B = n/2 \; ,$$

$$= \frac{2n_An_B(2n_An_B - n)}{n^2(n-1)} \qquad \text{otherwise;}$$

$$\gamma_1 = 0 \qquad \text{if } n_A = n_B = n/2 \,,$$

$$= \sqrt{\frac{n-1}{2(n-2)^2}} \times \frac{16n_A^2 n_B^2 - 4n_A n_B n(n+3) + 3n^3}{\sqrt{n_A n_B (2n_A n_B - n)^3}} \qquad \text{otherwise;}$$

$$\gamma_2 = \frac{(n-1)(3n^2-14n+12)}{n(n-2)(n-3)} - 3 \qquad \text{if } n_A = n_B = n/2 \,,$$

$$= \frac{(n-1)\left[n^4(8n_A n_B - 13) - 7n^5 + 2n^3(52n_A n_B + 3) - 120n^2 n_A^2 n_B^2 + 24nn_A^2 n_B^2(n_A n_B - 6) + 144n_A^3 n_B^3\right]}{2n_A n_B (2n_A n_B - n)^2 (n-2)(n-3)} - 3$$

otherwise.

Approximations. For the cases of equal samples sizes, $n_A = n_B$, the distribution of r is symmetrical, and the normal approximation (with a continuity correction) is very effective. In fact, $P(r) \approx \Phi(x)$, where Φ is the normal d.f. and $x = (r \pm ½ - \mu)/\sigma$. For the unequal sizes cases, the skewness index (γ_1) is proportional to $-(n_A - n_B)^2$, so that the distribution is strongly and negatively skewed when $n_A \neq n_B$, a fact that makes the normal approximation inadvisable. The laborious user may resort to the improved approximation below, based upon the same standardized x variable, and in which the computed γ_1 and γ_2 indices are incorporated:

$$P(r) \approx \Phi(x) - \varphi(x)\left[\frac{\gamma_1}{6}(x^2-1) + \frac{\gamma_2}{24}(x^3-3x) + \frac{\gamma_1^2}{72}(x^5-10x^3+15x)\right] .$$

This approximation formula, wherein Φ and φ denote respectively the normal d.f. and p.d.f. (*see* section on normal distribution), should do well for cases where $n_A > 30$ and $n_B > 30$.

Random numbers

- ✓ Table of 10000 random 0-9 digits
- ✓ Reading off the table
- ✓ Full examples
- ✓ Mathematical presentation
  - Generation of pseudo random variates
  - Generation of random samples

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Table of 10000 random 0-9 digits

97327	06015	78975	52016	30653	58926	46428	64581	18051	92654
79837	13931	49611	79736	27084	32281	67103	55444	79089	83924
36021	34149	57951	80741	99731	83446	30395	38191	33489	72363
45208	35102	85832	76460	42167	09026	06728	68010	28046	40262
97655	16843	68385	81445	88193	21349	70011	15227	62282	29853
16677	82612	58065	86901	63275	77118	39662	12203	29156	36880
90104	43577	21026	38295	22162	07788	34137	57223	05450	87913
83204	26764	45814	37348	80758	48664	06123	68839	28088	29119
10986	27363	07560	61695	29869	42842	27060	67718	97505	36180
52195	36158	87156	39575	25268	28202	94000	25336	42109	63115
77277	85455	54873	08102	69521	49656	36562	20026	35249	50049
15865	77111	66391	64351	79253	56873	68983	51894	85186	43997
58631	78443	08527	66405	56052	29207	98495	89406	68759	45502
68590	55592	25627	21460	05604	57569	05978	78133	24831	57512
96088	77989	04013	42378	16517	65105	12962	07835	19053	95878
81224	07617	80854	36660	96687	84639	27351	76496	96597	04941
00101	82871	28613	71972	09257	87877	81376	95180	69967	40177
44071	97630	20008	80292	50511	16853	39418	47380	25911	50201
93528	34617	07223	34222	69147	33135	24760	38340	12600	38154
73406	74862	17440	52629	93685	69154	22595	13600	45841	97119
19663	69144	14307	76098	40747	59099	06019	93991	90207	13284
27527	44678	31024	27241	69955	24778	72601	95227	03093	37156
64336	87231	92600	03433	55457	79414	02701	66839	15190	89598
31887	45720	31987	41521	23044	32308	86623	35122	17489	65755
85867	90172	96904	27847	65001	45692	41989	72318	94421	28512
10989	46658	93189	92649	98160	64707	12337	50443	52416	20524
10488	65765	14156	41947	91103	01793	71498	24246	61304	07591
46590	15380	81380	83632	53468	15426	16096	59592	86286	50115
93498	57286	86654	95643	40494	24210	25647	49919	63172	29280
25915	60929	48009	22931	37529	89944	86013	65127	77784	70442
25317	26059	39903	36940	66760	27040	06334	68114	16636	19962
48426	60405	66707	77806	09415	22741	29712	33986	57001	51233
95864	81114	48424	32633	31418	19058	45754	50974	82710	37101
39327	26586	81290	63628	48688	10789	05088	17782	44018	22420
76280	05735	72440	06250	88348	23868	78685	83531	70778	99687
37912	63702	64597	53501	01259	86554	10235	08038	01723	39183
57297	34111	47523	22287	89492	87228	28295	51750	74248	79225
77624	26145	49564	77930	22947	61816	16306	02429	02571	17731
24821	64226	01141	79682	68253	59947	13723	47003	06908	79812
60086	28624	86637	80512	10417	14133	42559	59682	24516	32309
72533	18856	02588	07885	46029	96391	52868	17236	48674	47285
91317	42042	94590	26645	31568	79126	63835	95509	62154	65868
33083	72212	66603	29562	66932	81793	47495	21953	81000	49840
18489	54409	53836	60940	31916	14241	97109	42848	05873	67664
59343	45514	43399	25962	20530	27864	63912	59287	55526	06064
79655	78758	40009	77269	83154	06039	54176	62238	46891	00227
48583	08725	25695	91483	15248	29066	60112	84032	35535	66303
47643	03365	16517	20781	67952	85300	25282	46635	59996	00211
11329	77697	31643	12272	50694	44895	55591	61401	37361	29216
18910	90874	44554	16148	20905	46317	34644	11519	00413	54757

Table of 10000 random 0-9 digits (cont.)

92673	47808	10149	10382	15268	09283	53302	88656	30171	35050
55448	15251	07216	34896	58332	36019	84105	44732	46382	07612
80470	06402	50989	37183	21633	33140	66169	12244	77246	51746
60132	84004	89052	36184	79139	99889	14480	96587	10744	44727
87856	73760	15249	76471	11280	01917	30148	77899	73280	37900
90609	07925	91605	99983	63113	50062	00177	39454	87068	70418
10916	55163	28567	99793	57218	04252	89375	33001	18165	26069
56513	19124	96966	20997	91049	88025	18435	95467	85197	64210
50468	94332	61922	43594	25840	23336	55816	17121	13814	75190
67240	25004	50858	48127	04841	29323	18791	10966	72547	15002
75316	96006	71015	32545	00847	48373	49730	22058	95843	21792
92883	55971	36285	85860	93280	49464	61288	14027	25315	35060
71191	22091	21798	59235	44965	15014	07480	32583	13137	45215
24595	48035	80296	20985	73354	50500	31137	15929	13050	81613
39416	03299	80342	86682	62180	70413	60175	68099	83611	67352
83814	72386	27557	49087	43261	45185	88328	01422	25490	44002
55085	50051	68009	95155	21727	24271	53977	91887	09375	84686
65781	69065	97228	58244	38989	96821	60700	19806	06547	21243
79221	08789	26749	57899	81638	06081	66520	05999	48912	13506
68396	04469	85773	04107	68931	69416	54973	22352	38968	13116
67908	23277	42592	48457	64481	80314	76010	13328	51594	83177
76166	83359	05913	41693	56205	36188	85984	75033	44434	03720
01948	64964	46293	69473	23947	68503	79049	52361	75736	31615
08752	34176	25573	03712	16334	20933	28969	05896	83860	04772
78199	52241	24802	55482	48968	91462	89727	48863	51022	05867
33674	51723	34200	71851	46417	48644	03247	92261	34006	37756
42319	65414	63656	12375	16710	85054	12888	89769	54863	58145
87643	97681	41933	66947	23325	21687	95160	85472	06301	31642
20319	76482	85991	32042	23961	59875	98372	98769	45909	66913
17689	17953	50685	20349	20874	94433	61855	49003	04993	38897
10068	24669	77767	21855	92553	16108	07307	83306	11501	18290
36372	32224	23484	50641	45175	39841	35932	23281	62184	68349
81882	42691	63841	64097	76757	51583	80542	08139	12666	25692
35240	39672	02775	49725	43345	33983	53408	42865	21787	15414
84384	04011	58246	88298	74984	96355	86165	54363	67408	98477
85169	83293	52294	98225	36242	03143	97697	44616	40275	82082
79462	14852	51709	05158	97877	29898	14793	60574	55226	24829
49355	18145	80492	70413	80037	39374	02247	64556	34543	61573
84450	91829	58522	94919	61702	86737	60661	95554	70419	24892
21178	84981	76999	65325	34011	28408	61348	87456	79346	61237
10453	33243	14863	77764	06818	30652	22251	22161	40001	46166
62179	72825	74982	15904	60887	37296	58934	76950	38969	43373
10184	05718	02467	42355	53506	59211	09976	73958	81490	31994
36290	63971	70529	29727	81089	83223	07744	10236	73923	84357
28723	70025	23813	71254	93877	96198	28051	50654	65337	28714
75806	30667	75617	94662	01467	93983	41821	37363	31324	61571
34105	94680	06860	14076	55090	62214	14885	49879	62064	04519
42559	64561	60812	49046	63335	49822	58427	40777	43659	82891
94441	12031	55053	69567	41189	26909	17473	85474	53411	78593
73935	65411	05378	73990	66472	42369	89685	12824	57821	29763

Table of 10000 random 0-9 digits (cont.)

76770	40079	38354	98494	05946	01269	45406	80994	05525	62119
14896	79608	07802	09853	12470	22439	94684	06117	49830	57437
20795	44841	78470	88922	61170	02310	47330	96330	81642	51390
44877	24051	15074	40003	49563	36733	66631	27317	03886	76605
30364	98810	49807	50492	94968	69981	10847	93226	02114	49571
08735	35662	84309	13020	82700	03158	80614	77228	36660	21989
04828	47554	69703	39986	48731	02977	76571	04008	75347	78422
02544	11028	21727	21226	48087	72183	08448	42367	03915	75293
50653	61678	81511	88767	65803	38170	69603	14063	68783	24999
34877	57624	49450	42525	46260	55633	59052	61267	38415	52997
73590	24204	05043	97397	44816	99744	09763	74259	70798	29832
35554	59694	16202	77786	07465	34241	77523	87934	62822	01002
50561	67900	59097	22145	58268	26137	49138	66126	25920	10251
22698	35211	61939	24805	46351	07766	86615	06640	32148	93511
01468	70499	60444	10482	85945	02270	47722	10904	69297	22403
45145	79138	54159	92739	90625	92070	52718	49210	23267	64738
24439	97424	49428	63468	01001	40938	90130	34300	12215	68492
36550	67577	54983	45097	53426	69801	15775	52736	28797	84444
55374	22539	43314	00596	37430	54560	67986	86089	73349	81797
20249	23504	42571	96243	07037	03357	94244	52742	80269	60353
10501	19685	10292	43926	29094	04450	37279	79007	02775	42433
40896	74371	64391	48143	29209	97349	74508	48862	58186	52700
43688	93599	06012	64797	44348	53995	15733	72576	83207	20841
91399	09878	66775	46497	73490	50487	69046	39608	86764	55262
22856	07755	02033	27494	30896	51067	38589	23842	53556	71010
72459	27923	83173	45646	25467	51443	29815	83057	53931	40646
98416	15365	57390	33900	36619	93393	04560	56649	94468	79674
35652	87402	52283	22270	13944	73211	83317	10312	34126	91039
40265	76501	39302	40157	24462	08982	30035	87724	42102	81005
40745	23356	42300	86036	21933	35012	18527	56861	31817	67311
37009	14198	57603	59909	69967	34274	62535	78512	45401	95730
71789	93173	21949	60702	27049	68180	99183	31138	75716	64976
90174	82125	94501	79303	75646	14986	68977	27049	78380	50286
45475	43983	19005	05182	87797	54898	40883	56024	79300	60826
58596	58252	02120	02319	83561	38189	57795	18432	28584	05424
15340	65152	49034	02884	84604	26218	10743	59846	22173	64265
92093	10926	45741	21383	81294	22578	98574	75344	50021	95217
31496	67932	22221	69890	82117	22811	83133	25038	25777	30893
54302	70573	94518	83415	76673	46569	31170	22770	78277	82748
57322	67704	57648	42377	38040	48191	45025	26398	15925	14609
96696	66118	49206	47449	81042	97376	56422	54613	18628	46827
43480	60880	87075	27426	76270	11101	68274	20656	42535	05497
00548	53503	17263	09630	20317	52906	35607	42815	33034	76373
12590	54629	35680	94105	21058	27076	75699	03641	86915	52089
53140	07042	19957	56283	60151	06995	14106	32293	92631	78612
17072	13978	18322	42053	57768	73207	97994	63008	24727	72374
86651	44444	53369	98867	54066	85664	67055	46201	25308	68616
17356	48073	75770	23959	31796	36611	14167	87011	53056	58916
43937	87743	56848	93113	44138	10229	19506	82568	34791	92132
24616	75049	16542	09523	62083	19167	29837	53071	74534	61956

Table of 10000 random 0-9 digits (cont.)

45727	31214	04406	16029	52403	19056	00822	59664	60753	21617
57691	52333	88274	91696	56338	73411	19313	26144	68337	99870
39582	44583	25709	79857	65126	01642	12342	10884	26196	34894
54899	01517	74485	83486	00034	38991	89129	91707	43869	41050
41375	08712	60802	28898	94555	16467	37867	11691	65625	52448
59132	27274	57568	55727	46112	12298	53051	22542	49187	74971
43924	13338	04256	12904	44948	75143	69801	06043	23033	74391
31712	51378	87441	87050	36427	57255	58842	46733	88461	35231
13482	57143	35552	48835	86623	29142	65138	41870	13753	21235
25186	92893	45226	37041	34030	33617	78726	86116	55798	25410
60117	83384	91099	16327	01619	21561	30049	43070	45249	29862
18634	82162	85452	92171	50476	55988	01959	71014	12105	04584
75781	68553	96102	69493	10174	99218	32356	01255	06475	58449
03555	76465	78464	23072	24832	84527	33437	88157	94606	79105
43749	47248	27593	01761	23952	47373	79445	40506	80440	19020
76437	39747	62773	25436	15914	77136	33671	15427	28711	65516
51443	79831	76861	42952	42977	21032	28539	84767	06332	59870
83990	37884	29233	48359	13562	49948	37631	73751	45540	98824
09188	65325	55130	75340	06077	60896	46372	18856	72809	11944
01248	52219	43459	01345	28678	37336	30629	80530	93414	62554
40953	74332	52671	10273	67906	26461	95957	24212	61946	39157
12439	10915	45166	47427	60628	58790	27300	46566	14158	76917
13662	02266	60154	10854	25860	94873	05556	23640	47701	37163
47682	67590	88748	24773	44508	21434	47850	06051	73883	04273
44505	07491	46875	70114	83577	68783	08228	37798	67020	21676
92920	43462	32042	94758	14954	96039	98194	39265	79375	12059
31128	77007	69679	27482	41333	60915	24543	23712	53957	95505
35775	17718	89876	46616	89547	80594	92213	35788	79234	39042
39876	86859	34730	03566	16232	21889	62773	20545	13009	65343
74005	88607	88852	36810	89691	88920	51178	60781	45376	38171
52220	26390	91729	90368	27486	26533	86495	23380	67268	83600
73882	12161	72084	92125	95268	77353	32215	78953	13816	47919
82022	72239	66196	34757	85996	27797	76951	80733	59799	86010
52746	20944	21823	96803	59260	92791	80874	08084	90290	82566
64724	27034	27387	04835	00403	99427	18273	19107	66984	92428
66699	97921	46402	12774	97520	51465	07443	67305	68777	70399
66645	99846	08435	91207	57769	20300	29594	49537	09597	43190
87722	69465	71574	63228	02954	44772	63370	77775	58437	32953
85541	77671	10247	37190	28075	54681	12055	34873	23475	62031
45373	75429	05463	57970	58302	24184	16939	33487	80855	91484
07183	21782	89721	38143	74911	25688	36109	77571	73423	96702
79710	18963	76960	40198	84065	75315	59381	69080	95454	65146
27985	35052	26064	44720	17681	42463	46128	21857	84516	41402
61439	74021	13171	01693	72988	11762	39728	19760	08057	44842
62101	07548	15762	41383	14252	14255	58269	88094	50767	97823
71485	26428	88966	04043	99255	25903	13758	15016	45753	67561
49504	58761	50662	37988	61106	44944	88253	61578	54107	51862
38014	80293	61050	90927	39453	22366	87492	06329	31284	85938
53860	39712	02819	21973	60411	72192	75063	34545	36320	34672
90758	53361	76986	90829	67768	07064	72529	11366	73207	92623

Reading off the table

The four-page table presents 10000 random decimal digits, from 0 to 9; each page offers 50 lines and 50 columns. A number may be completed in any format by placing two or more digits side by side, as is suitable; the order of retrieval for digits may be from left to right, from right to left, upward or downward, diagonally, etc. The starting point from which a series of numbers is to be formed should be randomly determined.

Illustration. To form a random number X in the range from 1 to 325 inclusively, three successive digits are needed. Suppose that we retain the first page of the table. Then, the tip of a pencil, after swivelling it round, points to line 16 (toward the bottom of page) and column 26 (toward the center). At that location, digits are (from left to right) 8-4-6, not acceptable because number 846 exceeds the wanted interval (1,325). Looking underneath (at line 17), we read 878 (refused), then 168 at line 18. The retained number is thus 168.

A somewhat more advanced method consists in splitting up the initial range of numbers, such as 000-999, into equal-sized intervals, such as 001-325, 326-650, 651-975, and in converting the read number if need be. The numbers outside the target intervals, like 000 and 976-999, are simply dropped. For example, re-using the indications above, the first group 8-4-6 lies in the 651-975 interval and becomes 196 (= 846−650). We would also obtain (on down-going lines) 228 (= 878−650), 168, 6 (= 331−325), etc.

Full examples

Example 1. A commission in a large metropolitan school board wishes to sound out the principals on the question of violence at school. It is decided that a survey be planned among the 455 principals, with a sample of 23 persons, about 5 % of the 455. How do we build up the sample? *Solution*: The persons (principals) being numbered 001 to 455, it is only necessary to obtain 23 numbers at random, in that range. Thus, we need three-digit numbers. Also, to speed up the operation, we shall exploit each three-digit number doubly, by reducing all number of 500 or more to the 0-499 range: thus, 637 would be converted to $637-500 \rightarrow 137$. Let us start, say, at page 2, line 11, and take up the three central digits of each column-block, from left to right, and then on the next (downward) lines. Now, we collect: $531-500 \rightarrow 31$, $600-500 \rightarrow 100$, 101, 254, 084, $837-500 \rightarrow 337$, $973-500 \rightarrow 473 > 455$ rejected, 205, $584-500 \rightarrow 84$ rejected (already retained), 179, 288, 597−500 → 97, 628−500 → 128, 586−500 → 86, 328, 946−500 → 446, 128 rejected (already retained), 402, 531−500 → 31 rejected (already retained), 506−500 → 6, 119, 209, 179 rejected (already retained), 923−500 → 423, 496 > 455 rejected, 501−500 → 1, 748−500 → 248, 258, 313. The ordered array of numbered people that should be included in the survey is thus: { 1, 6, 31, 84, 86, 97, 100, 101, 119, 128, 179, 205, 209, 248, 254, 258, 288, 313, 328, 337, 402, 423, 446 }.

Example 2. Gather five "random" standard normal (z) scores. *Solution*: Some published tables furnish directly collections of random or pseudo-random normal or other-type variables. With the present table, a customary way to proceed is: 1) to form random numbers u's, in the $(0,1)$ interval, and 2) to interpret these values u's as percentiles in the standard normal distribution, table 2 (in the section on the normal distribution) furnishing the corresponding z scores. That table 2 has but three digits of precision for the percentiles; therefore, we shall gather five three-digit numbers in the present table. Now, choosing page 3, line 25, column 25, we read in reverse, from right to left (why not!): 698, 034, 947, 233, 020. Looking up in table 2 (Normal distribution), we find, for $P = .698$, $z_1 = 0.5187$, and $z_2 = -1.8250$ ($P = .034$, corresponding to the negative of $P = 1 - 0.034 = .966$), $z_3 = 1.6164$ ($P = .947$), $z_4 = -0.7290$ ($P = .233$), and $z_5 = -2.0537$ ($P = .020$).

Mathematical presentation

Conventionally, "random numbers" are discrete numbers coming from the discrete uniform, or rectangular, distribution, and taking any one of the N values a, $a+\delta$, $a+2\delta$, ..., $b = a+(\mathrm{N}-1)\times\delta$, $\delta>0$, each with probability 1/N. The standard rectangular distribution, on values (1, 2, ⋯, N), is sometimes designated by $R(1,\mathrm{N})$. It is related to the uniform (or continous uniform) distribution $U(a,b)$, with p.d.f. $p(x) = 1/(b-a)$, $a \le x \le b$, or $U(0,1)$ in standard form, through the relation:

$$1 + \lfloor \mathrm{N} \times U(0,1) \rfloor \rightarrow R(1,\mathrm{N}) \ ;$$

the notation $\lfloor y \rfloor$ indicates the integral part of y (*see* also *Uniform* in the Mathematical complements).

Random numbers have many fields of application, especially in Monte Carlo simulation studies, all of these lying outside the scope of this book. In planning a future experiment or for sampling purposes in a survey, for instance, a short list of random numbers may be needed, and it is those uses we had in mind in preparing the table of 10000 random digits.

The digits listed in our table were produced sequentially, one line after another and from left to right, from the first line on the first page down to the last line on the fourth. Several checks were made in order to confirm the random character of the series. The following table shows the number of occurrences of each digit, the expected number being 1000.

Digit	0	1	2	3	4	5	6	7	8	9
Frequency	983	1001	1054	985	1056	995	1006	1010	958	952

The Chi-square test for goodness-of-fit (*see* Supplementary examples, n° 5)[1] yielded 10.796, with 9 *df*, a quite non-significant result; its bears out the hypothesis of an equal occurrence of all ten digits. We also checked the sequential independence of all 5000 pairs (c_{2i-1}, c_{2i}) and 3333 triples $(c_{3i-2}, c_{3i-1}, c_{3i})$, and obtained respectively $X^2 = 102.360$ ($df = 99$) and $X^2 = 1000.333$ ($df = 999$), both non-significant values. The average of all 10000 digits is 4.460, their variance 8.107, against their expected values $\mu = 4.5$ and $\sigma^2 = 8.25$, again two sample values which did not seriously drift away from their parametric targets[2].

Generation of pseudo random variates

Several computer programming languages and softwares propose at least one random number generating function, say the UNIF ("uniform") function. It customarily pertains to numbers from the continuous standard uniform distribution $U(0,1)$, also designated by $u \sim U(0,1)$: these numbers usually meet the double constraint: $0 \leq u < 1$. In order to generate random decimal digits, for instance, that is, digits ($c =$) 0, 1, 2, ···, 9, it is sufficient to write:

$$c \leftarrow \lfloor 10 \times \text{UNIF} \rfloor \ .$$

For producing discrete numbers in the range from a to b inclusively, *i.e.* $x \sim R(a,b)$, we would write: $x = a + \lfloor (b-a+1) \times \text{UNIF} \rfloor$. The authors offer a profusion of methods that allow the production of variates from most statistical distributions (*see* sections on the normal, χ^2, t and F distributions), and these methods all resort to the uniform random variate u.

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[1] This is an interesting special case of the Chi-square goodness-of-fit test, related to the model of a uniform (or equiprobable) distribution in k categories, with observed frequencies n_1, n_2, ···, n_k, $n = n_1 + n_2 + \cdots + n_k$. For this case, the test formula simplifies to $X^2 = kn^{-1}\sum n_j^2 - n$ and possesses $\nu = k-1$ *df*.

[2] As regards $\bar{X}$, we have $z(\bar{X}) = \sqrt{n}(\bar{X} - \mu)/\sigma = \sqrt{(10000)}(4.460 - 4.5)/2.8723 \approx -1.393$; using a bilateral test criterion and an approximate normal model, we estimate $2 \times \Pr(z \geq |-1.393|) \approx 0.164$. Next, for s^2, we can compute $X^2 = df \times s^2/\sigma^2 = 9999 \times 8.107/8.25 \approx 9825.684$. This result may be interpreted approximately as a value from χ^2 having $\nu = df = n-1 = 9999$. Using the expectation (ν) and variance (2ν) of χ^2_ν, we calculate, as a rough approximation, $z = (X^2 - \nu)/\sqrt{(2\nu)} = (9825.684 - 9999)/\sqrt{(2 \times 9999)} \approx -1.226$, and $2 \times \Pr(z \geq |-1.226|) \approx 0.220$.

As for methods that directly generate random uniform numbers u, the most widespread is the linear congruential method, that is:

$$y_n = (ay_{n-1} + c) \bmod m$$

and:

$$u_n = y_n / m \ .$$

The numbers y, a, c, and m are positive integers (the notation "y mod x" designates the remainder following integer division of y by x). Some optimality rules should be followed to fix parameters a, c and m in order to obtain a sequence $\{ y_n \}$ and, thus, a sequence $\{ u_n \}$ that display fair random characteristics. Some "popular" values, given here as illustrations, are:

$a = 16807$, $c = 0$, $m = 2^{31} - 1$;

$a = 524293$, $c = 0$, $m = 2^{35}$;

$a = 1220703125$, $c = 0$, $m = 2^{35}$;

$a = 3141592653$, $c = 2718281829$, $m = 2^{35}$.

Gentle (1998) documents and discusses all these questions.

Generation of random samples

The program outlines given below allow the generation of simple random samples, in different contexts. There exists a population comprising N elements, represented in an array POP[1..N], out of which a sample of n elements is to be drawn and transferred in a second array SAM[1..n].

A (Sampling without replacement, first variant)

For $j = 1$ **to** n **do** $k \leftarrow 1 + \lfloor (\mathrm{N}+1-j) \times \mathrm{UNIF} \rfloor$;
SAM[j] $\leftarrow$ POP[k] ;
POP[k] $\leftarrow$ POP[N+1$-j$] .

B (Sampling without replacement, second variant)

$j \leftarrow 0$; $i \leftarrow 0$;
While $j < n$ **do** **If** UNIF $< (n-j)/(\mathrm{N}-i)$ **then** $j \leftarrow j+1$; SAM[j] $\leftarrow$ POP[i+1] ;
$i \leftarrow i + 1$.

Remark : This variant is less efficient than the first one (because its temporal cost of execution is proportional to N rather than n); its merit, however, is to allow the user to accept or refuse inclusion of each element of the population in its turn.

C (Sampling with replacement)

For $j = 1$ **to** n **do** $k \leftarrow 1 + \lfloor N \times UNIF \rfloor$;
SAM[j] $\leftarrow$ POP[k] .

D (Generation of a random permutation of N elements)

For $j = 1$ **to** $N-1$ **do** $k \leftarrow j + \lfloor (N+1-j) \times UNIF \rfloor$;
Swap POP[j] , POP[k] .

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Supplementary examples

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– IMPORTANT NOTICE –

All examples and data presented herein are fictitious
and the conclusions drawn therefrom are only of a pedagogical value.

Example 1 [z test for a sample mean, with population variance (σ^2) known]

A biologist wants to determine whether absorption of chlorinated water by young rats influences or not their growth, up to the age of 10 days. The rats, all of the same strain, come from a single breeding farm; for the strain in question, it has been established that the mean body weight at 10 days was 150 g, with a s.d. of 12 g. Taking a sample of 20 newborn rats, the researcher puts them under a diet with chlorinated water and, at day 10 they are weighed, giving an average of 146.81 g, with s.d. 10.46. Were the rats' weights modified by the experimental diet, using a significance level of 5 %?

Solution: Let us assume that the observed variable, weight, is normally distributed. According to the null hypothesis (H_0) of no systematic variation or influence ("The diet with chlorinated water has no influence on the rats's growth, *i.e.* their weight"), each young rat's weight at 10 days is distributed normally, as $N(150, 12^2)$, and the sample's average weight ($\bar{X}$) similarly, as $N(150, 12^2/n)$, where $n = 20$. The formula:

$$z_{\bar{X}} = \frac{\sqrt{n}\,(\bar{X} - \mu)}{\sigma}$$

furnishes a standard score z distributed as $N(0,1)$. With our sample of $n = 20$ young rats, H_0 stipulates that $\mu = 150$ and $\sigma = 12$; then, $z_{\bar{X}} = \sqrt{20} \times (146.81 - 150)/12 \approx -1.189$. In order that the two-tailed test be significant at 5 %, the test statistic $z_{\bar{X}}$ must lie under the 2.5th or over the 97.5th percentile of the standard normal distribution. Looking at table 2 (Normal distribution), we find the appropriate critical values $z = \pm 1.960$. The observed difference ($z_{\bar{X}} = -1.189$) lies in the acceptance interval for H_0, so that the biologist cannot assert that chlorinated water influences in one way or another the growth of young rats.

Example 2 [t test for a sample mean, with population variance (σ^2) unknown]

Are college hockey players in outstanding physical shape? That is a question put to a researcher in exercise physiology, who gathers a sample of 27 members in a league, all 23 years old. The measure retained is $\dot{V}O_2$ max, *i.e.* the rate (in ml/min) of oxygen (O_2) consumed per minute, divided by body weight (in kg), taken at the peak of a process of maximal exertion: this measure, also called "(maximum) aerobic power", reflects the capacity of the organism to use oxygen during intensive muscular work. The 27 male subjects obtain an average measure of 49.31 $ml.min^{-1}.kg^{-1}$, with a s.d. of 5.57. Having consulted a variety of textbooks and scientific research, the researcher finds one with empirical norms adequate for this situation, the norms indicating a mean value of 47.6 $ml.min^{-1}.kg^{-1}$ for males in the age range 20-24.

Solution: Let us assume that the $\dot{V}O_2$ max variable is normally distributed, and that the published value 47.6 $ml.min^{-1}.kg^{-1}$ is "μ", the mean value appropriate for 23 years old males. The null hypothesis (H_0), to the effect that hockey players are representative elements of the population, is therefore equivalent to asserting that our mesure (X) is normally distributed around 47.6, or $X \sim N(47.6,\sigma^2)$. Since the population s.d. (σ) is not known, we *estimate* it with the sample s.d., s = 5.57. The appropriate test statistic is then:

$$t_{\bar{X}} = \frac{\sqrt{n}(\bar{X} - \mu)}{s} ,$$

and it refers to Student's t distribution, with $\nu = n-1$ *df.* The research question being directional ("Are they in an *outstanding*, or exceptionnaly high, good shape?"), the test should be unilateral, or one-tailed, say at the 5 % significance threshold. Using $n = 27$, $\nu = n-1 = 26$, and $P = 0.95$ (= 1−0.05), table 1 of the Student's t distribution furnishes the critical value $t_{26[.95]} = 1.706$. Now, the sample t, calculated as $\sqrt{27}\times(49.31-47.6)/5.57 \approx 1.595$, does not exceed the critical value. In conclusion, available data do not warrant the allegation that college hockey players are in outstanding physical shape, at least with respect to the $\dot{V}O_2$ max criterion.

Example 3 [z test for a correlation coefficient]

The height and weight of 12 children reared in a state orphanage were measured, and the Pearson correlation (r) between the two was 0.753; these children, having no illness or health problem, were aged 4 to 6. For this interval of age, some treatises in pediatry report a height-weight correlation of 0.92. Are our 12 children a representative sample of children of the same age, as depicted in the treatises, or did their particular situation hamper their development?

Solution: The question asked is equivalent to deciding if, in a population globally characterized by a correlation of $\rho = 0.92$, it is plausible to observe a sample of $n = 12$ individuals having a correlation of $r = 0.753$. Now, for answering that question, we cannot take advantage of the test no more than of the significance tables in the section on Student's t distribution, those being used to test whether the observed correlation differs really from 0 or not, *i.e.* whether the population correlation ρ is 0. In the present, general situation where $\rho \neq 0$, the appropriate test, also due to Fisher, demands a mathematical transformation of the value of r,

$$Q_r = \tanh^{-1} r = \tfrac{1}{2}\ln[(1+r)/(1-r)] \ .$$

The transformed value (Q_r) is quasi normally distributed, with approximate mean $Q_\rho+\rho/(2n-4)$ and variance $1/(n-3)$. The test statistic to be computed,

$$z_Q = \sqrt{(n-3)}[Q_r - Q_\rho - \rho/(2n-4)] \ ,$$

has a quasi standard normal distribution. Here, $r = 0.753$ becomes $Q_r \approx 0.980$, $\rho = 0.92$ becomes $Q_\rho = 1.589$, and $z_Q = \sqrt{9}\times(0.980 - 1.589 - 0.92/20) \approx -1.965$. For a two-tailed test at significance level of 5 %, table 2 of the normal distribution furnishes critical values $z = \pm 1.960$. Our observed value lying outside the acceptance interval of the null hypothesis, we must reject it. Despite the small size ($n = 12$) of our sample, we have been able to prove statistically that the 12 children from the state orphanage display an unusual degree of correlation between height and weight, as compared to pediactric norms, a fact that hints at a possible deficiency in their environment.

Example 4 [Confidence interval for a predicted value based on linear regression, using Student's t]

A technician in ergonomics must devise a method for allocating mail bags among postal workers, in sorting offices. He obtains a random sample of seven (filled) bags, weights each of them and counts the content. The measures obtained are:

X (weight, in kg)	14.0	16.5	21.6	18.8	19.5	12.0	16.4
Y (nbr. of pieces)	607	850	997	849	993	581	701

The sample statistics are: $\bar{X} = 16.97$, $s_X = 3.30$, $\bar{Y} = 796.86$, $s_Y = 171.22$ and $r_{XY} = 0.942$. Now, for the linear regression coefficients, we calculate: $b = r_{XY}s_Y/s_X \approx 48.88$, $a = \bar{Y} - b\bar{X} \approx -32.6$. The resulting equation:

$$\hat{Y} = 48.88\ X - 32.6$$

helps predict the number of mail pieces in a bag as a function of the bag's weight (in kg). For a bag weighting, say, 20 kg, what is the interval inside which the number of pieces may vary, with a confidence degree of $1-\alpha$?

Solution: The confidence interval for a simple linear prediction is calculated with:

$$\hat{Y} \pm t_{n-2[1-\alpha/2]} s_{\hat{Y}.x} \ ;$$

the confidence level of the interval is $1-\alpha$, and the confidence limits are based on percentiles of Student's t with $\nu = n-2$. We need to compute the *standard error of prediction* $s_{\hat{Y}.x}$; we propose to do so in two steps. First, the *standard error of estimation* s_e (or $s_{Y|X}$) may be obtained by:

$$s_e = s_Y \sqrt{\frac{n-1}{n-2}(1-r^2)} \ ,$$

then:

$$s_{\hat{Y}.x} = s_e \sqrt{1 + \frac{1}{n} + \frac{(x-\bar{X})^2}{(n-1)s_X^2}} \ .$$

For our data, using $x = 20$ kg, the predicted number of pieces is $\hat{Y} = 48.88 \times 20 - 32.6 \approx 945.0$. The standard error of estimation s_e is $171.22\sqrt{[6/5(1-0.942^2)]} \approx 62.948$; then, $s_{\hat{Y}.x} = 62.948\sqrt{[1 + 1/7 + (20-16.97)^2/(6\times 3{,}30^2)]} \approx 71.31$. Lastly, using $\nu = n-2 = 5$ and a confidence level of 0.95, the appropriate t-interval is $\pm t_{5[0.975]} = \pm 2.571$. Finally, we can write out the desired interval, $945.0 \pm 2.571 \times 71.31$, *i.e.* state that there should be between 762 and 1128 pieces of mail in a bag of 20 kg.

Example 5 [Chi-square test of goodness-of-fit (from Laurencelle 1998)]

Students on a Québec campus have been evaluated using Cooper's 12-min running test: the measure is the total distance run on a circular track, in meters (m), in 12 minutes. 282 male students were measured, aged 19 to 23, their results being summarized in the table on next page: column "f" is the number (or *frequency*) of people having obtained a distance within the indicated interval. Now, the corresponding histogram, a graphical equivalent of the frequency distribution, displays a slight positive asymmetry. Using a 5 % significance threshold, may we believe that these data pertain to the normal distribution model?

Solution: A major application of the Chi-sqare distribution concerns the analysis of frequency tables, in particular goodness-of-fit tests where the observed distribution of a variable is compared to the theoretical (or expected) distribution pertaining to a given model. The test statistic for this situation is, in general:

$$X^2 = \sum_{j=1}^{k} \frac{(f_j - ft_j)^2}{ft_j} \ .$$

Distance (in m)	f	ft	Distance (in m)	f	ft
≥ 3200	0	0.14	2300 - 2399	40	38.87
3100 - 3199	2	0.31	2200 - 2299	35	36.62
3000 - 3099	2	0.85	2100 - 2199	34	30.61
2900 - 2999	4	2.09	2000 - 2099	41	22.72
2800 - 2899	6	4.53	1900 - 1999	24	14.96
2700 - 2799	8	8.74	1800 - 1899	19	8.74
2600 - 2699	9	14.96	1700 - 1799	9	4.53
2500 - 2599	13	22.72	1600 - 1699	2	2.09
2400 - 2499	34	30.61	≤ 1599	0	1.30

Quantities ft_j, called "theoretical frequencies", are the expected numbers of data (people, elements, sample units) in each interval of values; they can be written down as $ft_j = n \times p_j$, where p_j is the probability that any datum (or sample unit) lies in the j^{th} interval, according to the chosen model. The *df* (ν) of this Chi-square are $k-m-1$, where k is the effective number of (non-zero) frequency intervals (or categories), and m is the number of parameters in the model that had to be estimated from the data.

For the normal model, we have to estimate the μ and σ parameters, which we do usually by substituting the sample mean $\bar{X}$ and s.d. s: for our n = 282 data of the frequency distribution, we calculated $\bar{X}$ = 2250.00 and s = 289.44. The determination of ft_j can be made approximately[1] using $ft_j \approx \frac{nL}{s}\varphi\left(\frac{X_{mp_j} - \bar{X}}{s}\right)$, where L is the interval's length, here L = 100, X_{mp_j} is the interval's midpoint, and φ is the standard normal p.d.f. For instance, the interval (2900-2999) has midpoint X_{mp} = 2950, the equivalent z score is $(2950-2250.00)/289.44 \approx 2.4185$, $\varphi(2.4185) = \exp[\ -(2.4185)^2/2\ ]/\sqrt{(2\pi)} \approx 0.2142$, and $ft = 282 \times 100/289.44 \times 0.02142 \approx 2.09$.

Values of ft_j were established for every interval and were transcribed in the frequency table. Note that it is recommended to group data and intervals so that no interval contains a ft

[1] A more rigorous method of determination consists in finding the probability integral corresponding to each interval's upper limit, in the normal model, and then, by substraction using the lower interval, in computing a percentage of occupancy of the interval. For instance, the exact limits of interval (2900 - 2999) are b_1 = 2899.5 and b_2 = 2999.5 ; their equivalent z scores are $z(b_1) = (2899.5-2250.00)/289.44 \approx 2.244$ and $z(b_2) \approx 2.589$. In table 1 of the normal distribution, we find the probability integral values P_1 = 0.98758 for $z(b_1)$ abd P_2 = 0.99519 for $z(b_2)$; the difference between the two is the expected occupancy, i.e. $p = P_2-P_1 = 0.00761$ and, after multiplication with n = 282, $ft_j = 282 \times 0.00761 \approx 2.15$.

value less than 2. Consequently, in our example, we combine all data going from 2900 upwards ($f_{\text{comb}} = 8, ft_{\text{comb}} = 3.35$) and those going from 1699 downwards ($f_{\text{comb}} = 2, ft_{\text{comb}} = 3.39$). Finally, the computation of the Chi-square statistic runs through $k = 15$ intervals and yields $X^2 = 24.120$;[2] the df are $15-2-1 = 12$. The table of percentiles of Chi-square, at line $\nu = 12$ and column 0.950, gives $\chi^2 = 21.03$. As the obtained value exceeds percentile 95, we can reject the normal model for our data. Besides, a thorough examination of the data rather suggests a skew model, very likely that of the "lognormal" distribution"[3].

Example 6 [z test for the difference of means between two groups]

An all-country survey has revealed that, for the age group of 20 to 29 years, men watch television for an average of 364.6 minutes per week, with a s.d. of 51.2, whereas women do it for 404.5 min, with a s.d. of 38.7. In a separate survey of some university campuses, 64 male and 59 female students were questioned about their TV habits, particularly the time spent weekly facing television, in min: the males obtained $\bar{X}_1 = 372.4$, $s_1 = 56.3$, and the females, $\bar{X}_2 = 380.8$, $s_2 = 41.4$. Is the difference between men and women enrolled in universities the same as that in the general population?

Solution: The test statistic:

$$z_{\bar{X}_1-\bar{X}_2} = \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\sqrt{\dfrac{\sigma_1^2}{n_1} + \dfrac{\sigma_2^2}{n_2}}}$$

has a standard normal distribution inasmuch as variable X (here, the number of minutes spent per week in front of a TV set) is approximately normally distributed. In the present context, we consider the means and s.d.'s from the all-country survey as aptly defining the parameters of the corresponding populations (for given age and sex). We may thus apply the above formula:

$$z_{\bar{X}_1-\bar{X}_2} = \frac{(372.4 - 380.8) - (364.6 - 404.5)}{\sqrt{\dfrac{51.2^2}{64} + \dfrac{38.7^2}{59}}} \approx 3.867 \ .$$

---

[2] If we had used the more rigorous, and fastidious, method of determination for the ft_j's, we would have obtained $X^2 \approx 23.84$, the relative difference with the other method being only ~ 1 %, and would have arrived at the same conclusion.

[3] A r.v. is said to obey the lognormal distribution if its logarithm is normally distributed, precisely if $\ln(X-C) \sim N(\mu, \sigma^2)$, for some C.

The obtained value, $z = 3.867$, exceeds by far the 1 % bilateral critical values of the standard normal distribution $\pm z_{[0.995]} = \pm 2.5758$ (*see* table 2 of the normal distribution). The statistical hypothesis of no difference is thus rejected: the mean time spent watching TV differs less between male and female university students than they do between age-equivalent men and women in the general population. (Note that the above test procedure is also appropriate to situations where $\mu_1 = \mu_2$ and/or $\sigma_1^2 = \sigma_2^2$.)

***Example** 7* [*t* test for the difference between two paired means]

An attitude questionnaire bearing on various dairy products was first administered to a group of eight consumers. 15 questions were asked, with Likert-type answering format, the answer ranging from 0 ("not at all favorable") to 5 ("very much favorable"); the total score could vary from 0 to 75, a high score meaning a favorable attitude. A videotape was then lent to each person, containing a presentation of the benefits of the said products for one's health. Finally, a week later, the questionnaire was re-administered. The research firm was asked to check whether the video publicity was effective, that is, whether the consumers' attitude may be swayed. The two arrays of obtained scores follow:

Consumer	1	2	3	4	5	6	7	8
X_1 (before)	34	29	55	43	67	38	69	50
X_2 (after)	52	47	59	46	69	55	71	62
$d = (X_2 - X_1)$	18	18	4	3	2	17	2	12

Solution: The computations required for performing the significance test are most simple when based on the *score differences*, $d_i = X_{1i} - X_{2i}$, for every subject i. Using the mean difference, $\bar{d} = 9.50$ and the corresponding s.d. $s_d = 7.48$, the formula[4]:

$$t_{\bar{d}} = \frac{\sqrt{n}\,\bar{d}}{s_d}$$

yields a Student's t with $\nu = n - 1$ *df.* Here, $t = \sqrt{8} \times 9.50 / 7.48 \approx 3.592$, and $\nu = 7$. At the 1 % bilateral significance level, table 1 of Student's t distribution indicates $\pm t_{7[0.99]} = \pm 2.998$.

[4] The extensive formula based on raw scores, $t = \sqrt{n}(\bar{X}_2 - \bar{X}_1)/\sqrt{(s_1^2 + s_2^2 - 2r_{12}s_1s_2)}$, is algebraically equivalent to the formula given in the text. A feature of the long formula is that of making obvious the specific advantage of the "repeated mesures" design, *i.e.* a research design where the same subjects (or sample units) are repeatedly measured under two or more experimental conditions. The said advantage emanates from the existence of a positive and usually high correlation (r_{12}) between such repeated series of scores, and the consequent drop in the standard error of the difference between observed means.

Therefore, we can reject the null hypothesis and conclude that consumers' attitude has been truly improved in favor of dairy products. (The reader may have noticed that all eight observed differences d_j are positive: this particular arrangement has, under H_0 and according to a simple binomial argument, a probability of occurrence equal to $(½)^8 \approx 0.0039$, allowing us to reject directly H_0 at the 0.01 significance level.)

Example 8 [t test for the difference between two paired variances]

A graduate student in psychology puts forward the hypothesis that stress can either facilitate or impede performance, depending on the people concerned. To test this hypothtesis, 15 male undergraduates were required to perform at a fine motor task under two experimental conditions: the task was to sort small colored sticks, one at a time, for three minutes. In one experimental condition, the task environment was neutral, while in the second, to create stress, the passing of time was marked off with a periodic tone, emitted at progressively higher sound levels. The order of conditions was randomized across subjects. The results of the 15 subjects, for the two conditions, are reproduced below.

Neutral condition	25	40	41	58	51	36	75	53	27	63	29	46	44	55	48
"Stress" condition	17	21	31	69	44	37	77	57	29	46	46	39	41	65	47

The novice researcher proposes that the stress condition would enhance the differences among subjects, hence that the scores' variance would be greater under this conditon.

Solution: The variances to be compared are "paired", an expression which means that both series come from the sample sample of subjects. This mutual dependence in the data proscribes the classical F test on the quotient of two variances. In its place, authors have developped the following test statistic:

$$t_{s_1^2-s_2^2} = \frac{s_1^2 - s_2^2}{2s_1s_2\sqrt{\dfrac{1-r^2}{n-2}}} ,$$

which is approximately distributed as Student's t with $\nu = n-2$ df. For our data, we calculated $n = 15$, $s_1^2 = 192.07$, $s_2^2 = 290.97$, $r_{12} \approx 0.808$, and $t \approx -1.280$. Using a one-tailed, 1 % significance threshold, table 1 of Student's t distribution furnishes $t_{13[0.01]} = -1.771$. The obtained difference does not exceed the critical value, so that we must keep the null hypothesis and conclude, with the psychology student, that the stress condition did not succeed at enhancing the performance differences among subjects.

Example 9 [z test for the difference between two independent correlation coefficients]

Intelligence, as measured with a test of Intellectual Quotient (IQ), is a proved determinant of scholastic achievement. The research bureau of a Québec school board randomly sampled 250 pupils in final grade of a primary school. Among these, 150 were of French or Anglo-French extraction and were tagged "group A"; as for "group B", it was composed of 100 pupils with parentage of diverse Far East ethnic groups (Chinese, Vietnamese, Laotian, etc.) and living in Québec for at most two generations. All pupils had their first years of primary education in the same school. Each of them was administered a IQ test, and then the correlation between the pupils' IQ and the average on their yearly report was computed. For group A, the value r_A = 0.565 was obtained, whereas group B got r_B = 0.482. Did intellectual abilities have the same import on achievement in both groups of children?

Solution: To answer this question, we must decide whether the correlation between IQ and the school report's average differs or not from one group to the other. To do that, we must again resort to the transformation of coefficient r already applied in example 3, namely $Q = \tanh^{-1} r = \frac{1}{2}\ln[(1+r)/(1-r)]$, first for $r_A \to Q_A$, and for $r_B \to Q_B$. Each transformed variable has a variance approximately equal to $1/(n-3)$, and the test statistic:

$$z_{Q_{r_A} - Q_{r_B}} = \frac{Q_{r_A} - Q_{r_B}}{\sqrt{\dfrac{1}{n_A - 3} + \dfrac{1}{n_B - 3}}}$$

is approximately distributed as a standard normal variable. Here, $Q(r_A) = Q(0.565) \approx 0.640$, $Q(r_B) \approx 0.526$, and $z = (0.640 - 0.526)/\sqrt{(1/147 + 1/97)} \approx 0.871$. The bilateral percentiles of the standard normal distribution for signifiance level 5 % are ±1.960. As the obtained test statistic lies within the tolerance interval, H_0 must be accepted: we could not prove that, for children in Québec having parental origins in the Far-East, intellectual abilities have a different impact on school achievement than for the native.

Example 10 [Chi-square test for the differences among k independent variances]

A given research design comprises five groups of measures issuing from as many samples of subjects. Before proceeding to analysis of variance (ANOVA) for the purpose of studying the differences among the groups' means, the researcher wishes to assess the homogeneity of variances condition. The relevant data are:

Groups	n_j	$\nu_j(=n_j-1)$	s_j^2
1	12	11	29.62
2	9	8	17.45
3	10	9	23.19
4	14	13	21.91
5	8	7	42.24
	53	48	

Solution: The F_{max} and Cochran's C tests for comparing k independent variances (see appropriate sections) both suppose that the df's (ν_j) among compared variances are equal, hence that groups' sizes are the same. M. S. Bartlett has proposed a Chi-square test to accommodate unequal group sizes. A symbolic formula is:

$$X^2 = \frac{A - B}{C} ,$$

where $A = (\Sigma\nu_j) \ln \hat{\sigma}^2$; $B = \Sigma\nu_j \ln s_j^2$; $C = 1 + [\Sigma\nu_j^{-1} - (\Sigma\nu_j)^{-1}]/[3(k-1)]$; $\hat{\sigma}^2 = \Sigma\, \nu_j s_j^2 / \Sigma\nu_j$. This statistic has an approximate Chi-square (χ^2) distribution with $\nu = k-1$. The component $\hat{\sigma}^2$ is a weighted average of the k groups' variances; for our example, it coincides with the within-group mean square (MS_{Within}) of ANOVA.

With our data, we calculate $\hat{\sigma}^2 = 1254.64 / 48 \approx 26.138$; $A = 48 \ln 26.138 \approx 156.643$; $B = 11 \ln 29.62 + 8 \ln 17.45 + \cdots + 7 \ln 42.24 \approx 154.775$, and $C = 1 + [1/11 + 1/8 + \cdots + 1/7 - 1/48] / [3\times4] \approx 1.04383$; whence $X^2 = (156.643 - 154.775) / 1.04383 \approx 1.790$. In the table of percentiles of the Chi-square distribution, at line $\nu = k-1 = 4$ and column "95", we get $\chi^2_{4[0.95]} = 9.49$; since the observed result does not exceed the critical value, the homogeneity condition may be said confirmed. (The reader may check that the F_{max} and C tests on the present data render both the same conclusion, even when the permissive significance criterion is applied.)

Example 11 [F tests on a set of contrasts among means, according to Scheffé's criterion]

Following an analysis of variance (ANOVA), it may happen that the researcher has to compare not only means among themselves singly but also in various combinations. To illustrate the case, let us take again Example 1 in the section on Studentized range, with $k = 5$ groups. Suppose here that we have three experimental groups (E1, E2, E3) and two control groups (C1, C2). The summary data are reproduced below.

Groups (n = 10)	Means	S.d.
E1	22.63	3.17
E2	18.90	2.86
E3	26.14	2.59
C1	23.35	3.06
C2	19.31	2.77

Recall that the reference mean square (MS_{error}), here $MS_{Within\text{-}groups}$, equals 8.395 and possesses $k\times(n-1) = 5\times9 = 45$ df. The researcher wants to compare, say, the experimental groups among themselves, also the control groups among themselves, both types of groups globally, plus E1 to C1, E2 to C2 and E3 to C1+C2. The design of these comparisons, also called contrasts, is given in the following table, together with the contrast coefficients[5] and intermediary calculations. Are these differences significant?

	Coefficients (c_j)							
Contrast	E1 22.63	E2 18.90	E3 26.14	C1 23.35	C2 19.31	$\Sigma c_j\bar{X}_j$	Σc_j^2	MS = $n(\Sigma c_j\bar{X}_j)^2/\Sigma c_j^2$
E1 vs E2	+1	−1				+3.73	2	69.565
E1 vs E3	+1		−1			−3.51	2	61.601
E2 vs E3		+1	−1			−7.24	2	262.088
C1 vs C2				+1	−1	+4.04	2	81.608
(E1,E2,E3) vs (C1,C2)	2	2	2	−3	−3	+7.36	30	18.057
E1 vs C1	+1			−1		−0.72	2	2.592
E2 vs C2		+1			−1	−0.41	2	0.841
E3 vs (C1,C2)			2	−1	−1	+9.62	6	154.241

[5] The fundamental rule for setting up any contrast is that its non-zero coefficients must sum to zero, *i.e.* $\Sigma c_j = 0$. The coefficients used here have an arbitrary "unit" scale, their normalized value being obtained through the subsequent computational formulae. For instance, for the comparison between (E1,E2,E3) vs (C1,C2), the difference ⅓(E1+E2+E3)−½(C1+C2) may be translated into coefficients +⅓,+⅓,+⅓,−½,−½, or +2,+2,+2,−3,−3, or, in normalized form, $+2/\sqrt{30},+2/\sqrt{30},+2/\sqrt{30},-3/\sqrt{30},-3/\sqrt{30}$, the squared values now summing to unity.

Solution: All these comparisons are apparently of an exploratory nature and are diverse in form, and they, as a group, lie outside the scope of Tukey's HSD method; moreover, the intricate (but nonetheless interesting) and numerous groupings of means ban the application of a planned comparisons approach (such as that used with the Dunn-Šidák criterion, *see* section on Student's t distribution). There remains Scheffé's multiple comparisons method. It consists in making up, for each comparison or contrast, a mean square like:

$$\mathrm{MS}_{\text{contrast}} = \frac{n\left(\sum_j c_j\bar{X}_j\right)^2}{\sum_j c_j^2} .$$

The sole condition that must be enforced is that $\sum c_j = 0$.[6] The steps and results of the calculations are illustrated in the last columns of the table. The decision to be rendered on each comparison is then dependent on a F quotient, $F = \mathrm{MS}_{\text{contrast}}/\mathrm{MS}_{\text{error}}$, which must exceed Scheffé's critical value, $F_{\text{Scheffé}} = (k-1)F_{k-1,df_{\text{err}}[1-\alpha]}$, in order to be significant at 100α %; the parameters ν_1 and ν_2 of F are thus $k-1$ and df_{error} (associated with $\mathrm{MS}_{\text{error}}$).

In the present example, the df's are respectively $k-1 = 4$ and $df_{\text{error}} = 45$. Looking at the F table of percentiles in the appropriate section, for $\alpha = 0.05$ and percentile 95, we find $F_{4,44} = 2.584$ and $F_{4,46} = 2.574$, averaging to $F_{4,45} \approx 2.579$. The appropriate critical value for our contrasts is thus $4 \times F_{4,45[0.95]} = 4 \times 2.579 \approx 10.316$, a comparatively high and conservative value. Our observed quotients $\mathrm{MS}_{\text{contrast}} / \mathrm{MS}_{\text{error}}$ are, from the top of the table downwards: $65.565/8.395 \approx 7.810$, and 7.338, 31.220, 9.721, 2.151, 0.309, 0.102, 18.373. Thus, only two comparisons find favor with Scheffé's criterion, that between E2 and E3, and that between E3 and C1+C2. The reader is invited to compare this yield of Scheffé's method to that of Tukey's HSD for the same data (*see* section on Studentized range distribution). Certainly, Scheffé's criterion is more strict but its domain has much more span (as to the number and form of possible contrasts), a fact that more than compensates its conservativeness.

[6] Quantity "n" in the numerator of the MS formula denotes the common number of data (and/or subjects) per mean. When the group sizes (n_j) are unequal, various ways out are possible, and we recommend the so-called "harmonic-mean" solution: in this solution, the common size "n" is replaced by the harmonic mean ($\tilde{n}$) of the k group sizes, i.e. $\tilde{n} = k/(n_1^{-1}+n_2^{-1}+\cdots+n_k^{-1})$.

Mathematical complements

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Entries are ordered alphabetically with respect to the (principal) name of the concept treated. Note that only main results and formulae are given, without demonstration.

Beta [*Beta* distribution $\beta_x(a,b)$, *Beta* function $B(a,b)$]

The standard *Beta* p.d.f, defined by $p(t) = [B(a,b)]^{-1}t^{a-1}(1-t)^{b-1}$, $a > 0$, $b > 0$ and $0 \le t \le 1$, where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\,dt$ is the *Beta* function, is used to represent the behavior of a doubly bounded r.v., under various distributional shapes depending upon parameters a and b; the *Beta* d.f. is simply the integral of $p(t)$, sometimes indicated by $\beta_x(a,b) = \int_0^x p(t)\,dt$. The uniform distribution is a special case, where $a = b = 1$. If y is a random variable from the $F(\nu_1,\nu_2)$ distribution, then $x=\nu_2/(\nu_2+\nu_1 y)$ has a standard *Beta* distribution with $a = \nu_2/2$, $b = \nu_1/2$, and $P(y) = 1-\beta_x(a,b)$.

The moments of the *Beta* distribution are: $\mu = a/(a+b)$, $\sigma^2 = ab/[(a+b+1)(a+b)^2]$, $\gamma_1 = 2(b-a)\surd(a+b+1)/[(a+b+2)\surd(ab)]$, $\gamma_2 = 6[(a+b+1)(a^2+b^2-3ab) - ab] / [ab(a+b+2)(a+b+3)]$. The mode sits at $(a-1)/(a+b-2)$ and the median equals ½ if $a = b$.

The *Beta* function, $B(a,b)$, can also be evaluated through its relation with the *Gamma* (Γ) function, thanks to the equation $B(a,b) = [\Gamma(a)\Gamma(b)]/\Gamma(a+b)$. In statistical work, the function is usually encountered having integral arguments a, b. Note that $B(a,b) = B(b,a)$.

Example: $B(3,4) = [\Gamma(3)\Gamma(4)]/\Gamma(7) = 2\times 6/720 \approx 0.01667$.

Binomial expansion

The expansion of the binomial $(a + b)^n$, attributed to Newton, serves as a device of general mathematical import in addition to its connection with the binomial probability distribution (*see* appropriate section). Noteworthy is the special case which consists of $a = 1$ and $b = x$. The expansion is then:

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots$$
$$= 1 + nx + \tfrac{1}{2}n(n-1)x^2 + \tfrac{1}{6}n(n-1)(n-2)x^3 + \ldots$$

Coefficients $\binom{n}{r}$, known as (numbers of) combinations, are also called "binomial coefficients" and go back to Pascal. The expansion is valid for all x and n. When n is a non-negative integer, the expansion gives $n+1$ terms; when n is negative or has a fractional part, the expansion is infinite and its sum converges if $|x| < 1$.

Examples: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$; $\sqrt{(0.9)} = (1 - 0.1)^{½} = 1 - ½\times0.1 + \frac{½(-½)}{2}\times0.1^2 + ... \approx 1 - 0.05 - 0.00125 = 0.94875$ (*i.e.* summing only the first three terms of the infinite expansion). Also, $(1 + x)^{-1} = 1 - x + x^2 - x^3 + ...$

Combinations, C(m,n) or $\binom{m}{n}$

The number of combinations, indicated by $C(m,n)$, $\binom{m}{n}$, ${}_mC_n$ or C_n^m, is the number of different groups of n objects that can be combined from m objects. Necessarily, $m \geq n \geq 0$, otherwise $C(m,n) = 0$. This function can be calculated using the factorial function and the equivalence $C(m,n) = m!/[n!(m-n)!]$. Other handy formulae are: $C(m,n) = m_{(n)}/n!$, where $m_{(n)}$ is a descending factorial, or $C(m+n,n) = (m+n)!/[m!n!]$.

Examples: $C(6,2) = 6!/[2!(6-2)!] = 6_{(2)}/2! = 6\times5/2 = 15$; $C(9,9) = 1$; $C(9,0) = 1$; $C(5,7) = 0$, by definition.

Correlation coefficient, ρ_{XY}, r_{XY}

The linear, product-moment, or Pearson's correlation coefficient ρ_{XY} (or ρ) between two random variables X and Y is defined by $\rho = E\{(X-\mu_X)(Y-\mu_Y)\}/\sigma_X\sigma_Y$, $-1 \leq \rho \leq 1$, and its absolute value reflects the degree of proportional (linear) relationship between X and Y. Its sample estimate, obtained from a bivariate series (X_i,Y_i), $i = 1,n$, is: $r_{XY} = \sum (x_i-\bar{X})(y_i-\bar{Y})/[(n-1)s_Xs_Y]$. Other computing formulae are available.

Distribution function, $P(x)$

Abbreviated as d.f., the distribution function, also called probability distribution function, is the probability integral of (random) variable X at x. It is related to the density function (p.d.f.) $p(x)$ through $P(x) = \int_{-\infty}^{x} p(y)\, dy$ or, equivalently for integer-valued variables, to the mass function (p.m.f.) through $P(x) = \sum_{-\infty}^{x} p(y)$. By definition, $P(-\infty) = 0 < P(x) < 1 = P(\infty)$. *See* also Probability density function.

Note the distinction between the abbreviations "d.f." (distribution function) and "*df*" (degrees of freedom).

Expectation (of a random variable), μ or E(X)

See Moments of a distribution

Exponential distribution, $E(\theta)$

The exponential p.d.f., defined by $p(t) = \theta e^{-\theta t}$, $\theta > 0$, $t > 0$, is used to model the behavior of a positive r.v. bounded at zero. The d.f. is $P(t) = 1 - e^{-\theta t}$. The exponential law is a frequent candidate to represent the waiting time for an event (e.g. the next accident, the next free slot in a queue) whose rate of occurence is $1/\theta$. The law $E(1)$ is a special case of the *Gamma* distribution $G_k(x)$ with $k = 1$, and it is logically related to the Poisson distribution.

The moments of the exponential distribution $E(\theta)$ are: $\mu = 1/\theta$; $\sigma^2 = 1/\theta^2$; $\gamma_1 = 2$; $\gamma_2 = 6$. The mode is zero, and the median equals $(\ln 2)/\theta \approx 0.6931/\theta$.

Factorial (function), $n!$

The factorial function of n, denoted by $n!$, represents the number of permutations, or linear arrangements, that can be formed with n different elements. It is easily shown that $n! = n \times (n-1)! = n(n-1)(n-2) \cdots 2 \cdot 1$; conventionally, $0! = 1$. For instance, $1! = 1$, $2! = 2 \times 1 = 2$, $3! = 3 \times 2 \times 1 = 6$, and so on. As was the case for the *Gamma* function to which it is related, one can use the following approximating formula: $n! \approx \sqrt{(2\pi)} n^{n+½} e^{-n} [1 + (12n)^{-1}]$, where $\sqrt{(2\pi)} \approx 2.5066$.

Example: $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40320$ or, with the approximating formula, $8! \approx \sqrt{(2\pi)} \times 8^{8.5} e^{-8} \times [1 + 1/(12 \times 8)] \approx 40318.05$.

Factorial (ascending $n^{(m)}$, descending $n_{(m)}$)

The *ascending* factorial function $n^{(m)}$ is the product of m increasing integers, calculated as $n(n+1) \cdots (n+m-1)$; the *descending* factorial function $n_{(m)}$ is the product of m decreasing integers, $n(n-1) \cdots (n-m+1)$. Note that $n_{(n)} = n!$, $n_{(0)} = n^{(0)} = 1$.

Examples: $5_{(3)} = 5 \times 4 \times 3 = 60$; $4^{(2)} = 4 \times 5 = 20$; $n_{(1)} = n^{(1)} = n$. By extension, we can also compute $4.12_{(3)} = 4.12 \times 3.12 \times 2.12 = 27.251328$ and $3.62^{(2)} = 3.62 \times 4.62 = 16.7244$.

Gamma [*Gamma* distribution $G_k(x)$, *Gamma* function $\Gamma(x)$]

The *Gamma* p.d.f., defined by $p(t) = [\Gamma(k)]^{-1}t^{k-1}e^{-t}$, $e \approx 2.7183$, $k > 0$, $t > 0$, where $\Gamma(k)$ is the *Gamma* function and is evaluated with $\int_0^\infty t^{k-1}e^{-t}\,dt$, is used to model the behavior of a positive r.v. bounded at zero. The d.f., $P(x) = \Pr(t \le x) = \int_0^x p(t)\,dt$, may be indicated by $G_k(x)$. The density function $p(t) = [\Gamma(k)]^{-1}\theta^k t^{k-1}e^{-\theta t}$, $\theta > 0$, is yet another form. Note that law $G_1(x)$, with $\theta = 1$, is in fact the exponential law $E(1)$, and that law $G_k(x)$, integer k, corresponds to a sum of k $E(1)$ random variables: this form is also called the Erlang distribution. Random variable χ^2_ν (Chi-square) is in fact a *Gamma* variable, with $k = \nu/2$ and $\theta = ½$.

The moments of the *Gamma* distribution (for any $\theta>0$) are: $\mu = k/\theta$; $\sigma^2 = k/\theta^2$; $\gamma_1 = 2/\sqrt{k}$; $\gamma_2 = 6/k$. The mode sits at $(k-1)/\theta$.

The *Gamma* function, $\Gamma(k)$, is defined for all k except $k = 0, -1, -2$, etc. It is recursive, according to $\Gamma(k) = (k-1)\Gamma(k-1)$, and, for integer k, it can be computed as a simple factorial, *i.e.* $\Gamma(n+1) = n!$. We can also approach its value with Stirling's expansion: $\Gamma(k+1) = \sqrt{(2\pi)}k^{k+½}e^{-k}[1+(12k)^{-1}]$. For the cases, frequently encountered in statistical work, where k is given in half-units, such as $k = 4.5$ or 11.5, then $\Gamma(k) = (k-1)(k-2) \cdots ½\sqrt{\pi}$; another formula is $\Gamma(n+½) = [1\cdot3\cdot5\cdot \cdots \cdot(2n-1)\sqrt{\pi}]/2^n$.

Examples: $\Gamma(5) = (5-1)! = 24$; $\Gamma(3.5) = 2.5\times1.5\times0.5\times\sqrt{\pi} = [1\cdot3\cdot5\sqrt{\pi}]/2^3 \approx 3.32335$, or else, approximately, $\Gamma(3.5) = \Gamma(2.5+1) \approx \sqrt{(2\pi)}\times2.5^3e^{-2.5}[1 + 1/(12\times2.5)] \approx 3.32211$.

Integration (analytic, direct)

By integration, or mathematical integration, is meant the process of calculating the area controlled by a function of (usually) one variables, say X, and delimited by two or more bounds. In statistical work, the function to be integrated has the usual form: $y = f(x)$, where $f(x)$ is a p.d.f.; the probability area is delimited vertically between zero (the horizontal axis) and y, the probability density, and horizontally by the bounds X $= a$ and X $= b$, and it is indicated by $\int_a^b f(x)\,dx$. If one can find or come up with the antiderivative of $f(x)$, *i.e.* a function $F(x)$ whose first derivative is $f(x)$, then the searched-for area is simply $F(b)-F(a)$. This is a standard form of direct, or analytic integration. Many other forms exist, especially numerical integration.

The reader is referred to any of the numerous textbooks and handbooks on calculus for more information.

Integration (numerical) (*see* also Simpson's rule, analytic Integration)

For lack of a proper integration formula or if the antiderivative of $f(x)$ is too complex to be determined, one can evaluate the searched-for area: 1) by segmenting the integration interval (a, b) in small equal sub-intervals; (2) by finding the approximate area comprised in each sub-interval; and (3) by summing the series of areas thus obtained. The approximation method in each sub-interval is *ad lib* (trapezoidal rule, Simpson's parabolic rule, etc.), the whole process being call numerical integration. The precision of the result depends essentially on how fine is the segmentation applied.

Example: Let $Q = \int_0^1 \sqrt{x}\, dx = 2/3$, this integral being now approximated with the trapezoidal rule. Using one trapezoid, $f(0) = \sqrt{0} = 0$, $f(1) = 1$, Q_1 = (area of a trapezoid) = (Base × Mean height) = $1 \times \frac{1}{2}(0+1) = 0.5$. Partitioning the (0, 1) interval in two, we must use $f(0) = 0$, $f(\frac{1}{2}) \approx 0.7071$ and $f(1) = 1$, whence $Q_2 = (\text{area})_1 + (\text{area})_2 = \frac{1}{2} \times \frac{1}{2}(0+0.7071) + \frac{1}{2} \times \frac{1}{2}(0.7071+1) = 0.60355$. We thus obtain successively $Q_3 \approx 0.63128$, $Q_4 \approx 0.64238, \ldots$, $Q_{n\to\infty} \to 2/3$.

Interpolation (linear, harmonic)

Interpolation techniques are applied, notably in mathematical tables, for estimating the unknown value of a function, $y_x = f(x)$, from two (or more) known values who embrace it, *i.e.* values y_a and y_b such that $a < x < b$. We keep here to the general method of linear interpolation. When it is applied to data pairs (x, y), it is called (strictly) *linear interpolation*; when applied to pairs $(1/x, y)$, it is called *harmonic interpolation*, and we speak of logarithmic interpolation for $(x, \ln y)$, and so on. For the simple case of strict linear interpolation, on data pairs (x, y), the estimation formula is:

$$y_x \approx y_a + (y_b - y_a) \times [(x-a)/(b-a)] \ .$$

Adaptation of this formula to various contexts is illustrated in the following.

Usage in statistical tables favors harmonic interpolation when a value must be estimated across two numbers of degrees of freedom (*df*), and it favors occasionally logarithmic interpolation when a value must be estimated across two extreme percentage points of a distribution. In most cases, though, simple linear interpolation, as formulated above, do the job.

Example 1 [Linear interpolation]. Let us try to estimate $\Phi(0.302)$, the standard normal d.f., using $\Phi(0.30) \approx 0.61791$ et $\Phi(0.31) \approx 0.62172$. Applying the linear interpolation formula above, we write $\Phi(0.302) \approx 0.61791 + (0.62172-0.61791)\times [(0.302-0.300) / (0.310-0.300)] = 0.61791 + 0.00381\times 0.2 = 0.61867$, here a result with 5-digits accuracy.

Example 2 [Harmonic interpolation]. Let us try to estimate $t_{50[0.95]}$ from $t_{30[0.95]} \approx 1.697$ and $t_{\infty[0.95]} \approx 1.645$. Here, using the reciprocal of argument x, *i.e.* $1/x$, we compute $t_{50} \approx 1.697 + (1.645-1.697)\times[(50^{-1}-30^{-1})/(\infty^{-1}-30^{-1})] = 1.697 - 0.052\times 0.4 = 1.676$, a result precise to 3 decimal places.

Mean (of a random variable), μ or E(X), $\bar{X}$

See Moments of a distribution [μ or E(X)]

See Moment estimates [$\bar{X}$]

Moments of a distribution [μ, σ^2, γ_1, γ_2]

The distribution of a r.v., which is governed by p.d.f. $p(X)$, can be characterized or described summarily by its *moments*. Simple moments (μ'_r) and central moments (μ_r) of order r correspond respectively to[1]:

$$\mu'_r = E[X^r] = \int_{-\infty}^{\infty} x^r\, p(x)\, dx \; ; \;\; \mu_r = E[(X-\mu)^r] = \int_{-\infty}^{\infty} (x-\mu)^r\, p(x)\, dx \; ,$$

where $\mu'_1 = E(X) = \mu$ denotes the (mathematical) expectation of X, its "mean". Central moments μ_r can be obtained from simple moments μ'_r thanks to the binomial expansion, with $\mu_r = E(X-\mu)^r = \mu'_r - r\mu'_{r-1}\mu + \binom{r}{2}\mu'_{r-2}\mu^2 - \ldots \pm \mu^r$. Note particularly that $\mu_1 = 0$, $\mu_2 = \mu'_2 - \mu^2$, $\mu_3 = \mu'_3 - 3\mu'_2\mu + 2\mu^3$, $\mu_4 = \mu'_4 - 4\mu'_3\mu + 6\mu'_2\mu^2 - 3\mu^4$.

The four foremost statistics used to characterize the distribution of a r.v. are:

μ : (= μ'_1) the expectation (or mean), which points to a central position, the centre of gravity of $p(X)$ on the axis of (values of) X;

σ^2 : (= μ_2) the variance, or squared standard deviation (σ), which indicates the amount of scatter of X, the spread of $p(X)$ around μ;

[1] The moment expressions are given here for *continous* random variables. Equivalent expressions, with the summation (Σ) operator replacing the integration ($\int$) operator, apply for *discrete* or integer-valued r.v.'s.

γ_1 : (= $\mu_3/\mu_2^{3/2} = \mu_3/\sigma^3$) the skewness (or asymmetry) index (also denoted by α_3 or $\sqrt{\beta_1}$), which indicates the degree of positive ($\gamma_1 > 0$) or negative skewness ($\gamma_1 < 0$), or the symmetry ($\gamma_1 = 0$) of $p(X)$;

γ_2 : (= $\mu_4/\mu_2^2 - 3$) the kurtosis index (also denoted by $\alpha_4 - 3$ or $\beta_2 - 3$), which indicates whether distribution $p(X)$ is more flattened ($\gamma_2 < 0$), more peaked ($\gamma_2 > 0$) or of a similar shape ($\gamma_2 = 0$) as compared to the normal p.d.f.

Indices γ_1 and γ_2 are sometimes named "shape coefficients" and "moment ratios".

Each moment index (or moment ratio) may be estimated on a random sample of n data values, allowing the calculation of simple moment estimates $\hat{\mu}'_r = \sum_i x_i^r / n$ and their derived values.

Moment estimates [$\bar{X}$, s^2, g_1, g_2]

A statistical series, or sample of n data points { $x_1, x_2, \cdots, x_n$ }, can be described by summary statistics similar to the moments of the distribution of a r.v.. These statistics are also called "moment estimates", "sample estimates" or "sample moments", and they are used to estimate corresponding moments of the distribution. Here are the most noteworthy:

$\bar{X}$: the (sample) mean, or average, calculated as $\sum x_i / n$. The sample mean is an unbiased estimate of the expectation, *i.e.* $E(\bar{X}) = E(X) = \mu$.

s^2 : the (sample) variance, calculated as $\sum (x_i - \bar{X})^2 / (n-1)$. The sample variance (with divisor $n-1$) is an unbiased estimate of the variance, *i.e.* $E(s^2) = \sigma^2$.

s : the (sample) standard deviation, equal to $\sqrt{s^2}$. The sample standard deviation is a biassed estimate of σ. In the case of the normal distribution, for instance, $E(s) \approx \sigma\{1 - 1/[4(n-1)]\}$.

g_1, g_2 : the sample estimates g_1 and g_2 for γ_1 and γ_2 are calculated respectively as:

$$g_1 = \frac{\sum (x_i - \bar{X})^3}{(n-1)\, s^3} \quad ; \quad g_2 = \frac{\sum (x_i - \bar{X})^4}{(n-1)\, s^4} - 3$$

Poisson distribution, Po(λt)

The Poisson distribution describes the probability of occurrence of a number x (= 0,1,2,...) of events during time interval t where the occurrence rate of an event is λ; the p.m.f. is $p(x) = e^{-\lambda t}(\lambda t)^x/x!$. The d.f. is the sum $P(x) = \sum_{y=0}^{x} p(y)$; it may also be derived from a relation with the Chi-square distribution, *i.e.* $P(x\,|\,\lambda t) = 1 - P_{\chi^2}(2\lambda t\,|\,\nu{=}2x{+}2)$. The p.m.f. is also written as $e^{-\theta}\theta^x/x!$, where $\theta = \lambda t$. The Poisson law is used to model the behavior of rare events, as to their number of occurences during some time interval. It can also be used to approximate the binomial distribution (*see* appropriate section), of parameters π and n, $\theta = n\pi$, when $\pi \to 0$. It is logically related to the exponential and *Gamma* distributions.

The moments of the Poisson distribution, using $\theta = \lambda t$, are: $\mu = \sigma^2 = \theta$, $\gamma_1 = 1/\sqrt{\theta}, \gamma_2 = 1/\theta$. The mode (Mo) and median (Md) are two integers such that $\theta - 1 \leq \text{Mo} \leq \theta$, and $\lfloor\theta\rfloor \leq \text{Md} \leq \lfloor\theta\rfloor + 1$ ($y = \lfloor x \rfloor$ is the integral part of x).

Probability density function, *p(x)*

Abbreviated as p.d.f., the probability density function is also referred to as density function, frequency function, and, for discrete or integer-valued variables, as probability mass function (p.m.f.); it is denoted by $p(x)$, sometimes $f(x)$. In a graph of the distribution of r.v. X, the p.d.f. $p(x)$ is the ordinate of the point whose abscissa is $X = x$.

Let $P(x) = \Pr(X \leq x)$, the probability that r.v. X lies in the interval bounded upward by value x. Then, $P(x+h) - P(x)$ denotes the probability that x occur in the interval $(x, x+h)$, and $[P(x+h) - P(x)]/h$ denotes a density of probability. The p.d.f. appears as a mathematical limit, *i.e.* $[P(x+h) - P(x)]/h \to p(x)$ as $h \to 0$; conversely, the p.d.f. $p(x)$ is the (first) derivative of the d.f. $P(x)$, or $dP(x) = p(x)\,dx$.

Probability distribution function, *P(x)*

See Distribution function.

Simpson's (parabolic) rule (*see* numerical Integration)

The integration method known as Simpson's rule is used to approximate the area of a surface under an arbitrary (continuous) function $f(x)$ in the (a, b) interval, by segmenting the interval in many sub-intervals and in mimicking the variation of $f(x)$ in each sub-interval with a 2nd degree linear function, *i.e.* a parabolic arc. Simpson's rule demands that the original interval be partitioned in $2n$ segments allowing the position of n arcs. In each segment, we get the triple of coordinates $\{(x_L, y_L), (x_C, y_C), (x_R, y_R)\}$ corresponding to the Left, Center and Right positions in the segment. Then, using $y = f(x)$, the area component for that segment is: (Area) = $[y_L + 4y_C + y_R] \times \frac{1}{6}(x_R - x_L)$. Easy to implement in a computer program, this method is the most efficient among the so-called simple methods of numerical integration.

Example 1. Let $Q = \int_0^1 \sqrt{x}\,dx = 2/3$. In order to position a (single) parabolic arc in the integration interval (0, 1), we must have $y_L = f(0) = 0$, $y_C = f(½) \approx 0.7071$, and $y_R = f(1) = 1$, whence $Q_1 = [0+4\times0.7071+1]\times\frac{1}{6}(1-0) = 0.63807$. For two arcs and using ordinates $f(0)$, $f(¼)$, $f(½)$, $f(¾)$ and $f(1)$, we get $Q_2 \approx [0+4\times0.5+0.7071]\times\frac{1}{6}(0.5)+[0.7071+4\times0.8660+1]\times\frac{1}{6}(0.5) = [0+4\times0.5+2\times0.7071+4\times0.8660+1]\times\frac{1}{6}(0.5) = 0.65652$. Likewise, we could calculate $Q_3 \approx 0.66115$, $Q_4 \approx 0.66308$, ···, $Q_{n\to\infty} \to 2/3$.

Example 2. Let $\Phi(1) \approx 0.841344746$. where $\Phi(z)$ is the standard normal d.f. (*see* section on the normal distribution), and $f(z) = \varphi(z) = \exp(-½z^2)/\sqrt{2\pi}$, the corresponding p.d.f. As the normal p.d.f. is symmetrical around $\mu = 0$ and $\Phi(0) = ½$, hence, if $z > 0$, $\Phi(z) = ½ + \int_0^z f(z)\,dz$. Using a single parabolic arc and ordinates $f(0) = 1/\sqrt{2\pi} \approx 0.39894$, $f(½) \approx 0.35207$, $f(1) \approx 0.24197$, we obtain $\Phi(1) \approx ½ + [0.39894+4\times0.35207+ 0.24197]\times\frac{1}{6}(1-0) = 0.84153$, a result precise to three decimal places.

Standard deviation (of a random variable), σ, s

See Moments of a distribution [σ]

See Moment estimates [s]

Uniform distribution, $U(a,b)$ and $U(0,1)$

A r.v. that takes any (continous) value in some interval (a, b) with equal probability is said to obey the uniform (or rectangular) distribution, with p.d.f. $p(x) = 1/(b-a)$, $a \le x \le b$, and d.f. $P(x) = (x-a)/(b-a)$. The special case with $a = 0$ and $b = 1$ defines the standard uniform distribution, denoted by $U(0,1)$; it is also a particular case of the *Beta* distribution, *i.e.* $\beta(1,1)$. It is also related to the (discrete) rectangular distribution (see below).

Notwithstanding its simplicity, the random variable $u \sim U(0,1)$ has a great importance in statistics and statistical applications (including sampling and Monte Carlo methods), because any r.v. x from distribution D can be transformed into r.v. u through its d.f.: effectively, if $f_D(x)$ and $F_D(x)$ are the p.d.f. and d.f. of x, then $y = F_D(x)$ is distributed as $U(0,1)$. This distribution is also the basis of algorithms and methods for generating r.v.'s pertaining to any other distribution function, be it by inverting the d.f., for instance $F_D^{-1}(u) \rightarrow x$, or by other means. Moreover, most computer languages and programming systems comprise one or a few generating functions for pseudo-random values of the uniform distribution, with function names such as "RND", "RANF", "RANDU", or our own symbolic function "UNIF".

The moments of the uniform distribution $U(a, b)$ are: $\mu = (a+b)/2$; $\sigma^2 = (a-b)^2/12$; $\gamma_1 = 0$; $\gamma_2 = -6/5$. The median (Md) equals μ, and there is no mode.

The set of values (1, 2, ..., N) are said to obey the so-called discrete, or integer-valued rectangular (or uniform) distribution, sometimes denoted by $R(1,N)$, if the probability of occurrence of each value (p.m.f.) is $p(x) = 1/N$. The d.f. is $P(x) = x/N$. The moments are: $\mu = (N+1)/2$; $\sigma^2 = (N^2-1)/12$; $\gamma_1 = 0$; $\gamma_2 = -\frac{6}{5}\cdot(N^2+1)/(N^2-1)$.

Variance (of a random variable), σ^2 or var(X), s^2

See Moments of a distribution [σ^2 or var(X)]

See Moment estimates [s^2]

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Index of examples

sorted by type of distribution

General index

Statistical Tables, Explained and Applied

This book contains several new or unpublished tables, such as one on the significance of the correlation coefficient *r*, one giving the percentiles of the $\breve{E}^2$ statistic for monotonic variation (with two structural models of variation), an extensive table for the number-of-runs test, three tables for the binomial sum of probabilities, and a table of coefficients for the re-conversion of orthogonal polynomials.

In the case of the more familiar tables, such as those of the normal integral, or Student's *t*, Chi-square and *F* percentiles, all values have been re-computed, occasionally with the authors' own algorithms, using the most accurate methods available today.

For each of the fifteen distributions in the book, the authors have gathered the essential information so that interested readers can handle by themselves all phases of the computations.

An appendix, containing supplementary examples that pertain to the various tables, helps to complete the authors' review of current hypothesis-testing procedures. A mini-dictionary of often-used concepts and methods, statistical as well as mathematical, completes the book.

Besides meeting the needs of practitioners of inferential statistics, this book should be helpful to statistics teachers as well as graduate students, researchers and professionals in scientific computing, who will all find it a rich source of essential data and references on the more important statistical distributions.

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